6 Linearized viscous sublayer. Flow past a shallow obstacle.

On eliminating λ_0 by means of an affine transformation, and simplifying notation, the triple-deck equations derived in the previous section can be written as,

$$u_t + uu_x + vu_y = -p_x + u_{yy},\ p_y = 0, \tag{6.1}$$

$$u_x + v_y = 0, \tag{6.2}$$

$$p = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial A(s,t)}{\partial s} \frac{ds}{s-x}, \tag{6.3}$$

$$x \to -\infty: \quad u \to y,\ v \to 0,\ p \to 0, \tag{6.4}$$

$$y \to \infty: \quad u = y + A(x,t) + ..., \tag{6.5}$$

$$y = f(x,t): \quad u = 0,\ v = f_t. \tag{6.6}$$

6.1 Prandtl transposition.

It is convenient to shift the y coordinate so that the lower boundary of the solution domain becomes in effect 'flat', i.e. to make the following change of the independent variables known as the Prandtl transposition:

$$\tilde{y} = y - f(x,t),\ \tilde{x} = x,\ \tilde{t} = t, \tag{6.7}$$

and introduce the new flow functions,

$$\tilde{u} = u,\ \tilde{v} = v - f_t - uf_x,\ \tilde{p} = p,\ \tilde{A} = A. \tag{6.8}$$

The change in the y-component of the velocity has the effect of placing the observer into a frame of reference moving with the wall vertically (the term f_t) and also rotated locally by an angle $\arctan(f_x)$ to align the flow field with the wall roughness.

Formally, by the chain rule,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} - f_t \frac{\partial}{\partial \tilde{y}},\ \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} - f_x \frac{\partial}{\partial \tilde{y}},\ \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{y}}, \tag{6.9}$$

and the sublayer formulation becomes

$$\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}} = -\tilde{p}_{\tilde{x}} + \tilde{u}_{\tilde{y}\tilde{y}},\ \tilde{p}_{\tilde{y}} = 0, \tag{6.10}$$

$$\tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} = 0, \tag{6.11}$$

$$\tilde{p} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial \tilde{A}(s,\tilde{t})}{\partial s} \frac{ds}{s-\tilde{x}}, \tag{6.12}$$

$$\tilde{x} \to -\infty: \quad \tilde{u} \to \tilde{y},\ \tilde{v} \to 0,\ \tilde{p} \to 0, \tag{6.13}$$

$$\tilde{y} \to \infty: \quad \tilde{u} = \tilde{y} + \tilde{A}(\tilde{x}, \tilde{t}) + f(\tilde{x}, \tilde{t}) + ..., \tag{6.14}$$

$$\tilde{y} = 0: \quad \tilde{u} = 0,\ \tilde{v} = 0. \tag{6.15}$$

As we can see, the governing equations are invariant under the Prandtl shift but the boundary conditions become simpler.

6.2 Linearization.

In what follows we drop the tilde and work with the formulation,

$$u_t + uu_x + vu_y = -p_x + u_{yy},\ p_y = 0, \tag{6.16}$$

$$u_x + v_y = 0, \tag{6.17}$$

$$p = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\partial A(s,t)}{\partial s} \frac{ds}{s - x}, \tag{6.18}$$

$$x \to -\infty: \quad u \to y,\ v \to 0,\ p \to 0, \tag{6.19}$$

$$y \to \infty: \quad u = y + A(x,t) + f(x,t) + ..., \tag{6.20}$$

$$y = 0: \quad u = 0,\ v = 0. \tag{6.21}$$

If the wall roughness is absent, $f(x,t) = 0$, then we can take $u = y,\ v = p = A = 0$, which represents the unperturbed flow in the near-wall part of the boundary layer. Suppose we have a shallow roughness,

$$f(x,t) = \delta f_1(x,t), \tag{6.22}$$

where the amplitude factor, δ, is small. Then we expect the flow functions to be perturbed by an amount of order δ so that

$$u = y + \delta u_1(x,y,t) + ...,\ v = \delta v_1(x,y,t) + ...,\ p = \delta p_1(x,t) + ...,\ A = \delta A_1(x,t) + ..., \tag{6.23}$$

and, ignoring terms of order δ^2, the formulation becomes,

$$u_{1t} + yu_{1x} + v_1 = -p_{1x} + u_{1yy},\ u_{1x} + v_{1y} = 0, \tag{6.24}$$

$$p_1 = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\partial A_1(s,t)}{\partial s} \frac{ds}{s - x}, \tag{6.25}$$

$$x \to -\infty: \quad u_1 \to 0,\ v_1 \to 0,\ p_1 \to 0, \tag{6.26}$$

$$y \to \infty: \quad u_1 = A_1(x,t) + f_1(x,t) + ..., \tag{6.27}$$

$$y = 0: \quad u_1 = 0,\ v_1 = 0. \tag{6.28}$$

6.3 Time-periodic perturbations.

Suppose the wall roughness performs periodic oscillations with frequency ω and x-dependent amplitude, $f_a(x)$, so that

$$f_1(x,t) = 2\cos(\omega t) f_a(x), \tag{6.29}$$

which we can write as

$$f_1(x,t) = e^{-i\omega t} f_a + c.c., \tag{6.30}$$

where *c.c.* denotes the complex conjugate (the minus sign in the exponent is not essential but proves a little more convenient). A time periodic response in the flow is described by

$$u_1 = e^{-i\omega t} u_a(x,y) + c.c.,\ v_1 = e^{-i\omega t} v_a(x,y) + c.c.,\ p_1 = e^{-i\omega t} p_a(x) + c.c.,\ A_1 = e^{-i\omega t} A_a(x) + c.c. \tag{6.31}$$

The formulation for the amplitude functions is

$$-i\omega u_a + y u_{ax} + v_a = -p_{ax} + u_{ayy},\ u_{ax} + v_{ay} = 0, \tag{6.32}$$

$$p_a(x) = -\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\partial A_a(s)}{\partial s} \frac{ds}{s-x}, \tag{6.33}$$

$$x \to -\infty: \quad u_a \to 0,\ v_a \to 0,\ p_a \to 0, \tag{6.34}$$

$$y \to \infty: \quad u_a = A_a(x) + f_a(x) + ..., \tag{6.35}$$

$$y = 0: \quad u_a = 0,\ v_a = 0. \tag{6.36}$$

This problem can be solved using Fourier transforms in x which we define, for a function $g(x)$ for example, by the relations,

$$\bar{g}(k) = \int_{-\infty}^{\infty} e^{-ikx} g(x) dx, \tag{6.37}$$

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \bar{g}(k) dk, \tag{6.38}$$

for the transform and the inverse transform, respectively.

On applying Fourier transforms to (6.32)-(6.36), we have

$$(iky - i\omega)\bar{u}_a + \bar{v}_a = -ik\bar{p}_a + \frac{d^2\bar{u}_a}{dy^2}, \tag{6.39}$$

$$ik\bar{u}_a + \frac{d\bar{v}_a}{dy} = 0, \tag{6.40}$$

$$\bar{p}_a = |k|\bar{A}_a, \tag{6.41}$$

$$y \to \infty: \quad \bar{u}_a = \bar{A}_a(k) + \bar{f}_a(k) + ..., \tag{6.42}$$

$$y = 0 : \quad \bar{u}_a = 0, \; \bar{v}_a = 0. \tag{6.43}$$

In these equations, the transformed velocity components are functions of y and k. The transform of the Hilbert integral can be obtained using generalized functions or, alternatively, by solving for the inviscid part of the triple-deck using Fourier transforms.

Hence, the problem has been reduced to solving ordinary differential equations.

Applying the wall conditions (6.43)to the momentum equation (6.39), we have

$$\frac{d^2\bar{u}_a}{dy^2} \,|_{y=0} = ik\bar{p}_a. \tag{6.44}$$

Then, differentiating (6.39) with respect to y and using (6.40) we obtain the equation,

$$(iky - i\omega)\tau = \frac{d^2\tau}{dy^2}, \tag{6.45}$$

for the shear, $\tau = d\bar{u}_a/dy$. Next, the substitution,

$$\eta = y(ik)^{1/3} - \frac{i\omega}{(ik)^{2/3}}, \tag{6.46}$$

reduces (6.45) to the Airy equation,

$$\eta\tau(\eta) = \frac{d^2\tau(\eta)}{d\eta^2}. \tag{6.47}$$

The Airy equation has two standard linearly independent solutions, $Ai(\eta)$ and $Bi(\eta)$. Let us specify the branch of the function $(ik)^\alpha$ for some real α by making a cut in the complex k-plane along the positive imaginary axis and taking $z^\alpha$to be real and positive for real and positive values of the complex variable z. In essence this means that $(ik)^{1/3} = k^{1/3}exp(i\pi/6)$ for $k > 0$ and $(ik)^{1/3} = |k|^{1/3}exp(-i\pi/6)$ for $k < 0$, but specifying the branch cut in the complex k-plane also helps to extend these last two relations into the entire k-plane. Then, from the general solution of (6.47),

$$\tau(\eta) = C_1 Ai(\eta) + C_2 Bi(\eta), \tag{6.48}$$

with arbitrary constants C_1, C_2. The term with $Bi(\eta)$ needs to be eliminated as growing exponentially in magnitude as $y \to +\infty$. Hence, $\tau(\eta) = C_1 Ai(\eta)$, so that

$$\frac{d\bar{u}_a}{dy} = \tau = C_1 Ai\left[y(ik)^{1/3} - i\omega(ik)^{-2/3}\right]. \tag{6.49}$$

Applying next the boundary conditions, we find, from (6.44), that

$$(ik)^{1/3} C_1 Ai'\left[-i\omega(ik)^{-2/3}\right] = ik\bar{p}_a, \tag{6.50}$$

and, from (6.42) combined with the condition of zero velocity at $y = 0$,

$$C_1 \int_0^\infty Ai\left[y(ik)^{1/3} - i\omega(ik)^{-2/3}\right] dy = \bar{A}_a(k) + \bar{f}_a(k). \qquad (6.51)$$

We can now eliminate C_1between (6.50) and (6.51) and use the pressure-displacement relation (6.41) to find the Fourier transform of the pressure function,

$$\bar{p}_a = \frac{|k| Ai'(-\zeta)\bar{f}_a}{D(\omega, k)}, \qquad (6.52)$$

where we denote

$$\zeta = \frac{i\omega}{(ik)^{2/3}}, \qquad (6.53)$$

and for the denominator in (6.52) we have

$$D(\omega, k) = (ik)^{1/3}|k| \int_0^\infty Ai(s - \zeta)ds - Ai'(-\zeta). \qquad (6.54)$$

Note that in deriving (6.54) we have changed the path of integration from the real positive semi-axis to a ray in the complex k-plane which is justified on the basis of the branch convention adopted earlier.

In principle, this completes calculations for the Fourier transforms of the solution in terms of the given wall roughness. The displacement function can now be found in transformed form from $\bar{A}_a = \bar{p}_a/|k|$, and we can also determine the velocity components, the wall shear and so on.

Taking the inverse Fourier transform we can also write down the linearized pressure function in physical variables,

$$p_1(x, t) = e^{-i\omega t} \int_{-\infty}^\infty e^{i(kx - \omega t)} \frac{|k| Ai'(-\zeta)\bar{f}_a}{D(\omega, k)} dk + c.c. \qquad (6.55)$$

The last formula is interesting in two respects. First of all, the flow response to a moving wall roughness appears as a superposition of harmonic travelling waves of the form $p_1 \backsim exp[i(kx - \omega t)]$ with the amplitudes determined by the wall roughness through the term $\bar{f}_a(k)$. Second, as is typical of evolution problems, the integrand in (6.55) contains a fraction with the denominator which is nothing else but the dispersion relation of the system. Recall that by the dispersion relation we understand a functional relationship between the wavenumber k and frequency ω of free, unforced waves which may exist in the flow even without an oscillating wall roughness. The dispersion relation is usually written as

$$D(\omega, k) = 0, \qquad (6.56)$$

for some function $D(\omega, k)$. In order to convince ourselves that the expression (6.54) indeed gives us the dispersion relation for the flow (and to avoid solving the free wave problem from the beginning) we can go back to (6.52) and write that formula as

$$D(\omega, k)\bar{p}_a = |k|Ai'(-\zeta)\bar{f}_a. \tag{6.57}$$

Now, if the wall is undisturbed, $\bar{f}_a = 0$, then a non-trivial solution for the pressure exists if $D(\omega, k) = 0$.

6.4 Properties of the dispersion relation.

For a free travelling wave, the dispersion relation $D(\omega, k) = 0$ serves to determine the frequency ω if the wavenumber k is given, or to find the wavenumber for a fixed wave frequency. With $D(\omega, k)$ given by (6.54), it is known that a countable set of free waves (wave modes) exists for any fixed $k \neq 0$, so that $\omega = \omega_n(k), n = 1, 2, 3, ...$. When k is real, the frequencies of free waves are complex valued. If we write $\omega = \omega_r + i\omega_i$ thus separating the real and imaginary parts then, from

$$e^{i(kx-\omega t)} = e^{i(kx-\omega_r t)}e^{\omega_i t}, \tag{6.58}$$

it follows that waves with $\omega_i < 0$ decay whereas waves with $\omega_i > 0$ grow. It turns out that in our case all wave modes except one are decaying. The diagram in Fig.1 (adapted from the original work by Terent'ev in 1981) shows the loci of the first three roots in the complex plane ω for real $k > 0$. At small positive k the roots are clustered around the origin. As k increases, the second, third (and all subsequent roots) move into the lower half plane and stay there for all k, hence indicating decaying modes with $\omega_i < 0$. However the first root crosses the real axis when $k = K_0 = 1.0005$ and $c = \omega/k = 2.2968$ approximately, which gives a neutral wave with $\omega = \Omega_0 = 2.298$. The wave with precisely $k = K_0, \omega = \Omega_0$is neutral. Waves with $k > 1.0005$ grow exponentially. We conclude that the triple deck theory describes instability in the flow. Instability persists for higher k, in fact as $k \to \infty$ for the first mode the growth rate ω_i approaches a finite limit.

6.5 Oscillating wall.

Let us return to the problem of the oscillating wall roughness. Now ω is real and the roots of the dispersion relation have complex valued wavenumbers $k = k_r + ik_i$. Waves with $k_i < 0$ grow in space as x increases since $exp[i(kx - \omega t)] = exp[i(k_r x - \omega t)]exp(-k_i x)$ for a real ω.

It takes some analysis to prove that in the triple deck flow no waves can grow upstream, as $x \to -\infty$. Also it turns out that wave modes again form a countable set, $k = k_n(\omega), n = 1, 2, 3, ...$, for each real $\omega > 0$. Modes with $n > 1$ always decay. The first mode decays downstream if $0 < \omega < \Omega_0$, it becomes neutral at $\omega = \Omega_0$and grows downstream exponentially when $\omega > \Omega_0$.

Discussion. Connection between triple-deck instability and Orr-Sommerfeld instability at finite Reynolds numbers.

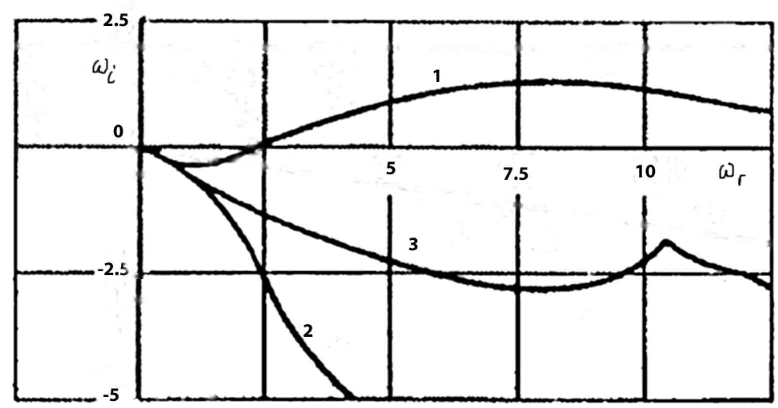

Figure 1: First three roots of the dispersion relation in the complex ω plane for real wavenumbers k.

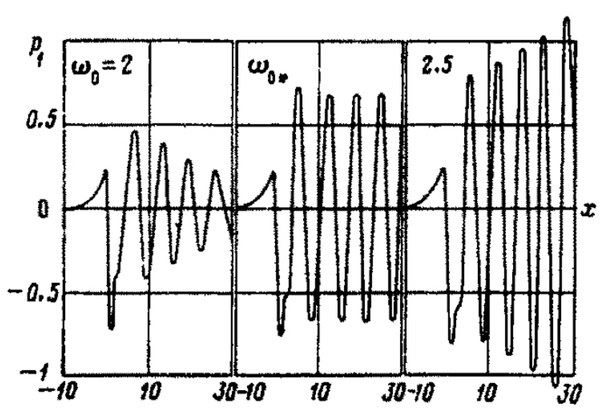

Figure 2: Pressure past the oscillator for subcritical ($\omega < \Omega_0$), neutral ($\omega = \Omega_0$) and supercritical ($\omega > \Omega_0$) frequencies. The oscillating section of the wall has a triangular shape located in $0 \leq x \leq 2$.

Discussion. How did Terent'ev manage to get a solution exponentially growing downstream at ($\omega > \Omega_0$) using Fourier transforms? Answer - the contour of integration needs to dip below the real axis to walk around the pole in the lower half plane k.

Exercise. Derive the formula (6.41) for the Fourier transform of the pressure-displacement relation by solving for the flow in the upper deck of the triple-deck, equations (5.7)-(5.9), using Fourier transforms and appropriate boundary conditions.

Exercise. Find the dispersion relation for the flow by solving linearized equations of motion in the viscous sublayer without wall roughness.

Reference. E.D. Terent'ev. The linear problem of a vibrator performing harmonic oscillations at supercritical frequencies in a subsonic boundary layer. PMM USSR, vol 48, No 2, pp 184-191, 1984.