CHAPTER 1

Basic homological algebra

In this chapter we will define cohomology via cochain complexes. We will restrict to considering modules over a ring and to giving a very constructive definition. At the end of the chapter we will look at a more axiomatic apporach. These notes do not contain detailed proofs, although we shall go through them during the lecture. Most of the results in this chapter are fairly standard and can be found in most textbooks on homological algebra. These are, for example, the Springer Graduate Text by Hilton and Stammbach [**9**], the old classic by Rotman [**25**] or the newer and slightly more modern book by Weibel [**29**]. There are two very good places with online lecture notes: Peter Kropholler's notes on cohomology [**10**] and Daniel Murfet's collection of lecture notes [**23**]. Some of the proofs in this chapter follow those of Peter Kropholler and I recommend these as reference.
(I suspect that there are still a number of typos and other inaccuracies in these notes. So, if you find any, please let me know.)

1. Modules

In this section we will quickly review the basic definitions of modules over a ring. In general we denote a ring by R and assume that R has a unit.

Definition 1.1. Let R be a ring. A left R-module is an abelian group $(M,+)$ together with a multiplication

$$\begin{array}{ccc} R\times M & \to & M \\ (r,m) & \mapsto & rm \end{array}$$

satisfying the following axioms:

(M1) $r(m+n) = rm + rn$ for all $r \in R$ and $m,n \in M$
(M2) $(r+s)m = rm + sm$ for all $r,s \in R$ and $m \in M$
(M3) $(rs)m = r(sm)$ for all $r,s \in R$ and $m \in M$
(M4) $1_R m = m$ for all $m \in M$.

We usually write M_R - or M if it is clear which ring is meant. Right R-modules are defined analogously. If R is commutative a left R-module can be made into a right R-module by defining the multiplication by $(m,r) \mapsto rm$.

Example 1.2.

1. Let k be a field. Then any k-vector space is a k-module.
2. Any additive abelian group A can be viewed as a $\mathbb{Z}$-module.
3. *The regular module:* Left multiplication makes any ring R into an R-module by $(r,s) \mapsto rs$. We call R the left regular module.

DEFINITION 1.3. Let M be an R-module. An R-submodule is an abelian subgroup N such that for all $r \in R, n \in N: \; rn \in N$.

EXERCISE 1.4. Let V be a finite dimensional k-vector space and denote by $End_k(V)$ the ring of endomorphisms. Prove that

(1) V is a left $End_k(V)$-module via
$$\begin{array}{ccc} End_k(V) \times V & \to & V \\ (\phi, v) & \mapsto & \phi(v). \end{array}$$
(2) V has no $End_k(V)$-submodules except 0 and V. Such a module is called *simple*.

DEFINITION 1.5. Let M and N be R-modules. A map $\alpha : M \to N$ is called R-linear or an R-module homomorphism if

- $\alpha(m + m') = \alpha(m) + \alpha(m')$ for all $m, m' \in M$
- $\alpha(rm) = r\alpha(m)$ for all $m \in M, r \in R$.

Let M and N be R-modules. We denote by $\mathrm{Hom}_R(M, N)$ the set of all R-linear maps $\alpha : M \to N$.

REMARK 1.6. $\mathrm{Hom}_R(M, N)$ is an abelian group with addition defined pointwise. Furthermore $End_R(M) = \mathrm{Hom}_R(M, M)$ is a ring where multiplication is defined by composition of maps.

LEMMA 1.7. *For every R-module M there is a natural isomorphism:*
$$\phi : \qquad \mathrm{Hom}_R(R, M) \longrightarrow M$$
defined by $f \mapsto f(1)$.

<u>Proof:</u> Exercise. □

Naturality means that for every R-module homomorphism $\alpha : M \to N$ the following diagram commutes,
$$\begin{array}{ccc} \mathrm{Hom}_R(R, M) & \xrightarrow{\phi_M} & M \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Hom}_R(R, N) & \xrightarrow{\phi_N} & N \end{array}$$
where $\alpha_*(f) = \alpha \circ f$ and $\alpha \circ \phi_M = \phi_N \circ \alpha_*$.

DEFINITION 1.8. **Direct product and direct sum of modules:** Let I be an index set and for each $i \in I$ let M_i be an R-module. Define a new R-module, the direct product of the M_i, by
$$M = \prod_{i \in I} M_i.$$
The elemenets $m \in M$ are families $(m_i)_{i \in I}$, where addition is defined component wise. The R-module structure is given by $r(m_i)_{i \in I} = (rm_i)_{i \in I}$.
Denote by M_0 the submodule of M consisting of those families $(m_i)_{i \in I}$, for which $m_i = 0$ for all but finitely many $i \in I$. We call M_0 the direct sum of the M_i, denoted by
$$M_0 = \bigoplus_{i \in I} M_i.$$

Remark 1.9. For every $i \in I$ there are natural projections:

$$\begin{array}{cccc} \pi_i: & M & \twoheadrightarrow & M_i \\ & (m_j)_{j\in I} & \mapsto & m_i \end{array}$$

and natural injections

$$\begin{array}{cccc} \iota_i: & M_i & \hookrightarrow & M_0 \\ & m_i & \mapsto & (m_j)_{j\in I}, \end{array}$$

where

$$m_j = \begin{cases} m_i & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.10. *Let X be an R-module and let $\phi_i : X \to M_i$ be an R-module homomorphism for every $i \in I$. Then there exists a unique R-module homomorphism $\phi : X \to \prod_{i\in I} M_i$, such that for all $i \in I$ $\pi_i \circ \phi = \phi_i$. In particular,*

$$Hom_R(X, \prod_{i\in I} M_i) \cong \prod_{i\in I} Hom_R(X, M_i).$$

Proposition 1.11. *Let Y be an R-module and let $\psi_i : M_i \to Y$ be an R-module homomorphism for every $i \in I$. Then there is a unique R-module homomorphism $\psi : \bigoplus_{i\in I} M_i \to Y$ such that for every $i \in I$, $\psi \circ \iota_i = \psi_i$. In particular,*

$$Hom_R(\bigoplus_{i\in I}, Y) \cong \prod_{i\in I} Hom_R(M_i, Y).$$

Proof: We leave this an an exercise. Define $\psi((M_i)_{i\in I}) = \sum_{i\in I} \psi_i(m_i)$. $\square$

Remark 1.12. If I is a finite set, then $\prod_{i\in I} M_i \cong \bigoplus_{i\in I} M_i$.

Exercise 1.13. Let M be an R-module and I be a set. Suppose that for each $i \in I$, M_i is a submodule of M. Further assume:

(1) M is generated by the M_i. (i.e. Each $m \in M$ can be expressed as $m = \sum_{i\in I} m_i$, where all but a finite number of the m_i are zero.)
(2) For all $j \in I$ let N_j be the submodule generated by all M_i with $i \neq j$. Then $N_j \cap M_j = \{0\}$. for all $j \in I$.

Show that

$$M \cong \bigoplus_{i\in I} M_i.$$

2. Exact sequences and diagram chasing

Let us begin with some notation and basic facts. Let $\alpha : M \to N$ be an R-module homomorphism. The kernel of α is defined to be the following subset of M: $ker(\alpha)$=$\{m \in M \,|\, \alpha(m) = 0\}$, and the image of α is defined to be the follwing subset of N: $im(\alpha)$=$\{\alpha(m) \,|\, m \in M\}$. Recall, that $ker(\alpha) = \{0\} \iff \alpha$ is a monomorphism, i.e. an injective homomorphism. It is an epimorphism, i.e. a surjective homomorphism if $im(\alpha) = N$. The cokernel of α is defined to be

$$coker(\alpha) = N/im(\alpha).$$

DEFINITION 2.1. A sequence

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\alpha_{i+1}} M_i \xrightarrow{\alpha_i} M_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

$(i \in \mathbb{Z})$ of linear maps is called exact at M_i if $im(\alpha_{i+1}) = ker\alpha_i$.
The sequence is called exact if it is exact at every $M_i (i \in \mathbb{Z})$.

EXERCISE 2.2. Show that:

(1) $0 \longrightarrow L \xrightarrow{\alpha} M$ is exact if and only if α is a monomorphism.

(2) $M \xrightarrow{\beta} N \longrightarrow 0$ is exact if and only if β is an epimorphism.

(3) $0 \longrightarrow L \xrightarrow{\alpha} M \longrightarrow 0$ is exact iff α is an isomomorphism.

REMARK 2.3. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0.$$

In particular, α is a monomorphism, β is an epimorphism and $im(\alpha) = ker(\beta)$. Hence $N \cong M/\alpha(L)$. Conversely, if $N \cong M/L$, then there is a short exact sequence

$$L \hookrightarrow M \twoheadrightarrow N.$$

DEFINITION 2.4. A short exact sequence

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

(1) splits at N if there exists an R-module homomorphism $\tau : N \to M$ such that $\beta \circ \tau = id_N$.
(2) splits at L if there exists an R-module homomorphism $\sigma : M \to L$ such that $\sigma \circ \alpha = id_L$.

THEOREM 2.5. *Let*

$$\mathrm{E}: \quad 0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence. Then the following are equivalent:

(1) *E splits at L;*
(2) *E splits at N;*
(3) *There exist R-module homomorphisms $\sigma : M \to L$ and $\tau : N \to M$ such that $\sigma \circ \alpha = id_L, \beta \circ \tau = id_N$ and $\alpha \circ \sigma + \tau \circ \beta = id_M$.*

Furthermore, any of the above conditions implies

$$M \cong L \oplus N$$

and we say the short exact sequence E splits.

Proof: $(3) \Rightarrow (1), (2)$ is trivial, $(1) \Rightarrow (3)$ we'll do in lecture and $(2) \Rightarrow (3)$ is an exercise. Now assume (3) and define

$$\begin{array}{rccc} \Theta: & M & \to & L \oplus N \\ & m & \mapsto & (\sigma(m), \beta(m)) \end{array}$$

and show this is an isomorphism. $\square$

Let us get back to the groups $\mathrm{Hom}_R(M, N)$: Let $\alpha \in \mathrm{Hom}_R(M, N)$ and let $\xi : N \to X$ be an R-module homomorphism. We then define

$$\xi_* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, X)$$

by $\xi_*(\alpha) = \xi \circ \alpha$. In other words, $\mathrm{Hom}_R(M, -)$ is a covariant functor. Now let $\psi : Y \to M$ be an R-module homomorphism. We define

$$\psi^* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(Y, N)$$

by $\psi^*(\alpha) = \alpha \circ \psi$. We say $\mathrm{Hom}_R(-, N)$ is a contravariant functor.

THEOREM 2.6. *Let X and Y be R-modules and let*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence. Then the following sequences are exact:

(1) $0 \longrightarrow \mathrm{Hom}_R(Y, L) \xrightarrow{\alpha_*} \mathrm{Hom}_R(Y, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Y, N)$

(2) $0 \longrightarrow \mathrm{Hom}_R(N, X) \xrightarrow{\beta^*} \mathrm{Hom}_R(M, X) \xrightarrow{\alpha^*} \mathrm{Hom}_R(L, X)$.

Proof: We leave (1) as exercise and do (2) in class. □

We say $\mathrm{Hom}_R(-, X)$ and $\mathrm{Hom}_R(Y, -)$ are left exact functors. Neither β_* nor α^* have to be surjective. We'll come back to conditions on X and Y for Hom to be an exact functor.

EXAMPLE 2.7. Consider the following short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

and let $X = \mathbb{Z}$. Then $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$ but $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \neq 0$ and the map $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ is not surjective.

Let us finish this section with two important results, which we'll come back to and apply a little later. We shall nevertheless prove them now in great detail as the methods used are essential to homological algebra, namely diagram chasing. But first let us say what we mean with a commutative diagram. Consider the following diagram of R-modules and R-module homomorphisms:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

We say this diagram commutes if $\beta \circ \alpha = \delta \circ \gamma$.

Proposition 2.8. **[The 5-Lemma]**
Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

be a commutative diagram with exact rows. Then

(1) *If β, δ are monomorphisms, α is an epimorphism, then γ is a monomorphism.*
(2) *If β, δ are epimorphisms and ϵ is a monomorphism, then γ is an epimorphism.*
(3) *If β, δ are isomorphisms, α is an epimorphism and ε is a monomorphism, then γ is an isomorphism.*

Proof: (3) obviously follows from (1) and (2). For (1), see Peter Kropholler's notes [**10**] and part (2) is an exercise, and a very good and useful one. □

Proposition 2.9. **[The Snake-Lemma]** *Let*

$$\begin{tikzcd} & A \arrow[r, "\theta"] \arrow[d, "\alpha"'] & B \arrow[r, "\phi"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\theta'"] & B' \arrow[r, "\phi'"] & C' & \end{tikzcd}$$

be a commutative diagram with exact rows. Then there is a natural exact sequence

$$ker(\alpha) \xrightarrow{\theta_*} ker(\beta) \xrightarrow{\phi_*} \ker(\gamma) \xrightarrow{\delta} coker(\alpha) \xrightarrow{\theta'_*} coker(\beta) \xrightarrow{\phi'_*} coker(\gamma).$$

Moreover, if θ is a monomorphism and ϕ' is an epimorphism, θ_* is a monomorphism and ϕ'_* is an epimorphism.

Before we embark of the proof of this Lemma, let us first note that both kernel and cokernel are functorial, i.e. whenever there's a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

we can insert kernels and cokernels into this diagram and have induced maps γ_* and β_* making the new, bigger diagram commute:

$$\begin{tikzcd} ker(\alpha) \arrow[r, tail] \arrow[d, "\gamma_*"'] & A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[r, two heads] \arrow[d, "\beta"] & coker(\alpha) \arrow[d, "\beta_*"'] \\ ker(\delta) \arrow[r, tail] & C \arrow[r, "\delta"] & D \arrow[r, two heads] & coker(\delta) \end{tikzcd}$$

Proof: The main part is the construction of the connecting map δ, which we will do in detail. The we need to prove exactness at each of 4 places. We also have to check naturality of δ. For a very detailed account see [**10**]. □

3. Projective modules

Projective modules are basically the bread and butter of homological algebra, so let's define them. But first, let's do free modules:

Definition 3.1. Let F be an R-module and X be a subset of F. We say F is free on X if for every R-module A and every map $\xi : X \to A$ there exists a unique R-module homomorphism $\phi : F \to A$ such that $\phi(x) = \xi(x)$ for all $x \in X$.

In other words F is free if there's a unique R-module homorphism ϕ making the following diagram commute:

$$\begin{tikzcd} & F \arrow[dd, dotted, "\phi!"] \\ X \arrow[ur, "i"] \arrow[dr, "\xi"'] & \\ & A \end{tikzcd}$$

A very hard look at this diagram now gives us the following lemma.

Lemma 3.2. *Let F and F' be two modules free, on X. Then $F \cong F'$.*

This gives us uniqueness, i.e. we can talk of *the* free module on X. The following gives us existence and a little bit more.

Exercise 3.3. Let X be a set and consider the R-module

$$E = \bigoplus_{x \in X} R.$$

For each $x \in X$ consider the following map

$$\begin{array}{rcl} s_x : \quad X & \to & R \\ s_x(y) & = & \begin{cases} 1 & \text{x=y} \\ 0 & otherwise \end{cases} \end{array}$$

Let $S = \{s_x \mid x \in X\}$. Show

1. E is free on S.
2. For every free module F on X there is an isomorphism $\Theta : E \to F$, such that $\Theta(s_x) = x$ for all $x \in X$.
3. F is free on X if and only if every element $f \in F$ can be written uniquely as $f = \sum_{x \in X} a_x x$ where $a_x \in R$ and all but a finite number of $a_x = 0$.

Example 3.4.

1. Let k be a field. Then every k-module is free. This is nothing other than saying that every k-vector space has a basis. (For the infinite dimensional case we need Zorn's Lemma).
2. A free $\mathbb{Z}$-module is the same as a free abelian group.
3. $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Lemma 3.5. *Every R-module M is the homomorphic image of a free R-module.*

PROPOSITION 3.6. *Let P be an R-module. Then the following statements are equivalent:*

(1) $\mathrm{Hom}_R(P,-)$ *is an exact functor*
(2) P *is a direct summand of a free module.*
(3) *Every epimorphism* $M \twoheadrightarrow P$ *splits.*
(4) *For every epimorphism* $\pi : A \twoheadrightarrow B$ *of R-modules and every R-module map* $\alpha; P \to B$ *there is an R-module homomorphism* $\phi : P \to A$ *such that* $\pi \circ \phi = \alpha$.

DEFINITION 3.7. Every R-module satisfying the conditions of Proposition 3.6 is called a projective R-module.

REMARK 3.8. Every free R-module is projective, but not every projective is free. Take, for example $R = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $P = \mathbb{Z}/2\mathbb{Z}$ but $(0,1)P = 0$, which contradicts uniqueness of expression of elements in a free module.

LEMMA 3.9. *A direct sum* $P = \bigoplus_{i \in I} P_i$ *is projective iff each* P_i, $i \in I$ *is projective.*

REMARK 3.10. Let k be a field and V a vector space of countable dimension. Then there is a k-vector space isomorphism $V \cong V \oplus V$. Hence, for the ring $R = End_k(V)$ we have the following chain of isomorphisms:

$$R = End_k(V) = \mathrm{Hom}_k(V,V) \cong \mathrm{Hom}_k(V \oplus V, V) \cong \mathrm{Hom}_k(V,V) \oplus \mathrm{Hom}_k(V,V) = R \oplus R$$

and there are free modules on a set of n elements which are isomorphic to free modules on a set with m elements where $n \neq m$.

LEMMA 3.11. [**Eilenberg-Swindle**] *Let R be a ring and P be a projective module. Then there exists a free module such that*

$$P \oplus F \cong F.$$

Let us prove one more important result before we return to define cohomology.

DEFINITION 3.12. Let M be an R-module. A projective resolution of M is an exact sequence

$$\cdots \longrightarrow P_{i+1} \xrightarrow{d_i} P_i \xrightarrow{d_{i+1}} \cdots \xrightarrow[1]{d} P_1 \xrightarrow[0]{d} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where every P_i, $i \geq 0, i \in \mathbb{Z}$, is a projective module.

We also use the short notation

$$\mathbf{P}_* \twoheadrightarrow M.$$

LEMMA 3.13. [**Schanuel's Lemma**] *Let*

$$K \hookrightarrow P \twoheadrightarrow M$$

and

$$K' \hookrightarrow P' \twoheadrightarrow M$$

be two short exact sequences such that P and P' are projective. Then

$$K \oplus P' \cong K' \oplus P.$$

In particular, K is projective if and only if K' is projective.

As a direct consequence of the proof we can now prove inductively the following result about projective resolutions (exercise):

PROPOSITION 3.14. *Let*

$$\mathbf{P}_* \twoheadrightarrow M \quad \text{and} \quad \mathbf{P}'_* \twoheadrightarrow M$$

be two projective resolutions of M *and denote by* $K_n = ker(P_n \to P_{n-1})$ *and* $K' = ker(P'_n \to P'_{n-1})$ *the* n*-th kernels respectively. Then, for all* $n \geq 0$

$$K_n \oplus P'_n \oplus P_{n-1} \oplus ... \cong K'_n \oplus P_n \oplus P'_{n-1} \oplus ....$$

In particular K_n *is projective iff* K'_n *is projective.*

DEFINITION 3.15. Let M be an R-module. We say M has finite projective dimension over R, $\mathrm{pd}_R M < \infty$, if M admits a projective resolution $\mathbf{P}_* \twoheadrightarrow M$ of finite length. In particular, there exists an $n \geq 0$ such that

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resulution of n. The smallest such n is called the projective dimension of M.

REMARK 3.16. The long Schanuel's Lemma 3.14 implies that if $\mathrm{pd}_R M \leq n$ for some n, then in every projective resolution $\mathbf{Q}_* \twoheadrightarrow M$, the kernel
$K_{n-1} = ker(Q_{n-1} \to Q_{n-2})$ is projective.
Projective modules have projective dimension equal to 0.

4. Cochain complexes

In this section we will give a first definition of the Ext-groups. We will later see a more axiomatic approach. The approach in this section will be very hands-on.
Let $\mathbf{d}_* : \mathbf{P}_* \twoheadrightarrow M$ be a projective resolution of the R-module M and let N be an arbitrary R-module. Apply the functor $\mathrm{Hom}_R(-M)$ to this projective resolution and we obtain a sequence of abelian groups (careful, it's not necessarily exact: see Lemma 2.6):

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N) \to \cdots \\ \cdots \to \mathrm{Hom}_R(P_i,N) \to \mathrm{Hom}_R(P_{i+1},N) \to \cdots \end{aligned}$$

EXERCISE 4.1. Let us denote by $\delta_i = d_i^* : \mathrm{Hom}_R(P_i,N) \to \mathrm{Hom}_R(P_{i+1},N)$ for all $i \geq 0$ and by $\delta_{-1} : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(P_0,N)$. Show that for all $i \geq -1$, $\delta_i\delta_{i-1} = 0$, i.e. composition of two consecutive maps in the above sequence is zero.

DEFINITION 4.2. A cochain complex is a family $\mathbf{C} = (C^q, \delta^q)$ of abelian groups C^q together with homomorphisms $\delta^q : C^q \to C^{q+1}$ such that for all $q \in \mathbb{Z}$,

$$\delta^{q+1}\delta^q = 0.$$

The kernel $Z^q(C) = ker(\delta^q) \subseteq C^q$ is called the group of $q - Cocycles$.
The image $B^q(C) = im(\delta_{q-1}) \subseteq C^q$ is called the group of $q - Coboundaries$

Since, by definition, $\delta^{q+1}\delta^q = 0$, it follows that $B^q \subseteq Z^q \subseteq C^q$ and we can make the following definition:

DEFINITION 4.3. The quotient group

$$\mathrm{H}^q(C) = Z^q(C)/B^q(C)$$

is called the q-th cohomology group of $\mathbf{C}$.

REMARK 4.4. Let $\mathbf{C}$ be exact at C^q. Then $B^q = Z^q$ implying $\mathrm{H}^q(C) = 0$.

We can dow define cohomology of the cochain complex $\mathrm{Hom}_R(\mathbf{P}_*, N)$ above, but it still remains to show that it won't change when we choose a different projective resolution of M.

DEFINITION 4.5. Let $\mathbf{C} = (C^q, \delta_q)$ and $\mathbf{C}' = (C'^q, \delta'_q)$ be two cochain complexes. A cochain map $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ is a system $\mathbf{f} = (f_q)_{q\in\mathbb{Z}}$ of homomorphisms $f_q : C^q \to C'^q$ such that $\delta'_{q-1} f_q = f_{q-1}\delta_q$ for all $q \in \mathbb{Z}$.

That is, we have a ladder of commutative squares:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{q-1} \arrow[r, "\delta_{q-1}"] \arrow[d, "f_{q-1}"] & C^q \arrow[r, "\delta_q"] \arrow[d, "f_q"] & C^{q+1} \arrow[r, "\delta_{q+1}"] \arrow[d, "f_{q+1}"] & \cdots \\
\cdots \arrow[r] & C'^{q-1} \arrow[r, "\delta'_{q-1}"] & C'^q \arrow[r, "\delta'_{q-1}"] & C'^{q+1} \arrow[r, "\delta'_{q+1}"] & \cdots
\end{tikzcd}$$

and we often forget the subscripts to our maps, i.e. we just write

$$f\delta = \delta f.$$

DEFINITION 4.6. Let $\mathbf{f}$ and $\mathbf{g} : (C^q, \delta)_{q\in\mathbb{Z}} \to (C'^q, \delta)'_{q\in\mathbb{Z}}$ be two cochain maps. We say $\mathbf{f}$ and $\mathbf{g}$ are homotopic if there is a system $\mathbf{\Psi} = (\Psi_q)_{q\in\mathbb{Z}}$ of homomorphisms $\Psi_q : C^q \to C'^{q-1}$ such that for all $q \in \mathbb{Z}$:

$$\delta'_{q-1} \circ \Psi_q + \Psi_{q+1} \circ \delta_q = f_q - g_q.$$

In particular, if we have two commutative ladders as above, we can fill in Ψ as follows,

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, phantom] & C^{q-1} \arrow[r, "\delta"] \arrow[dl, dotted, "\Psi"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C^q \arrow[r, "\delta"] \arrow[dl, dotted, "\Psi"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C^{q+1} \arrow[r, "\delta"] \arrow[dl, dotted, "\Psi"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & \cdots \arrow[dl, dotted, "\Psi"'] \\
\cdots \arrow[r] & C'^{q-1} \arrow[r, "\delta'"] & C'^q \arrow[r, "\delta'"] & C'^{q+1} \arrow[r, "\delta'"] & \cdots
\end{tikzcd}$$

such that, in short,

$$\delta'\Psi + \Psi\delta = f - g.$$

LEMMA 4.7.

(1) *Every cochain map* $\mathbf{f} : \mathbf{C} \to \mathbf{C}'$ *induces a homomorphism of abelian groups*

$$f^q_* = \mathrm{H}^q(f) : \mathrm{H}^q(C) \to \mathrm{H}^q(C').$$

(2) *Let* $\mathbf{f}$ *and* $\mathbf{g} : \mathbf{C} \to \mathbf{C}'$ *be homotopic cohcain-maps. Then* $f_* = g_*$.

To apply this to projective resolutions, let $\mathbf{P}_* \twoheadrightarrow M$ be a projective resolution and consider the deleted projective resolution

$$\cdots \to P_i \to P_{i-1} \to P_1 \to P_0 \to 0.$$

Note that the sequence is still exact everywhere except at P_0. But note also that $M = coker(P_1 \to P_0)$. We denote by $\mathrm{Hom}_R(P_*, N)$ the cochain complex resulting from applying $\mathrm{Hom}_R(-, N)$ to the deleted resolution.

THEOREM 4.8. *Let* $\mathbf{P}_* \twoheadrightarrow M$ *and* $\mathbf{Q}_* \twoheadrightarrow M$ *be projective resolutions of the* R*-module* M*. Then for all* $n \in \mathbb{Z}$ *and all* R*-modules* N*,*

$$\mathrm{H}^n(\mathrm{Hom}_R(\mathbf{P}_*, N)) \cong \mathrm{H}^n(\mathrm{Hom}_R\, \mathbf{Q}_*, N)).$$

DEFINITION 4.9. Let M and N be R-modules and $\mathbf{P}_* \twoheadrightarrow M$ be a projective resolution of M. We define

$$\mathrm{Ext}^n_R(M,N) \cong \mathrm{H}^n(\mathrm{Hom}_R(\mathbf{P}_*, N)).$$

By the above theorem 4.8, this definition is independent of the choice of projective resolution of M. Please note, that for all $n < 0$, the n-th Ext-group vanishes.

EXERCISE 4.10. Let M and N be R-modules. Prove

(1) $\mathrm{Ext}^0(M,N) \cong \mathrm{Hom}_R(M,N)$.
(2) For every projective R-module P and all $n \geq 1$, $\mathrm{Ext}^n(P,N) = 0$.

5. Long exact sequences in cohomology

Let $\mathbf{C} = (C,d)_{n\in\mathbb{Z}}$ and $\mathbf{C}' = (C',d')_{n\in\mathbb{Z}}$ be two cochain complexes and let $\mathbf{f} : \mathbf{C} \to \mathbf{C}'$ be a cochain map. We say $\mathbf{f}$ is a monomorphism(epimorphism/isomorphism) if for each $n \in \mathbb{Z}$ the maps $f_n : C^n \to C'^n$ are monomorphisms (epimorphisms/isomorphisms). Therefore it makes perfect sense to talk about short exact sequences of cochain complexes. In particular,

$$\mathbf{C}'' \hookrightarrow \mathbf{C} \twoheadrightarrow \mathbf{C}'$$

is a short exaxt sequence of cochain complexes if, for all $n \in \mathbb{Z}$,

$$C''^n \hookrightarrow C^n \twoheadrightarrow C'^n$$

is a short exact sequence of abelian groups.
We could have defined these terms in a more sophisticated manner, noting that the category of cochain complexes is an abelian category and so notions of monomorphisms, epimorphisms and isomorphisms have a category theoretical definition. We would also have to prove that our naive definition agrees with this definition.

THEOREM 5.1. *For every short exact sequence*

$$0 \longrightarrow \mathbf{C}'' \xrightarrow{\alpha} \mathbf{C} \xrightarrow{\beta} \mathbf{C}' \longrightarrow 0$$

of cochain complexes there are natural connecting maps δ such that there is a long exact sequence in cohomology:

$$\cdots \xrightarrow{\delta} \mathrm{H}^n(C'') \xrightarrow{\alpha_*} \mathrm{H}^n(C) \xrightarrow{\beta_*} \mathrm{H}^n(C') \xrightarrow{\delta} \mathrm{H}^{n+1}(C'') \xrightarrow{\alpha_*} \mathrm{H}^{n+1}(C) \longrightarrow \cdots$$

The connecting map $\delta : \mathrm{H}^n(C') \to \mathrm{H}^{n+1}(C'')$ natural means that for every commutative diagram of cochain complexes with exact rows

$$\begin{tikzcd}
\mathbf{C}'' \arrow[r, tail] \arrow[d, "\phi''"] & \mathbf{C} \arrow[r, two heads] \arrow[d, "\phi"] & \mathbf{C}' \arrow[d, "\phi'"] \\
\mathbf{D}'' \arrow[r, tail] & \mathbf{D} \arrow[r, two heads] & \mathbf{C}'
\end{tikzcd}$$

the following diagram commutes for every $n \in \mathbb{Z}$:

$$\begin{tikzcd}
\mathrm{H}^n(C') \arrow[r, "\delta"] \arrow[d, "\phi'_*"'] & \mathrm{H}^{n+1}(C'') \arrow[d, "\delta''_*"] \\
\mathrm{H}^n(D') \arrow[r, "\delta"] & \mathrm{H}^{n+1}(D'').
\end{tikzcd}$$

Exercise 5.2. Let

$$\begin{tikzcd} \mathbf{C}'' \arrow[r, tail] \arrow[d, "\phi''"] & \mathbf{C} \arrow[r, two heads] \arrow[d, "\phi"] & \mathbf{C}' \arrow[d, "\phi'"] \\ \mathbf{D}'' \arrow[r, tail] & \mathbf{D} \arrow[r, two heads] & \mathbf{C}' \end{tikzcd}$$

be a commutative diagram of cochain complexes. Prove that whenever any two of the cochain maps ϕ'', ϕ, ϕ' induce an isomorphism in cohomology, then so does the third.

Next we would like to derive long exact sequences for $\mathrm{Ext}_R(M, N)$. To do this we need to build short exact sequences of projective resolutions, for which the following lemma is an essential step.

Lemma 5.3. **[Horseshoe-Lemma]** *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules and let $K'' \hookrightarrow P'' \twoheadrightarrow M''$ and $K' \hookrightarrow P' \twoheadrightarrow M'$ be short exact sequences with P'' and P' projective. Then there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & K'' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P'' \arrow[r, "\eta"] \arrow[d, two heads, "\pi''"] & P'' \oplus P \arrow[r, "\mu"] \arrow[d, two heads] & P' \arrow[r] \arrow[d, two heads, "\pi'"] & 0 \\ 0 \arrow[r] & M'' \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & M' \arrow[r] & 0 \end{tikzcd}$$

where $\eta(\pi'') = (p'', 0)$ and $\mu(p'', p') = p'$ are the natural inclusion and projection repectively.

Proof: Since P' is projective, there exists a $\lambda : P' \to M$ such that $\beta\lambda = \pi'$. Define

$$\begin{array}{rccc} \pi : & P'' \oplus P & \to & M \\ & (p'', p') & \mapsto & \alpha\pi''(p'') + \lambda(p') \end{array}$$

$\square$

Corollary 5.4. *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. Then there is a short exact sequence of projective resolutions*

$$\mathbf{P}''_* \hookrightarrow \mathbf{P}_* \twoheadrightarrow \mathbf{P}'_*.$$

And now we can apply Theorem 5.1 to Ext:

Theorem 5.5. *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. And let N be an arbitrary R-module. Then there are long exact sequences in cohomology*

(1)

$$\cdots \to \mathrm{Ext}^n(N, M'') \to \mathrm{Ext}^n(N, M) \to \mathrm{Ext}^n(N, M') \to \mathrm{Ext}^{n+1}(N, M'') \to \cdots$$

(2)

$$\cdots \to \mathrm{Ext}^n(M', N) \to \mathrm{Ext}^n(M, N) \to \mathrm{Ext}^n(M'', N) \to \mathrm{Ext}^{n+1}(M', N) \to \cdots$$

LEMMA 5.6. **[Dimension shifting]** *Let $K \hookrightarrow P \twoheadrightarrow M$ be the beginning of a projective resolution of M and let N be an R-module. Then for all $n \geq 1$,*

$$\mathrm{Ext}^n(K, N) \cong \mathrm{Ext}^{n+1}(M, N).$$

Proof: Apply Theorem 5.5 and Exercise 4.10. □

Now let us have a more detailed look at projective dimension. Recall, Definition 3.15 that a module M is said to have $\mathrm{pd}_R M \leq n$ if there is a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

of length n. We say $\mathrm{pd}_R M = n$ if there is no projective resolution of shorter length. Let's summarise all we know so far:

PROPOSITION 5.7. *Let M be an R-module. Then the following statments are equivalent:*

(1) $\mathrm{pd}_R M \leq n$.
(2) $\mathrm{Ext}^i_R(M, -) = 0$ *for all* $i > n$
(3) $\mathrm{Ext}^{n+1}_R(M, -) = 0$
(4) *Let* $0 \to K_{n-1} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *be an exact sequence with* P_i *projective for all* $0 \leq i \leq n-1$. *Then* K_{n-1} *is projective.*

EXERCISE 5.8. Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. Prove the following:

(1) $\mathrm{pd}M' \leq sup\{\mathrm{pd}M, \mathrm{pd}M'' + 1\}$.
(2) $\mathrm{pd}M \leq sup\{\mathrm{pd}M'', \mathrm{pd}M'\}$.
(3) $\mathrm{pd}M'' \leq sup\{\mathrm{pd}M, \mathrm{pd}M' - 1\}$.

(This is an exercise in applying Theorem 5.5)

Let us finish this section with a very useful observation:

PROPOSITION 5.9. *Let M be an R-module such that $\mathrm{pd}M = n$. Then there exists a free R-module F such that*

$$\mathrm{Ext}^n(M, F) \neq 0.$$

6. Injective modules

DEFINITION 6.1. Let I be an R-module. We say I is injective if for every injective R-module homomorphism $\iota : A \hookrightarrow B$ and every R-module homomorphism $\alpha : A \to I$ there exists an R-module homomorphism $\beta : B \to I$ such that $\beta\iota = \alpha$.

PROPOSITION 6.2. *Let I be an R-module. Then the following are equivalent:*

(1) *I is injective.*
(2) $\mathrm{Hom}_R(-, I)$ *is exact.*
(3) *Every injective R-module homomorphism $I \hookrightarrow B$ splits, where B an arbitrary R-module.*

Proof: This is an exercise. □

PROPOSITION 6.3. *Let I be an injective module. Then for every R-module M and all $n \geq 1$,*

$$\mathrm{Ext}^n(M, I) = 0.$$

Proof: Let $\mathbf{P}_* \twoheadrightarrow M$ be a projective resolution of M. Then $\mathrm{Hom}_R(\mathbf{P}_*, I)$ is exact in degree $n > 0$. □

The reader will have noticed that injective modules have dual properties to those of projective modules. We have seen how to construct projective modules (3.3) and that every module has a projective mapping onto it (3.5). Analogous results hold but are slightly more complicated. One can show how to build injectives and that every module maps into an injective. A thorough account of these facts can be found in Rotman's book [**25**, pages 65–71].

EXERCISE 6.4. Prove the following:

(1) Let $\{E_j \mid j \in J\}$ be a family of injective modules, then $\prod_{j\in J} E_j$ is injective.
(2) Every summand of an injective module is injective.

7. An axiomatic approach to cohomology

We have seen before that we can set up cohomology using category theoretical language, such that Ext is just an example. Let us begin by recalling a few of the definitions in category theory. A detailed account of all the main results we might be needing later can be found in Hilton-Stammbach [**9**, Chapter II, Sections 1–6.]. A category $\mathfrak{C}$ consists of three sets of data:

- A class of objects $A, B, C, \ldots$
- To each pair of objects A, B of $\mathfrak{C}$ a set of morphisms $\mathfrak{C}(A, B)$ $(f : A \to B)$ from A to B
- to each triple of objects A, B, C of $\mathfrak{C}$ a law of composition $\mathfrak{C}(A, B) \times \mathfrak{C}(B, C) \to \mathfrak{C}(A, C)$ $((f, g) \mapsto g \circ f)$.

satisfying the following three axioms:

(1) The sets $\mathfrak{C}(A, B)$ and $\mathfrak{C}(A', B')$ are disjoint unless $A = A'$ and $B = B'$.
(2) given $f : A \to B$, $g : B \to C$ and $h : C \to D$ then $h(gf) = (hg)f$.
(3) To each object A there is a morphism $1_A : A \to A$ such that for any $f : A \to B$ and $g : C \to A$, $f1_A = f$ and $1_A g = g$.

Here is a small, by no means exhaustive list of categories:

- The category $\mathfrak{S}$ of sets and functions
- The category $\mathfrak{G}$ of groups and group homomorphisms
- The category $\mathfrak{Ab}$ of abelian groups and homomorphisms
- The category $\mathfrak{Top}$ of topological spaces and continuous functions
- The category $\mathfrak{V}_k$ of vector spaces over a field k and linear transformations
- The category $\mathfrak{R}$ of rings and ring homomorphisms
- The category $\mathfrak{Mod}_R$ of (left) R-modules and linear maps.

We now need a transformation from one category to another. This is called a *functor*. A *functor* $F : \mathfrak{C} \to \mathfrak{D}$ is a rule associating to each object $A \in \mathfrak{C}$ an object $FA \in \mathfrak{D}$ and to every morphism $f \in \mathfrak{C}(A, B)$ a morphism $Ff \in \mathfrak{D}(FA, FB)$ such that

$$F(fg) = (Ff)(Fg), \qquad F(1_A) = 1_{FA}.$$

Let us look at a few examples:

- The embedding of a subcategory $\mathfrak{C}_0$ into $\mathfrak{C}$ is a functor.
- Underlying every R-module M there is a set. Hence we get the forgetful functor $\mathfrak{Mod}_R \to \mathfrak{S}$.
- $\mathfrak{Mod}_R(A, B) = \mathrm{Hom}_R(A, B)$ can be given the structure of an abelian group. Fix A, then we obtain a functor $\mathrm{Hom}_R(A, -) : \mathfrak{Mod}_R \to \mathfrak{Ab}$ by $\mathrm{Hom}_R(A, -)(B) = \mathrm{Hom}_R(A, B)$.
- Similarly, we have functors $\mathrm{Ext}^n_R(A, -) : \mathfrak{Mod}_R \to \mathfrak{Ab}$.

A quick check shows that $\mathrm{Hom}_R(-, A)$ is not a functor, but we can repair this easily. For every category $\mathfrak{C}$ define the opposite category $\mathfrak{C}^{opp}$, which has the same objects as $\mathfrak{C}$ but the morhisms sets are defined to be $\mathfrak{C}^{opp}(A, B) = \mathfrak{C}(B, A)$. Now we can see, that both $\mathrm{Hom}_R(-, A)$ and $\mathrm{Ext}^n_R(-, A)$ are functors from the opposite category $\mathfrak{Mod}_R^{opp}$ to $\mathfrak{Ab}$. We also say these are contravariant functors from $\mathfrak{Mod}_R$ to $\mathfrak{Ab}$.

Let us finally come to the notion of a natural transformation. Naturality is an important concept we already have spent some time explaining before:

Let F and G be two functors $\mathfrak{C} \to \mathfrak{D}$. Then a *natural transformation* $t : F \to G$ is a rule assigning to each object $A \in \mathfrak{C}$ a morphism $t_A : FA \to GA$ such that for any morphism $f : A \to B$ in $\mathfrak{C}$ the following diagram commutes:

$$\begin{tikzcd}
FA \arrow[r, "t_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "t_B"] & GB.
\end{tikzcd}$$

We have seen a natural transformation in Lemma 1.7: For every R-module M there is a natural isomorphism: $\phi : \mathrm{Hom}_R(R, M) \to M$ defined by $f \mapsto f(1)$. The map ϕ is a natural transformation from $\mathrm{Hom}_R(R, -)$ to the identity functor.

From now on let R and S denote two rings and we consider the two categories $\mathfrak{Mod}_R$ and $\mathfrak{Mod}_S$. In most of our applications $S = \mathbb{Z}$ and $\mathfrak{Mod}_S = \mathfrak{Ab}$.

Definition 7.1. A cohomological functor from $\mathfrak{Mod}_R$ to $\mathfrak{Mod}_S$ is a family $(U^n)_{n\in\mathbb{Z}}$ of functors $U^n : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ together with natural connecting maps $\delta : U^n(M') \to U^{n+1}(M'')$ for each short exact sequence $M" \hookrightarrow M \twoheadrightarrow M'$ and each $n \in \mathbb{Z}$ such that the following axiom holds:

AXIOM (Long exact sequence)

For each short exact sequence $0 \longrightarrow M'' \xrightarrow{\iota} M \xrightarrow{\pi} M' \longrightarrow 0$ there is a long exact sequence

$$\cdots \xrightarrow{\delta} U^n(M'') \xrightarrow{\iota_*} U^n(M) \xrightarrow{\pi_*} U^n(M') \xrightarrow{\delta} U^{n+1}(M'') \xrightarrow{\iota_*} \cdots$$

We also require the following optional axiom

AXIOM (Coeffaceability)

U^n is zero for all $n < 0$ and $U^n(I) = 0$ for all injective R-modules and all $n \geq 1$.

It is extremely important to understand the significance of the maps δ to be natural. Let

$$\begin{tikzcd} M'' \arrow[r, tail, "\iota"] \arrow[d, "\phi''"'] & M \arrow[r, two heads, "\pi"] \arrow[d, "\phi"'] & M' \arrow[d, "\phi'"] \\ N'' \arrow[r, tail, "\eta"] & N \arrow[r, two heads, "\rho"] & N' \end{tikzcd}$$

be a commutative diagram of R-modules with exact rows. Then the naturality of δ ensures we get a commutative ladder

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & U^n(M'') \arrow[r, "\iota_*"] \arrow[d, "\phi''_*"'] & U^n(M) \arrow[r, "\pi_*"] \arrow[d, "\phi_*"'] & U^n(M') \arrow[r, "\delta"] \arrow[d, "\phi'_*"'] & U^{n+1}(M'') \arrow[r, "\iota_*"] \arrow[d, "\phi''_*"'] & \cdots \\ \cdots \arrow[r, "\delta"] & U^n(N'') \arrow[r, "\eta_*"] & U^n(M) \arrow[r, "\rho_*"] & U^n(M') \arrow[r, "\delta"] & U^{n+1}(M'') \arrow[r, "\eta_*"] & \cdots \end{tikzcd}$$

Commutativity of the squares not involving δ follows directly from functoriality of each U^n.

EXAMPLE 7.2. $\mathrm{Ext}^*(M,-)$ is a coeffaceable cohomological functor from $\mathfrak{Mod}_R$ to $\mathfrak{Ab}$. We have shown the long exact sequence axiom in Theorem 5.5 and coeffaceability follows from Proposition 6.3.

THEOREM 7.3. *Let U^* and V^* be two cohomological functors from $\mathfrak{Mod}_R$ to $\mathfrak{Mod}_S$ such that U^* is coeffaceable. Then any natural map $\nu^0 : U^0 \to V^0$ extends uniquely to a natural map $\nu^* : U^* \to V^*$*

EXERCISE 7.4. Use Theorem 7.3 to show that the definition of $\mathrm{Ext}^*(M,-)$ is independent of the choice of projective resolution of M. Hint: Use the fact that $\mathrm{H}^0(\mathrm{Hom}(\mathbf{P}_*, N)) \cong \mathrm{Hom}_R(M,N)$ for any projective resolution $\mathbf{P}_* \twoheadrightarrow \mathbf{M}$.

EXERCISE 7.5. Use Theorem 7.3 to show that composition of maps

$$\mathrm{Hom}_R(B,C) \times \mathrm{Hom}_R(A,B) \to \mathrm{Hom}_R(A,C)$$

extends to a biadditive product, the *Yoneda product*

$$\mathrm{Ext}^n_R(B,C) \times \mathrm{Ext}^m(A,B) \to \mathrm{Ext}^{m+n}_R(A,C).$$

8. Chain complexes

A chain complex (C_*, d) of R-modules is a family $(C_n)_{n\in\mathbb{Z}}$ of R-modules together with maps $d : C_n \to C_{n-1}$ such that composition of two consecutive maps is zero, i.e. $dd = 0$. We write $Z_n = ker(C_n \to C_{n-1})$ for the n-cycles and $B_n = im(C_{n+1} \to C_n)$ for the n-cycles and the n-th homology of $\mathbf{C}_*$ is defined to be

$$H_n(\mathbf{C}_*) = Z_n/B_n.$$

Alternatively, we may say that a chain complex (C_*, d) of R-modules is a family $(C_n)_{n\in\mathbb{Z}}$ of R-modules together with maps $d : C_n \to C_{n-1}$ such that $(\hat{C}^n)_{n\in\mathbb{Z}} = (C_{-n})_{n\in\mathbb{Z}}$ is a cochain complex and $H_n(\mathbf{C}_*) = H^{-n}(\hat{\mathbf{C}}_*)$. All theorems we have established for cohomology work for homology without requiring seperate proof. In particular, every short exact sequence of chain complexes

$$\mathbf{C}''_* \hookrightarrow \mathbf{C}_* \twoheadrightarrow \mathbf{C}'_*$$

gives rise to a long exact sequence in homology with natural connecting maps:

$$\cdots \to H_n(C'') \to H_n(C) \to H_n(C') \to H_{n-1}(C'') \to H_{n-1}(C) \to \cdots .$$