3 Fluid flow at a large Reynolds number.

In our first application of matched asymptotic expansions to problems of fluid dynamics, let us consider a two-dimensional flow past a solid body, B, at a large Reynolds number. In non-dimensional variables we have the Navier-Stokes equations derived in Section 1 (see equations (1.71)-(1.74)) which, omitting over bars, can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + Re^{-1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + Re^{-1}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.3}$$

As boundary conditions we take a uniform stream far from the body,

$$u \to 1,\ v \to 0,\ p \to 0 \text{ as } x^2 + y^2 \to \infty, \tag{3.4}$$

and the no-slip conditions on the surface of the body,

$$u = v = 0 \text{ at } \mathbf{r} = \partial B, \tag{3.5}$$

$\mathbf{r}$ being the position vector in the (x, y)-plane.

One needs to be a little careful with formulating far-field conditions, especially when the solid surface extends to infinity downstream, however in every particular case this should not be an issue.

3.1 Outer limit - inviscid flow.

Let $Re \to \infty$. We attempt solution for the flow as an expansion,

$$u = u_0(x, y, t) + ...,\ v = v_0(x, y, t) + ...,\ p = p_0(x, y, t) + ..., \tag{3.6}$$

where the neglected terms are small but at this stage we do not know how small. The Navier-Stokes equations suggest correction terms of order Re^{-1}but in fact the corrections prove to be $O(Re^{-1/2})$. On substitution into (3.1)-(3.3), we arrive at the system of inviscid flow Euler equations,

$$\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} = -\frac{\partial p_0}{\partial x}, \tag{3.7}$$

$$\frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} = -\frac{\partial p_0}{\partial y}, \tag{3.8}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0. \tag{3.9}$$

In the free stream, the conditions (3.4) hold giving,

$$u_0 \to 1,\ v_0 \to 0,\ p_0 \to 0 \text{ as } x^2 + y^2 \to \infty. \tag{3.10}$$

As for the conditions on the body surface, the situation is somewhat complicated. By taking the large Reynolds number limit we have lost the highest derivatives in the governing equations, so some of the boundary conditions at $\mathbf{r} = \partial B$ need to sacrificed. Near the body surface, the fluid velocity has a component normal to the wall, $\mathbf{v}_n$say, and a tangential velocity, $\mathbf{v}_t$. For an inviscid flow, it is 'natural' to assume that the fluid cannot cross the body surface, hence

$$\mathbf{v}_n = 0 \text{ at } \mathbf{r} = \partial B, \tag{3.11}$$

whereas the magnitude of the tangential velocity remains unspecified.

In applications, the wall conditions in the inviscid flow can vary. For example, if the volume of the body expands in time, the no-penetration condition turns into a statement that the normal to the boundary velocity component in the fluid matches the normal wall velocity. The key message is that for an inviscid flow we are not allowed to restrict the fluid velocity along the boundary: inviscid fluid can slide along solid walls. The magnitude of this slip velocity is determined by the inviscid flow itself.

3.2 Flow past a flat plate.

To be more specific, we now choose the solid body to be a semi-infinite flat plate aligned with the incoming flow, i.e. the solid wall is given by

$$y = 0,\ x \geq 0. \tag{3.12}$$

Due to symmetry, we can restrict ourselves to the flow in the upper half-plane. The wall conditions for the Euler equations (3.7)-(3.9) reduce now to

$$v = 0 \text{ at } y = 0. \tag{3.13}$$

An obvious solution of (3.7)-(3.9) with (3.10) and (3.13) is

$$u_0 = 1,\ v_0 = 0,\ p_0 = 0. \tag{3.14}$$

3.3 The inner expansion - boundary layer.

The solution in the inviscid, outer, region gives a unit slip velocity at the wall. This slip needs to be reduced to zero at the wall, as required by the wall conditions (3.5) for the Navier-Stokes equations. We therefore expect an inner solution to develop in a slender region surrounding the wall. An estimate for the thickness of this region

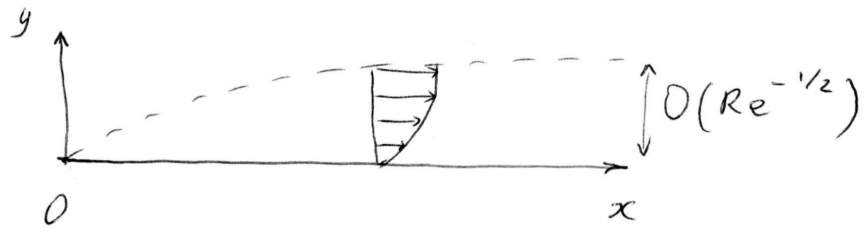

Figure 1: Boundary layer on a flat plate.

follows from the balance between advective terms on the left in the momentum equation (3.1) and a leading viscous term on the right,

$$u\frac{\partial u}{\partial x} \sim Re^{-1}\frac{\partial^2 u}{\partial y^2}. \tag{3.15}$$

With $u = O(1)$ and the x-scale also or $O(1)$ this gives $y = O(Re^{-1/2})$ for the thickness of the boundary layer. Also, from the balance of terms in the continuity equation (3.3) we deduce the estimate $v = O(Re^{-1/2})$. We therefore define the scaled inner variable,

$$y = Re^{-1/2}Y, \tag{3.16}$$

and attempt solution in the boundary layer in the form,

$$u = U(x,Y,t)+O(Re^{-1/2}), v = Re^{-1/2}V(x,Y,t)+O(Re^{-1}), p = P(x,Y,t)+O(Re^{-1/2}). \tag{3.17}$$

On substitution into (3.1)-(3.3) we obtain the standard set of the boundary-layer equations,

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial x} + \frac{\partial^2 U}{\partial Y^2}, \tag{3.18}$$

$$\frac{\partial P}{\partial Y} = 0, \tag{3.19}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0. \tag{3.20}$$

Next, for the boundary conditions. At the wall we impose no-slip conditions,

$$U = 0,\ V = 0 \text{ at } Y = 0 \text{ for } x \geq 0. \tag{3.21}$$

At the outer edge of the boundary layer the flow functions need to be matched with those in the outer region. We then have,

$$U \to 1,\ P \to 0 \text{ as } Y \to \infty, \tag{3.22}$$

and no matching is required for the normal velocity V as we are not considering terms of order $Re^{-1/2}$in the outer region.[1]

At this stage we can already find the pressure function. From (3.19), $P = P(x,t)$, and the second matching condition in (3.22) shows that $P = 0$.

We also need some conditions upstream. As we are formulating the problem for the upper half plane, ahead of the solid plate we require flow symmetry hence

$$\frac{\partial U}{\partial Y} = V = 0 \text{ at } Y = 0 \text{ for } x < 0. \tag{3.23}$$

Given also the fact that far upstream the flow remains unperturbed, we conclude that the uniform stream, $U = 1, V = 0$, is a valid solution inside the boundary layer upstream of the leading edge of the plate. We can say that he boundary layer does not propagate information about the existence of a flat plate upstream of the leading edge. This feature is related to the parabolic nature of the boundary-layer equations (as in the conventional diffusion equation) which holds as long as $U > 0$. To summarize here, the upstream condition, $U \to 1$, can be formulated right at the leading edge of the plate, at $x = 0$, rather than in the incoming stream where $x \to -\infty$.

Nothing in the formulation so far suggests a time-dependent solution, so we drop the time derivative, use the fact that the pressure does not vary in the boundary layer and collect the equations and the boundary conditions in the following formulation applicable in the region $x \geq 0, Y \geq 0$:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2},\ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0, \tag{3.24}$$

$$U \to 1 \text{ as } Y \to \infty \text{ for } x \geq 0, \tag{3.25}$$

$$U = 0,\ V = 0 \text{ at } Y = 0 \text{ for } x \geq 0, \tag{3.26}$$

$$U = 1 \text{ at } x = 0 \text{ for } Y > 0. \tag{3.27}$$

The boundary layer-formulation (3.24)-(3.27) does not have a characteristic length scale. This is a typical indication of a self-similar form of the solution which can be derived as follows. The continuity equation is satisfied by introducing the stream-function, $\Psi(x,Y)$, such that $U = \partial\Psi/\partial Y, V = -\partial\Psi/\partial x$, leading to the formulation,

$$\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial x \partial Y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial Y^2} = \frac{\partial^3 \Psi}{\partial Y^3}, \tag{3.28}$$

[1] However the very fact that the normal velocity inside the boundary layer is of order $Re^{-1/2}$ indicates an outer expansion proceeding in powers of $Re^{-1/2}$ rather than Re^{-1}as suggested by the form of the Navier-Stokes equations.

$$\Psi \sim Y \text{ as } Y \to \infty \text{ for } x \geq 0, \tag{3.29}$$

$$\Psi = \frac{\partial \Psi}{\partial Y} = 0 \text{ at } Y = 0 \text{ for } x \geq 0, \tag{3.30}$$

$$\Psi = Y \text{ at } x = 0 \text{ for } Y > 0. \tag{3.31}$$

Let $\Psi = x^\alpha f(\eta)$ with $\eta = Y/x^\beta$ with constant α and β. Then

$$\frac{\partial \Psi}{\partial Y} = x^{\alpha-\beta} f'(\eta), \ \frac{\partial \Psi}{\partial x} = x^{\alpha-1}\left[(\alpha-\beta)f'(\eta) - \beta\eta f(\eta)\right], \tag{3.32}$$

with similar expressions for the other derivatives. Substitution into (3.28) yields,

$$x^{2\alpha-2\beta-1}\left[(\alpha-\beta)(f')^2 - \alpha f f''\right] = x^{\alpha-3\beta} f'''. \tag{3.33}$$

Since x and η are independent, we require

$$2\alpha - 2\beta - 1 = \alpha - 3\beta. \tag{3.34}$$

The boundary condition at the outer edge of the boundary layer (3.29) reads $x^\alpha f(\eta) \sim x^\beta \eta$ as $\eta \to \infty$. From this single condition two requirements emerge: $\alpha = \beta$ and also $f(\eta) \sim \eta$ as $\eta \to \infty$. Perhaps surprisingly, the initial condition (3.31) gives no extra information and simply reasserts the condition at the outer edge. The wall conditions require that f and its derivative vanish at the wall. Hence $\alpha = \beta = 1/2$ and the similarity part of the streamfunction is determined from

$$f''' + \frac{1}{2} f f'' = 0; \ \ f(0) = f'(0) = 0; \ \ f(\eta) \sim \eta \text{ as } \eta \to \infty. \tag{3.35}$$

This formulation is known as the Blasius solution for a flat-plate boundary layer.

The graph of the velocity profile, $U = f'(\eta)$, is shown in the figure.

Properties. We have

$$f(\eta) = \frac{\lambda_0}{2}\eta^2 + ... \text{ as } \eta \to 0, \tag{3.36}$$

with $\lambda_0 = 0.332$ approximately. This means that the near-wall velocity in the boundary layer is given by

$$U(x, Y) = \lambda_0 \frac{Y}{x^{1/2}} + ... \text{ , as } Y \to 0, \tag{3.37}$$

hence the wall shear decreases downstream as the boundary layer thickens with growing x.

Exercise. Falkner-Skan boundary layer. The boundary-layer equations (3.18-3.20) are solved subject to the no-slip conditions (3.21) and with the outer-edge condition written as

$$U \to U_0 x^m \text{ as } Y \to \infty. \tag{3.38}$$

Suggest a similarity form for the solution in the boundary layer. Do such boundary layers have a physical meaning?

Discussion. What changes if the plate has finite length, for example $0 \leq x \leq 1$? What is the flow field in the near wake? In the far wake?

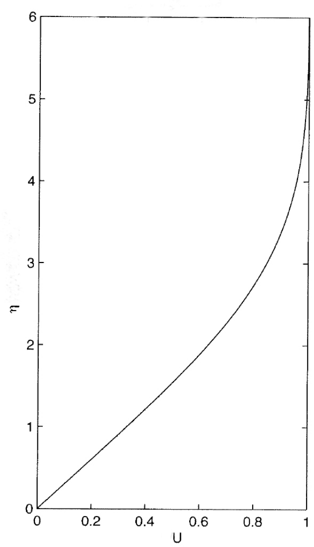

Figure 2: Velocity profile in the Blasius boundary layer.