7 Other multi-deck interactive flows.

We continue to work with non-dimensional Navier-Stokes equations for an incompressible fluid which, in two dimensions, have the form,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + Re^{-1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + Re^{-1}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{7.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{7.3}$$

7.1 Near-wall viscous jets. Double-deck.

If fluid is injected along a solid boundary into a quiescent environment, a near-wall jet is formed, of thickness $O(Re^{-1/2})$ in non-dimensional variables with a typical jet profile as shown in Figure 1. We are not concerned with the jet formation and evolution but consider instead a double-deck interactive flow initiated by a local wall roughness. The main new feature, compared with the standard triple-deck flow, is the pressure variation across the main part of the jet.

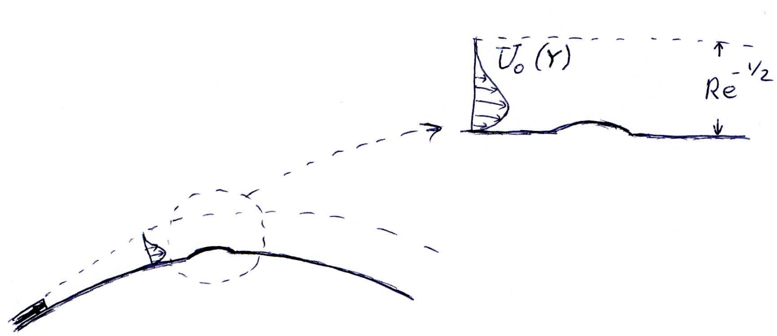

Figure 1: Wall jet near a local obstacle.

Let $x = 1 + Re^{-3/7}X$, $t = Re^{-2/7}T$.
In the **main part** of the jet, $y = Re^{-1/2}Y$,

$$u = U_0(Y) + Re^{-1/7}u_1(X, Y, T) + ..., \tag{7.4}$$

$$v = Re^{-2/7}v_1(X, Y, T) + ..., \tag{7.5}$$

$$p = Re^{-2/7}p_1(X, Y, T) + ..., \tag{7.6}$$

For the streamwise momentum and continuity, we have the equations,

$$U_0 \frac{\partial u_1}{\partial X} + v_1 \frac{dU_0}{dY} = 0, \ \frac{\partial u_1}{\partial X} + \frac{\partial v_1}{\partial Y} = 0, \tag{7.7}$$

with a solution containing the displacement function, $A(X, T)$, of the form,

$$u_1 = A(X, T) \frac{dU_0(Y)}{dy}, \ v_1 = -\frac{\partial A(X, T)}{\partial X} U_0(Y). \tag{7.8}$$

For the normal momentum,

$$U_0 \frac{\partial v_1}{\partial X} = -\frac{\partial p_1}{\partial Y}, \tag{7.9}$$

the solution is

$$p_1 = -\frac{\partial^2 A(X, T)}{\partial X^2} \int_Y^\infty U_0^2(s) ds + C(X, T). \tag{7.10}$$

From the decay condition at the outer edge of the boundary layer, $C(X, T) = 0$, then as $Y \to 0$,

$$p_1 \to -\kappa \frac{\partial^2 A(X, T)}{\partial X^2} \text{ where } \kappa = \int_0^\infty U_0^2(s) ds. \tag{7.11}$$

Also $U_0 = \lambda_0 Y + ...$, therefore

$$u_1 = \lambda_0 A(X, T) + ... \text{ as } Y \to 0. \tag{7.12}$$

In the **viscous sublayer**, $y = Re^{-1/2-1/7} y_2 = Re^{-9/14} y_2$, $y_2 = O(1)$.

$$u = Re^{-1/7} u_2(X, y_2, T) + ..., \tag{7.13}$$

$$v = Re^{-5/14} v_2(X, y_2, T) + ..., \tag{7.14}$$

$$p = Re^{-2/7} p_2(X, y_2, T) + ..., \tag{7.15}$$

From Navier-Stokes, $p_2 = p_2(X, T)$, and

$$\frac{\partial u_2}{\partial T} + u_2 \frac{\partial u_2}{\partial X} + v_2 \frac{\partial u_2}{\partial y_2} = -\frac{\partial p_2}{\partial X} + \frac{\partial^2 u_2}{\partial y_2^2}, \quad \frac{\partial u_2}{\partial X} + \frac{\partial v_2}{\partial y_2} = 0. \tag{7.16}$$

Matching with the flow in the main part of the jet and including the conditions in the incoming stream, we have

$$u_2 = \lambda_0 y_2 + \lambda_0 A(X, T) + ... \text{ as } y_2 \to \infty, \tag{7.17}$$

$$p_2 = -\kappa \frac{\partial^2 A(X, T)}{\partial X^2}, \tag{7.18}$$

$$u_2 \to \lambda_0 y_2 \text{ as } X \to -\infty. \tag{7.19}$$

We also need to include no-slip conditions at the wall (for example if the wall has an irregularity). These we can written as,

$$u_2 = v_2 = 0 \text{ at } y_2 = f(X), \tag{7.20}$$

assuming, for simplicity, that the flow boundary is stationary.

Exercise.

Find an affine transformation, $x_2 = a_1 x^*$, $y_2 = a_2 y^*$, $f = a_2 f^*$, $u_2 = a_3 u^*$, $v_2 = a_4 v^*$, $A = a_5 A^*$, $p_2 = a_6 p^*$, $T = a_7 T^*$ with constants a_{1-7} such that the parameters λ_0 and κ are scaled out, that is the interaction problem takes the form,

$$\frac{\partial u^*}{\partial T^*} + u^*\frac{\partial u^*}{\partial X^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial X^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \quad \frac{\partial u^*}{\partial X^*} + \frac{\partial v^*}{\partial y^*} = 0. \tag{7.21}$$

$$u^* = y^* + A^*(X^*, T^*) + ... \text{ as } y^* \to \infty, \tag{7.22}$$

$$p^* = -\frac{\partial^2 A^*(X^*, T^*)}{\partial X^{*2}}, \tag{7.23}$$

$$u^* \to y^* \text{ as } X^* \to -\infty. \tag{7.24}$$

$$u^* = v^* = 0 \text{ at } y^* = f^*. \tag{7.25}$$

Solution. From (7.17), $a_5 = a_2$ and $a_3 = \lambda_0 a_2$. In the continuity equation, require $a_4 = \lambda_0 a_2^2/a_1$. Then for the time-derivative term in (7.16) require $a_7 = a_1/(\lambda_0 a_2)$ and the balance between advection and pressure gradient terms gives $a_6 = (\lambda_0 a_2)^2$ and the balance with the viscous term leads to $a_1 = \lambda_0 a_2^3$. The relation unused so far is (7.18), where we find $\lambda_0^2 a_1^2 a_2 = \kappa$. Solving the last two equations for a_1, a_2we find

$$a_1 = \kappa^{3/7}\lambda_0^{-5/7}, \ a_2 = \kappa^{1/7}\lambda_0^{-4/7}, \tag{7.26}$$

and the rest of the constants now follow.

Upstream influence.

Suppose that the wall roughness is shallow, $f^* = \delta f_l$, with $\delta \ll 1$. We drop the asterisks, omit time dependence, write the solution as a small perturbation to the base near-wall flow,

$$\{u, v, p, A\} = \{y, 0, 0, 0\} + \delta\{u_l(X, y), v_l(X, y), p_l(X), A_l(X)\} + O(\delta^2), \tag{7.27}$$

and linearize the formulation (7.21)-(7.25) to get

$$y\frac{\partial u_l}{\partial X} + v_l = -p_l'(X) + \frac{\partial^2 u_l}{\partial y^2}, \ \frac{\partial u_l}{\partial X} + \frac{\partial v_l}{\partial y} = 0, \tag{7.28}$$

$$u_l \to A_l(X) \text{ as } y \to \infty, \tag{7.29}$$

$$p_l = -A_l''(X), \tag{7.30}$$

$$u_l \to 0 \text{ as } X \to -\infty, \tag{7.31}$$

$$u_l = -f_l(X), v_l = 0 \text{ at } y = 0. \tag{7.32}$$

The problem (7.28)-(7.32) is solved using Fourier transform in X, as in the previous section. We have,

$$iky\bar{u}_l + \bar{v}_l = -ik\bar{p}_l + \bar{u}_{lyy}, \; ik\bar{u}_l + \bar{v}_{ly} = 0, \tag{7.33}$$

hence

$$iky\bar{u}_{ly} = \bar{u}_{lyyy} \text{ and therefore } \bar{u}_{ly} = C(k)Ai(y(ik)^{1/3}). \tag{7.34}$$

From the momentum equation,

$$-C(k)(ik)^{1/3}|Ai'(0)| = ik\bar{p}_l, \tag{7.35}$$

using the fact that $Ai'(0)$ is a negative real number. Next, applying wall conditions (7.32),

$$\bar{u}_l = -\bar{f}_l(k) + C(k)\int_0^y Ai(s(ik)^{1/3})ds. \tag{7.36}$$

Now letting $y \to \infty$ and using (7.29) together with the known formula, $\int_0^\infty Ai(s)ds = 1/3$, we obtain a relation between transformed pressure and displacement functions,

$$\bar{A}_l = -\bar{f}_l - \frac{(ik)^{1/3}}{3|Ai'(0)|}\bar{p}_l. \tag{7.37}$$

The last formula is useful int hat it applies to any viscous sublayer flow irrespective of the pressure-displacement law.

Now using the interaction formula (7.30), we have, for the pressure,

$$\bar{p}_l = -\frac{3|Ai'(0)|(ik)^2}{(ik)^{7/3} - 3|Ai'(0)|}\bar{f}_l(k). \tag{7.38}$$

Localized obstacle and upstream influence.

We shall take the obstacle shape as a delta-function,

$$f_l(X) = \delta(X), \text{ then } \bar{f}_l = 1. \tag{7.39}$$

Taking the inverse Fourier transform,

$$p_l(X) = -\frac{a}{2\pi}\int_{-\infty}^{\infty} \frac{(ik)^2 e^{ikX}}{(ik)^{7/3} - a}dk. \tag{7.40}$$

where $a = 3|Ai'(0)|$. We aim to evaluate the response of the flow upstream of the obstacle, $X < 0$, which is done by closing the contour of integration in (7.40) in the lower half plane and using the residue theorem, with the pole at $k = -ia^{3/7}$.The result is

$$p_l(X) = \frac{3}{7}a^{9/7}e^{a^{3/7}X} \text{ for } X < 0. \tag{7.41}$$

We conclude that the flow 'feels' the presence of an obstruction at distances proportional to the length scale of the interaction region.

Discussion. Self-induced separation. Numerical solutions by Smith & Duck, 1977. Note that the wall shear ahead of interaction region is 1/2 in their paper.

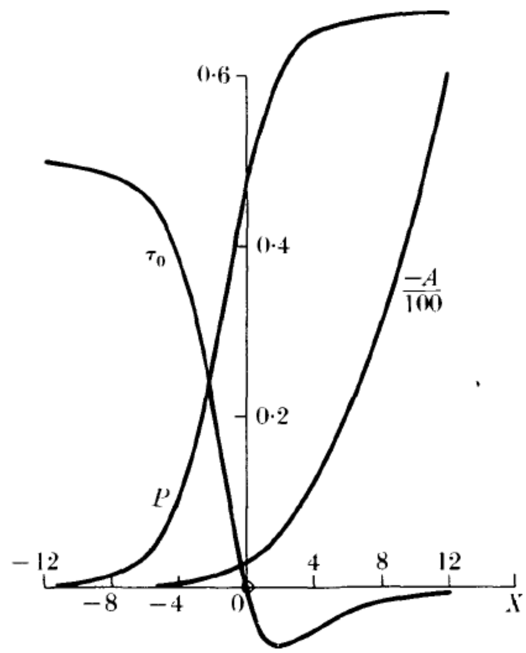

FIG. 2. Solution curves for τ_0, P and A against X (with $X_{-\infty} = -12{\cdot}364$).

Figure 2: Self-induced separation in a wall jet.

7.2 Supersonic self-induced separation.

The type of interaction which leads to self-induced separation in a supersonic flow was described by Stewartson & Williams and Neiland around 1970. In scaled variables it is governed by equations (7.21)-(7.25) only the pressure-displacement relation (7.23) changes to

$$p^* = -\frac{\partial A^*(X^*, T^*)}{\partial X^*}. \tag{7.42}$$

The technique of linearizing the sublayer equations can be used to prove the existence of upstream influence in the flow (see the exercise at the end of this section) and hence deduce the possibility of self-induced processes in the boundary layer. Self-induced separation is illustrated in Figure 3 taken from the Stewartson & Williams paper.

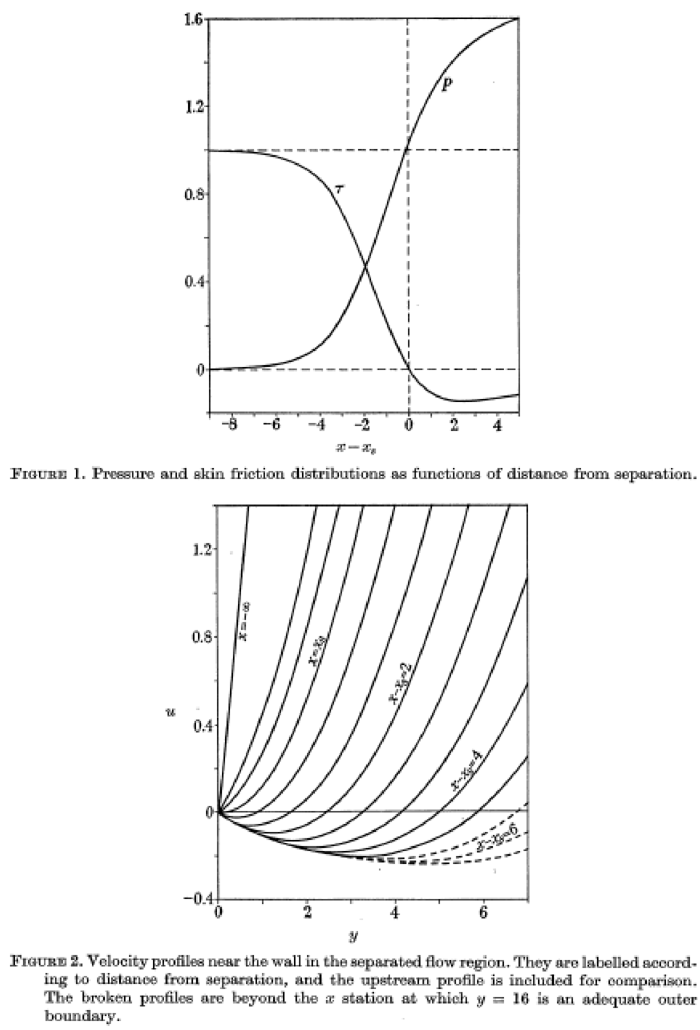

Figure 3: Supersonic separation.

7.3 Condensed flow.

In several situations we observe an interactive flow characterized by the absence of displacement effect in the viscous sublayer, that is the formula (7.23) is replaced by

$$A^* = 0. \tag{7.43}$$

The pressure in the boundary layer remains unknown therefore we have a non-classical boundary layer behaviour.

We shall consider a three-dimensional version of the flow given by the boundary-layer equations,

$$\frac{\partial u^*}{\partial T^*} + u^* \frac{\partial u^*}{\partial X^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial Z^*} = -\frac{\partial p^*}{\partial X^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{7.44}$$

$$\frac{\partial w^*}{\partial T^*} + u^* \frac{\partial w^*}{\partial X^*} + v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial Z^*} = -\frac{\partial p^*}{\partial Z^*} + \frac{\partial^2 w^*}{\partial y^{*2}}, \tag{7.45}$$

$$\frac{\partial u^*}{\partial X^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial Z^*} = 0, \tag{7.46}$$

$$u^* = y^* + A^*(X^*, T^*) + ..., \; w^* \to 0 \text{ as } y^* \to \infty, \tag{7.47}$$

$$u^* \to y^* \text{ as } X^* \to -\infty. \tag{7.48}$$

$$u^* = v^* = w^* = 0 \text{ at } y^* = f^*. \tag{7.49}$$

with the interaction law (7.43). Dropping the asterisks, upon linearization (as is done in (7.27)), and also omitting time dependence we have

$$y \frac{\partial u_l}{\partial X} + v_l = -\frac{\partial p_l}{\partial X} + \frac{\partial^2 u_l}{\partial y^2}, \tag{7.50}$$

$$y \frac{\partial w_l}{\partial X} = -\frac{\partial p_l}{\partial Z} + \frac{\partial^2 w_l}{\partial y^2}, \tag{7.51}$$

$$\frac{\partial u_l}{\partial X} + \frac{\partial v_l}{\partial y} + \frac{\partial w_l}{\partial Z} = 0, \tag{7.52}$$

$$u_l \to 0, \; w_l \to 0 \text{ as } X \to -\infty \text{and as } y \to \infty, \tag{7.53}$$

$$u_l = f_l(X, Z), \; v_l = 0, \; w_l = 0 \text{ at } y = 0. \tag{7.54}$$

The momentum and continuity equations can be reduced to a single equation for the normal velocity,

$$y \frac{\partial^2 v_l}{\partial X \partial y} = \nabla p_l + \frac{\partial^3 v_l}{\partial y^3}, \tag{7.55}$$

with the Laplacian of the pressure, $\nabla p_l = (\partial^2/\partial X^2 + \partial^2/\partial Z^2)p_l$. It follows immediately that the pressure in the flow is governed by a Poisson equation,

$$\nabla p_l = F(f_l), \tag{7.56}$$

where F is some functional of the wall roughness, f_l. Hence, for a localized in space wall roughness, condensed flow generates disturbances upstream as well as on the sides of the roughness. It is easy to verify that the two-dimensional analogue of the condensed flow problem does not support influence. Hence there is a significant difference between two- and three-dimensional condensed flow interactions. Verification of these properties is left as an exercise.

Exercise. Derive a linearized solution for the supersonic flow with viscous-inviscid interaction (7.42) past a shallow obstacle and hence show the existence of upstream influence. Note that to prove the existence of upstream influence there is no need to compute inverse Fourier transforms for the entire linearized solution.

Exercise. Consider interactive flows with the pressure-displacement relations $p = A$ and $p = -A$. Which of these two interactions supports upstream influence in a 2-D case?

Exercise. Complete the solution of the linear problem (7.50)-(7.54) and verify the properties of upstream propagation of disturbances stated in the text. Do we need any boundary conditions in the downstream part of the flow, as $X \to \infty$?

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