4 Sobolev spaces, trace theorem and normal derivative

Throughout, $\Omega \subset \mathbb{R}^n$ will be a sufficiently smooth, bounded domain.

We use the standard Sobolev spaces

$$H^0(\mathbb{R}^n) := L_2(\mathbb{R}^n), \quad H^0(\Omega) := L_2(\Omega), \quad H^k(\mathbb{R}^n), \quad H^k(\Omega) \quad (k \text{ positive integer})$$

Note that all these spaces are based on the use of weak derivatives up to order $k$. We will use the Fourier transform to redefine the norms in these spaces. Recall that the Fourier transform $F$ is defined by (there are different normalisations possible)

$$\hat{v}(\xi) := Fv(\xi) := \int_{\mathbb{R}^n} e^{-i2\pi \xi \cdot x} v(x) \, dx \quad (\xi \in \mathbb{R}^n).$$

Since

$$|\hat{v}(\xi)| = |\int_{\mathbb{R}^n} e^{-i2\pi \xi \cdot x} v(x) \, dx| \leq \int_{\mathbb{R}^n} |e^{-i2\pi \xi \cdot x} v(x)| \, dx = \int_{\mathbb{R}^n} |v(x)| \, dx$$

it follows that $\hat{v}$ is well-defined whenever $v \in L_1(\mathbb{R}^n)$. The inversion formula for the Fourier transform is

$$F^{-1}\hat{v}(x) := \int_{\mathbb{R}^n} e^{i2\pi \xi \cdot x} \hat{v}(\xi) \, d\xi.$$

One finds the following properties:

- If $v, \hat{v} \in L_1(\mathbb{R}^n)$ then $F^{-1}Fv = v = FF^{-1}v$ whenever $v$ is continuous.

- $F$ generalises to a bounded linear mapping

$$(F \varphi, Fv) = (\varphi, v) = (F^{-1} \varphi, F^{-1}v) \quad \forall \varphi, v \in L_2(\mathbb{R}^n),$$

i.e., $F$ is a unitary isomorphism. This property is known as Plancherel’s theorem. The symbol $(\cdot, \cdot)$ denotes the $L_2$ inner product on $\mathbb{R}^n$ and will be used throughout, also for its extension by duality. When referring to the inner product on a subset of $\mathbb{R}^n$, e.g. on $\Omega$, we add this subset as an index, e.g. $(\cdot, \cdot)_\Omega$.

- A conclusion from Plancherel’s theorem is the relation

$$\|v\|_{L_2(\mathbb{R}^n)} = \|\hat{v}\|_{L_2(\mathbb{R}^n)} \quad \forall v \in L_2(\mathbb{R}^n).$$

**Example 4.1** Consider the one-dimensional case, i.e., $n = 1$.

There holds $\|v\|_{L_2(\mathbb{R})} = \|F(v')\|_{L_2(\mathbb{R})}$, and for any $v \in H^1(\mathbb{R})$ with compact support we obtain

$$F(v')(\xi) = \int_{\mathbb{R}} e^{-i2\pi x \xi} v'(x) \, dx = v(x)e^{-i2\pi x \xi} \bigg|_{x = -\infty}^{x = \infty} - \int_{\mathbb{R}} -i2\pi \xi e^{-i2\pi x \xi} v(x) \, dx = i2\pi \xi \hat{v}(\xi).$$
Therefore,\[ \|v'\|_{L^2(\mathbb{R})} = \|i2\pi \hat{v}(\xi)\|_{L^2(\mathbb{R})} = 2\pi\|\hat{v}(\xi)\|_{L^2(\mathbb{R})} \]
and\[ \|v\|^2_{H^1(\mathbb{R})} = \|v\|^2_{L^2(\mathbb{R})} + \|v'\|^2_{L^2(\mathbb{R})} = \|\hat{v}\|^2_{L^2(\mathbb{R})} + 4\pi^2\|\hat{v}(\xi)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (1 + 4\pi^2\xi^2)|\hat{v}(\xi)|^2 \, d\xi, \]
so that\[ \|v\|_{H^1(\mathbb{R})} \quad \text{and} \quad \left( \int_{\mathbb{R}} (1 + \xi^2)|\hat{v}(\xi)|^2 \, d\xi \right)^{1/2} = \|(1 + |\xi|^2)^{1/2}\hat{v}\|_{L^2(\mathbb{R})} \]
are equivalent norms.

This example easily generalises to higher dimensions (\( n > 1 \)). Moreover, it leads us to the definition of Sobolev spaces on \( \mathbb{R}^n \) for any positive real order.

**Definition 4.1** For \( s > 0 \) we define\[ H^s(\mathbb{R}^n) := \left\{ v \in L^2(\mathbb{R}^n); \|(1 + |\xi|^2)^{s/2}\hat{v}\|_{L^2(\mathbb{R}^n)} < \infty \right\} \]
with norm\[ \|v\|_{H^s(\mathbb{R}^n)} := \|(1 + |\xi|^2)^{s/2}\hat{v}\|_{L^2(\mathbb{R}^n)}. \]

As in Example 4.1 one sees that, for integer \( s \), this norm is equivalent to the usual one (based on derivatives). For non-integer \( s \), \( H^s(\mathbb{R}^n) \) is called a fractional order Sobolev space.

We are now in a position to analyse the trace operator in the half-space case. Consider the situation given in Figure 4.1. For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we denote \( x' := (x_1, \ldots, x_{n-1}) \). Then we define for \( v \in C_0^\infty(\mathbb{R}^n) \) its trace onto the hyperplane \( \mathbb{R}^{n-1} \times \{0\} \) by\[ \gamma_0 v(x') := v(x', x_n = 0), \quad x' \in \mathbb{R}^{n-1}. \]

**Theorem 4.1** (trace theorem, half-space case) For \( s > 1/2 \) there exists a unique extension of \( \gamma_0 \) to a bounded linear operator\[ \gamma_0 : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1}). \]

**Proof.** By density it suffices to consider \( v \in C_0^\infty(\mathbb{R}^n) \). By the Fourier inversion formula we find that\[ \gamma_0 v(x') = \int_{\mathbb{R}^n} e^{i2\pi x \cdot \xi} \hat{v}(\xi) \, d\xi \bigg|_{x_n = 0} = \int_{\mathbb{R}^n} e^{i2\pi x' \cdot \xi'} \hat{v}(\xi) \, d\xi = \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \hat{v}(\xi', \xi_n) \, d\xi_n \right) e^{i2\pi x' \cdot \xi'} \, d\xi'. \]
Therefore,\[ \mathcal{F}(\gamma_0 v)(\xi') = \int_{\mathbb{R}} \hat{v}(\xi', \xi_n) \, d\xi_n = \int_{\mathbb{R}} (1 + |\xi|^2)^{-s/2}(1 + |\xi|^2)^{s/2}\hat{v}(\xi', \xi_n) \, d\xi_n \]

19
and an application of the Cauchy-Schwarz inequality yields

\[ |F(\gamma_0 v)(\xi')|^2 \leq \int_{\mathbb{R}} (1 + |\xi|^2)^{-s} \, d\xi_n \int_{\mathbb{R}} (1 + |\xi|^2)^{2s} |\hat{v}(\xi', \xi_n)|^2 \, d\xi_n. \]

Now, by the substitution \( \xi_n = (1 + |\xi'|^2)^{1/2} t, \)

\[ M_s(\xi') := \int_{\mathbb{R}} (1 + |\xi'|^2)^{-s} \, d\xi_n = \int_{\mathbb{R}} \frac{d\xi_n}{(1 + |\xi'|^2 + |\xi_n|^2)^s} = \frac{1}{(1 + |\xi'|^2)^{s-1/2}} \int_{\mathbb{R}} \frac{dt}{(1 + t^2)^s} < \infty \quad \text{iff} \quad s > 1/2. \]

Therefore, we can bound

\[ (1 + |\xi'|^2)^{s-1/2} |F(\gamma_0 v)(\xi')|^2 \leq C_s \int_{\mathbb{R}} (1 + |\xi|^2)^2 |\hat{v}(\xi)|^2 \, d\xi_n \]

for a constant \( C_s \) depending on \( s \), and integration with respect to \( \xi' \) yields

\[ \|\gamma_0 v\|_{H^{s-1/2}(\mathbb{R}^n-1)} \leq C_s^{1/2}\|v\|_{H^s(\mathbb{R}^n)}. \]

So far we have dealt with Sobolev spaces on \( \mathbb{R}^n \). For boundary value problems on Lipschitz domains this is obviously not enough.

**Definition 4.2** Let \( \Omega \subset \mathbb{R}^n \) be a Lipschitz domain. For \( s \geq 0 \) we introduce the following spaces:

\[ H^s(\Omega) := H^s(\mathbb{R}^n)|_{\Omega} \quad \text{with norm} \quad \|v\|_{H^s(\Omega)} := \inf_{V|\Omega=v} \|V\|_{H^s(\mathbb{R}^n)}, \]

\[ H^s_0(\Omega) := C^\infty_0(\Omega)|_{H^s(\Omega)} \quad \text{with norm} \quad \|\cdot\|_{H^s(\Omega)}, \]

\[ \square \]
and 
\[ \tilde{H}^s(\Omega) := \{ v \in H^s(\Omega); v^0 \in H^s(\mathbb{R}^n) \} \quad \text{with norm} \quad \| v \|_{\tilde{H}^s(\Omega)} := \| v^0 \|_{H^s(\mathbb{R}^n)} \]

where \( v^0 \) denotes the extension of \( v \) by 0 onto \( \mathbb{R}^n \setminus \Omega \).

For \( s < 0 \) we define
\[ H^s(\Omega) := \left( \tilde{H}^{-s}(\Omega) \right)' \quad \text{(dual space)} \quad \text{with operator norm} \]

and
\[ \tilde{H}^s(\Omega) := \left( H^{-s}(\Omega) \right)' \quad \text{(dual space)} \quad \text{with operator norm.} \]

Remark 4.1 One can show that, for \( s > 0 \), \( \tilde{H}^s(\Omega) = H^s_0(\Omega) \) if \( s \neq \text{integer} + 1/2 \). In the cases \( s = \text{integer} + 1/2 \) the spaces are different, \( \tilde{H}^s(\Omega) \subset H^s_0(\Omega) \) in general.

Without going into the details, we mention that on a Lipschitz surface or boundary \( \Gamma \) all the above spaces can be defined analogously when \( |s| \leq 1 \). To this end one uses a partition of unity and local transformations onto subsets of \( \mathbb{R}^{n-1} \). Higher order spaces require more regularity of \( \Gamma \).

The trace theorem can be generalised to Lipschitz domains.

**Theorem 4.2** (trace theorem, general form)
Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain with boundary \( \Gamma \).
(i) For \( 1/2 < s < 3/2 \), \( \gamma_0 \) has a unique extension to a bounded linear operator \( \gamma_0 : H^s(\Omega) \to H^{s-1/2}(\Gamma) \).

(ii) For any \( s \in (1/2, 3/2) \) and any \( v \in H^{s-1/2}(\Gamma) \) there exists \( V := \mathcal{E}v \in H^s(\Omega) \) such that \( \gamma_0(V) = v \) and
\[ \| \mathcal{E}v \|_{H^s(\Omega)} \leq C_s(\Omega) \| v \|_{H^{s-1/2}(\Gamma)} \quad \forall v \in H^{s-1/2}(\Gamma) . \]

**Remark 4.2** Part (ii) of Theorem 4.2 means that \( \gamma_0 \) has a right-inverse:
\[ v = \gamma_0 V = \gamma_0 \mathcal{E}v \]
which is continuous, and that
\[ \gamma_0 : H^s(\Omega) \to H^{s-1/2}(\Gamma) \]
is surjective, i.e., \( \gamma_0\left( H^s(\Omega) \right) = H^{s-1/2}(\Gamma) \). Of course, this right-inverse \( \mathcal{E} \) is an extension operator.

Having the trace operator at hand we can now make an interpretation of the Dirichlet boundary condition. Studying the Poisson equation with Dirichlet boundary condition
\[ -\Delta u = f \quad \text{in} \quad \Omega, \quad u|_\Gamma = g \]
we conclude two things. First, the equation \( u|_\Gamma = g \) means that \( \gamma_0 u = g \) in \( H^{1/2}(\Gamma) \) since the variational formulation of the Poisson equation is posed in \( H^1(\Omega) \) (subject to the Dirichlet condition). Second, the Dirichlet condition makes sense only for \( g \in H^{1/2}(\Gamma) \). If \( g \not\in H^{1/2}(\Gamma) \) then there does not exist a solution \( u \in H^1(\Omega) \) of the given boundary value problem. This is a conclusion of the surjectivity of the trace operator.

Besides the trace operator \( \gamma_0 \), in §1 we were concerned about the definition of the normal derivative \( \partial_n v \) of a function \( v \in H^1(\Omega) \). We now deal with this operator.

The origin for the definition of the normal derivative is the first Green’s formula, in the form

\[
\int_{\Omega} -\Delta v \, w = \int_{\Omega} \nabla v \cdot \nabla w - \int_{\Gamma} \partial_n v \, w.
\]

This leads us to the definition of \( \partial_n v \) for \( v \in H^1(\Omega) \) by

\[
\langle \partial_n v, w \rangle_\Gamma := \int_{\Omega} \nabla v \cdot \nabla W + \int_{\Omega} \Delta v W
\]

where \( W \in H^1(\Omega) \) is any extension of \( w \in H^{1/2}(\Gamma) \). The notation \( \langle \Phi, \varphi \rangle_\Gamma \) means the application of the functional \( \Phi \) to \( \varphi \) defined on \( \Gamma \), in this case it is the duality between \( H^{-1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \). For \( \Phi, \varphi \in L_2(\Gamma) \) it is simply the \( L_2(\Gamma) \)-inner product between \( \Phi \) and \( \varphi \).

**Lemma 4.1**

\( \partial_n : \{ v \in H^1(\Omega); \Delta v \in \tilde{H}^{-1}(\Omega) \} \rightarrow H^{-1/2}(\Gamma) \)

is well-defined and continuous when defining

\[
\int_{\Omega} \Delta v W := (\Delta v, W)_\Omega
\]

as duality between \( \tilde{H}^{-1}(\Omega) \) and \( H^1(\Omega) \).

**Proof.**  (i) First we show that the definition of \( \langle \partial_n v, w \rangle_\Gamma \) is independent of the extension \( W \) of \( w \). Let \( W_1, W_2 \in H^1(\Omega) \) be two extensions of \( w \), i.e., \( \gamma_0 W_1 = \gamma_0 W_2 = w \). Then

\[
\int_{\Omega} \nabla v \cdot \nabla (W_1 - W_2) + \int_{\Omega} \Delta v (W_1 - W_2) = 0
\]

by the second Green identity since \( W_1 - W_2 \in H^1_0(\Omega) \). This proves that \( \langle \partial_n v, \gamma_0(W_1 - W_2) \rangle_\Gamma = 0 \) as wanted.

(ii) Now we show the boundedness of \( \partial_n \). Let \( E : H^{1/2}(\Gamma) \rightarrow H^1(\Omega) \) denote the extension
operator from Theorem 4.2(ii). We estimate

$$\|\partial_n v\|_{H^{-1/2}(\Gamma)} = \sup_{w \in H^{1/2}(\Gamma) \setminus \{0\}} \frac{\langle \partial_n v, w \rangle_{\Gamma}}{\|w\|_{H^{1/2}(\Gamma)}} \leq C \sup_{w \in H^{1/2}(\Gamma) \setminus \{0\}} \frac{\langle \partial_n v, w \rangle_{\Gamma}}{\|Ew\|_{H^1(\Omega)}}$$

$$= C \sup_{w \in H^{1/2}(\Gamma) \setminus \{0\}} \frac{\int_{\Omega} \nabla v \cdot \nabla w + \int_{\Omega} \Delta v \, Ew}{\|Ew\|_{H^1(\Omega)}} \leq C \sup_{W \in H^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \nabla v \cdot \nabla W + \int_{\Omega} \Delta v \, W}{\|W\|_{H^1(\Omega)}}$$

$$\leq C \sup_{W \in H^1(\Omega) \setminus \{0\}} \frac{\|v\|_{H^1(\Omega)} \|W\|_{H^1(\Omega)} + \|\Delta v\|_{\tilde{H}^{-1}(\Omega)} \|W\|_{H^1(\Omega)}}{\|W\|_{H^1(\Omega)}}$$

$$= C \left( \|v\|_{H^1(\Omega)} + \|\Delta v\|_{\tilde{H}^{-1}(\Omega)} \right).$$

\[\square\]

**Remark 4.3** $\Delta v \in L_2(\Omega)$ implies $\Delta v \in \tilde{H}^{-1}(\Omega)$. 