PROBLEMS 4

Q1 Symmetric stable processes.
A symmetric stable process $X = (X_t)$ with index $\alpha \in (0, 2]$ is defined by its characteristic function,
$$E\exp\{isX_t\} = \exp\{-ts^\alpha\}.$$  
Show that for $c > 0$ the process $X_c$, where $X_c(t) := c^{-1}X(c^\alpha t)$, is again a symmetric stable process with index $\alpha$, and so has the same distribution as $X$. Deduce that, as for Brownian motion, the sample path of $X$ is a fractal, and so is the zero set $Z$ of $X$.

Q2 Time inversion.
If $B$ is Brownian motion and $X_t := tB(1/t)$, show that $X$ has mean 0 and covariance
$$\text{cov}(X_s, X_t) = \text{stcov}(B(1/s), B(1/t)) = st \min(1/s, 1/t) = \min(t, s) = \min(s, t),$$
the same covariance as Brownian motion. Show also that $X$ has continuous paths, and is a Gaussian process. Deduce that $X$ is Brownian motion. (We say that $X$ is obtained from $B$ by time inversion).

Q3 Zero set of Brownian motion.
Deduce from Q2 and the fact that Brownian motion has zeros at times increasing to infinity that Brownian motion has zeros at times decreasing to zero. That is, if $B$ is Brownian motion started at 0 at time 0, then $B$ is zero at (random) times $t_n \downarrow 0$, with probability 1.

Note. 1. This means that it is impossible, even in principle, to draw a Brownian path! The best we can do is to draw it approximately.
2. Brownian paths have other properties that may seem bizarre or counter-intuitive at first glance. For instance, it can be shown that, a.s., a Brownian path is (not only continuous but also) nowhere differentiable. Such behaviour is in fact typical, or generic: in a sense that can be made precise, most continuous functions are nowhere differentiable.
3. It is hard to construct explicit examples of such things. This opens up an important use of Brownian motion in Analysis: a Brownian path may have a property a.s. for which it is hard to find specific examples, so giving a non-constructive existence proof.

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