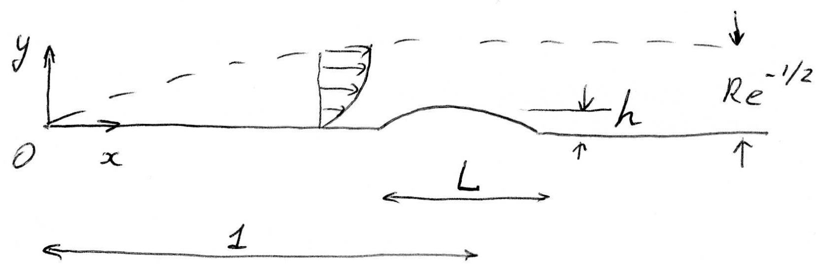

Figure 1: Flow past a hump on a flat plate.

4 Flow past a wall roughness. Estimates.

In this section the geometry of the solid body becomes more complicated. We introduce an irregularity (a bump or indentation) on the otherwise smooth surface and examine how the boundary layer flow reacts to such a disturbance in the wall shape.

The roughness is centered at $x = 1$ and has a characteristic length scale $L \ll 1$ and height scale h. The flow arriving at the location of the roughness is divided into two asymptotic layers: the inviscid current and the boundary layer of thickness $O(Re^{-1/2})$. Flow response to the wall obstacle depends on the obstacle geometry which can be more or less arbitrary, however not every choice of the roughness parameters leads to new or interesting flow responses. For instance, flow past a sufficiently elongated obstacle is not dissimilar to the flow past a body in a uniform stream: the inviscid flow determines the slip velocity on the surface of the body, which is then corrected inside the boundary layer. Also, flow past a long and very shallow roughness is likely to be just a small, linear, correction to the base Blasius flow.

Our task, therefore, will be to establish the length scale L short enough to cause deviations from the standard boundary-layer hierarchy of the boundary-layer theory. As for the obstacle height, we will be looking for marginally non-linear responses in the flow, i.e. h needs to be large enough to provoke first appearance of non-linearity in the flow.

In the inviscid flow just ahead of the obstacle, the base flow is

$$u = 1 + ..., v = O(Re^{-1/2}), p = O(Re^{-1/2}). \tag{4.1}$$

Consider first the inviscid part of the flow near the obstacle with characteristic dimensions $\Delta x \sim \Delta y \sim L$. When the flow traces the obstacle of height h and length L, the normal velocity component triggered by the wall geometry is of order $\Delta v \sim h/L$ assuming a sufficiently shallow roughness with $h \ll L$. From the continuity equation written as an order of magnitude balance,

$$\frac{\Delta u}{\Delta x} \sim \frac{\Delta v}{\Delta y}, \tag{4.2}$$

we have a correction to the streamwise velocity estimated as $\Delta u \sim h/L$. Then from the momentum balance,

$$u\frac{\partial u}{\partial x} + ... \sim \frac{\partial p}{\partial x}, \text{ or } (1 + ...)\frac{\Delta u}{\Delta x} + ... \sim \frac{\Delta p}{\Delta x}, \tag{4.3}$$

we obtain an estimate for the pressure induced by the wall roughness, $\Delta p \sim h/L$.

Now we move on to the flow underneath the inviscid region and evaluate the effect of the extra pressure Δp. We can reasonably expect that the part of the flow most affected by the additional pressure will be closest to the wall, where the velocity of the fluid particles is small (and where even a small change in the pressure can have a relatively strong nonlinear effect). Since the incoming boundary-layer thickness is of $O(Re^{-1/2})$, the flow velocity in the innermost part of the boundary layer is of order

$$u \sim yRe^{1/2}. \tag{4.4}$$

Consider once again the balance of terms in the streamwise momentum equation, this time keeping an eye on the leading viscous term alongside the advection pressure terms, i.e.

$$u\frac{\partial u}{\partial x} \sim \frac{\partial p}{\partial x} \sim Re^{-1}\frac{\partial^2 u}{\partial y^2}. \tag{4.5}$$

As a reaction to the pressure $\Delta p \sim h/L$, the flow velocity changes by δu and the change will be nonlinear if $\delta u \sim u$. Balance of the advection and pressure gradient gives then

$$u \sim \delta u \sim \sqrt{\Delta p} = \sqrt{\frac{h}{L}}. \tag{4.6}$$

In addition, viscous forces come into play if

$$u\frac{\delta u}{\Delta x} \sim Re^{-1}\frac{\delta u}{(\delta y)^2}, \tag{4.7}$$

where δy is the thickness of the local near-wall viscous layer (or rather a sublayer inside the base-flow boundary layer). Taking $y \sim \delta y$ in the estimate (4.4) and applying this estimate to (4.6),(4.7) we have, for an obstacle of length L, the following estimates for the flow in the viscous sublayer,

$$u \sim L^{1/3},\ \Delta p \sim L^{2/3},\ y \sim L^{1/3}Re^{-1/2}, \tag{4.8}$$

and the obstacle height for the flow to be nonlinear evaluated as

$$h \sim L^{5/3}. \tag{4.9}$$

Clearly, when $L = O(1)$ we recover the standard boundary-layer estimates for the flow past a solid body with both spatial dimensions of $O(1)$.

Now we introduce into consideration one more effect. From mass conservation it follows that fluid filaments become thinner when the fluid accelerates (and vice versa, deceleration leads to a thickening of liquid layers). This can be seen from the continuity equation written for the viscous sublayer as

$$\frac{\delta u}{\delta x} \sim \frac{\delta v}{\delta y}. \tag{4.10}$$

When the streamwise velocity changes, a normal velocity component is induced and the streamlines acquire an additional slope, ϕsay, estimated as

$$\phi \sim \frac{\delta v}{\delta u} \sim \frac{\delta y}{\delta x} = \frac{Re^{-1/2}}{L^{2/3}}, \tag{4.11}$$

where we have used the earlier estimate (4.8) for $\delta y \sim y$.

To recap, the wall roughness induces a pressure correction in the inviscid flow due to changes in the wall slope. The additional pressure acts on the viscous sublayer, changing the flow velocity and consequently changing once again the slope of the streamlines. If the two changes in the inclination of streamlines are comparable in magnitude, the flow in the viscous sublayer begins to interact with the flow in the inviscid region outside the boundary layer. Such viscous-inviscid interaction takes place when $\phi \sim h/L$, i.e. when $L \sim Re^{-3/8}$, using (4.9).

For the interaction regime the flow quantities in the viscous sublayer are estimated as

$$y \sim Re^{-5/8},\ u \sim \delta u \sim Re^{-1/8},\ v \sim Re^{-3/8},\ \Delta p \sim Re^{-1/4}. \tag{4.12}$$