Corollary 6.5. *Let I be a closed ideal in a C*-algebra B. Then $I = I^*$.*

Proof. Let $u_\lambda \in I$ be an approximate unit in I. Then $\lim_\lambda u_\lambda x = x \Rightarrow \lim_\lambda x^* u_\lambda = x^*$ lies in I as I is closed. $\square$

Theorem 6.6. *Let I be a closed ideal in a C*-algebra A. Then A/I is a C*-algebra with respect to the quotient norm.*

Proof. As $I = I^*$, we can define an involution on A/I by setting $\dot{x}^* := (x^*)\dot{}$. It remains to show that $\|\dot{x}^*\dot{x}\| = \|\dot{x}\|^2$. Let (u_λ) be an approximate unit in I, and let $x \in A$. Then $\|\dot{x}\| = \lim_\lambda \|x - u_\lambda x\|$. This is because given $\varepsilon > 0$, we can find $z \in I$ with $\|x+z\| \le \|\dot{x}\| + \varepsilon$, and we can find $\lambda_0 \in \Lambda$ with $\|z - u_\lambda z\| < \varepsilon \; \forall \lambda \ge \lambda_0$. Hence

$$\|\dot{x}\| \le \|x - u_\lambda x\| \le \|(1-u_\lambda)(x+z)\| + \|(1-u_\lambda)z\| \le \|1-u_\lambda\| \, \|x+z\| + \varepsilon \le \|x+z\| + \varepsilon \le \|\dot{x}\| + 2\varepsilon \; \forall \lambda \ge \lambda_0.$$

Therefore, for all $z \in I$, we have

$$\|\dot{x}\|^2 = \lim_\lambda \|x - u_\lambda x\|^2 = \lim_\lambda \|(1-u_\lambda)x^*x(1-u_\lambda)\| = \lim_\lambda \|(1-u_\lambda)(x^*x+z)(1-u_\lambda)\| \le \|x^*x+z\|.$$

Taking the infimum over all $z \in I$, we obtain $\|\dot{x}\|^2 \le \|\dot{x}^*\dot{x}\|$. This suffices to prove the C*-identity by Remark 4.4, 2). $\square$

Remark 6.7. Assume that A is a separable C*-algebra, i.e., A has a countable dense subset. Then there is a countable approximate unit $(u_n)_{n\in\mathbb{N}}$ (i.e., a sequence) with $u_1 \le u_2 \le \dots$.

Proof. Let $\{x_i \colon i \in \mathbb{N}\} \subseteq A$ be dense, and (u_λ) an approximate unit in A. Choose inductively λ_n such that $\lambda_n \ge \lambda_{n-1}$ and $\|u_\lambda x_i - x_i\| < \frac{1}{n}$, $\|x_i u_\lambda - x_i\| < \frac{1}{n}$ for all $\lambda \ge \lambda_n$, $1 \le i \le n$. Then set $u_n := u_{\lambda_n}$. $\square$

Corollary 6.8. *Let $\varphi : A \to B$ be a *-homomorphism between two C*-algebras A and B.*

(a) *Let $z \in A$ be normal and $f \in C_0(\mathrm{Sp}\,(z))$. Then $\varphi(f(z)) = f(\varphi(z))$.*
(b) *If φ is injective, then φ is isometric.*
(c) *$\varphi(A)$ is a C*-algebra, and $\varphi(A) \cong A/\ker(\varphi)$ as C*-algebras.*

Proof. (a) Let p_n be polynomials in $x, \overline{x}$ such that $\lim_n p_n = f$ in $C_0(\mathrm{Sp}\,(z))$. Then $\lim_n p_n(z) = f(z)$ in A. Moreover, as $\mathrm{Sp}\,(\varphi(z)) \subseteq \mathrm{Sp}\,(z)$, $f(\varphi(z))$ is well-defined and we have $\lim_n p_n(\varphi(z)) = f(\varphi(z))$ in B. Hence, as φ is continuous by Theorem 4.9,

$$\varphi(f(z)) = \lim_n \varphi(p_n(z)) = \lim_n p_n(\varphi(z)) = f(\varphi(z)).$$

(b) It is enough to show that $\|\varphi(x^*x)\| = \|x^*x\|$. Assume that φ is not isometric, i.e., there is $x \in A$ with $\|\varphi(x^*x)\| < \|x^*x\|$. Define $f : \mathbb{R} \to \mathbb{R}$ by setting $f \equiv 0$ on $(-\infty, \|\varphi(x^*x)\|)$, $f \equiv 1$ on $(\|x^*x\|, \infty)$ and extend f linearly on $[\|\varphi(x^*x)\|), \|x^*x\|]$. Then $\|f(x^*x)\| = \|f\|_\infty = 1$ as $\|x^*x\| \in \mathrm{Sp}\,(x^*x)$ (since $r(x^*x) = \|x^*x\|$). Thus $f(x^*x) \neq 0$. Also, $\|f(\varphi(x^*x))\| = 0 \Rightarrow f(\varphi(x^*x)) = 0$. As $\varphi(f(x^*x)) = f(\varphi(x^*x))$ by (a), this shows that φ is not injective.

(c) Define $\dot{\varphi} : A/\ker(\varphi) \to B$ by setting $\dot{\varphi}(\dot{x}) := \varphi(x) \; \forall x \in A$. This is a well-defined *-homomorphism, which is injective, hence isometric by (b). The image of $\dot{\varphi}$ is equal to $\varphi(A)$, which therefore must be complete, hence closed, so that it is a C*-algebra. $\square$

7. Positive linear functionals

Definition 7.1. *Let A be a C*-algebra, $\varphi : A \to \mathbb{C}$ a linear map (also called functional). φ is called positive (written $\varphi \ge 0$) if $\varphi(x) \ge 0 \; \forall x \ge 0$.*

Remark 7.2. $\varphi \ge 0 \Leftrightarrow \varphi(x^*x) \ge 0 \; \forall x \in A$. Moreover, φ preserves the partial order, i.e., $x \le y \Rightarrow \varphi(x) \le \varphi(y)$.

Examples 7.3. *(a) $A = C[0,1]$. Then for any $t \in [0,1]$, $\varphi(f) = f(t)$ is positive. Also $\varphi(f) = \int_0^1 f(t)dt$ is positive.*

(b) More generally, let $A = C(X)$, where X is compact Hausdorff. Then positive functionals on A are in bijection with Radon measures on X, $\varphi \leftrightarrow \mu$, where $\varphi(f) = \int_X f(x)d\mu(x)$.

(c) $A = M_n(\mathbb{C})$. Then the trace $M_n(\mathbb{C}) \to \mathbb{C}$, $(a_{ij}) \mapsto \sum_i a_{ii}$ is positive.

(d) Let H be a Hilbert space, $A = \mathcal{L}(H)$. For every $\xi \in H$, the functional $\varphi_\xi(x) := \langle x\xi, \xi\rangle$ is positive.

Theorem 7.4. *Let A be a C*-algebra and φ a positive functional on A. Then φ is bounded.*

Proof. First we show that φ is bounded on $S = \{x \in A \colon x \geq 0, \|x\| \leq 1\}$. Suppose not, i.e., there exists $(a_n) \subseteq S$ with $\varphi(a_n) \geq 2^n$. Define $a := \sum_n 2^{-n} a_n$. Then a is positive since A_+ is a closed convex cone. But then we have for every $N \in \mathbb{N}$ that $\varphi(a) \geq \sum^N 2^{-n}\varphi(a_n) \geq N$. ↯

So we know that $|\varphi(x)| \leq C\|x\|$ for all positive $x \in A$. Now take $z \in A$ arbitrary. Write

$$z = \mathrm{Re}(z) + \mathrm{i}\cdot\mathrm{Im}(z) = \mathrm{Re}(z)_+ - \mathrm{Re}(z)_- + \mathrm{i}\cdot\mathrm{Im}(z)_+ - \mathrm{i}\cdot\mathrm{Im}(z)_-$$

as a linear combination of four positive elements with norm bounded by $\|z\|$. Hence, by the above, we have $|\varphi(z)| \leq 4C\|z\|$. □

Proposition 7.5. *Let A be a C*-algebra, φ a positive functional on A. Then $\varphi(x^*) = \overline{\varphi(x)}$, and $|\varphi(x)|^2 \leq \|\varphi\|\,\varphi(x^*x)$ $\forall x \in A$.*

Proof. The first claim follows directly by writing arbitrary elements in A as linear combinations of four positive elements. To prove the second claim, define $\langle x,y\rangle := \varphi(y^*x)$. This is a sesquilinear form with $\langle x,x\rangle \geq 0$. By the Cauchy-Schwartz inequality, we obtain $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$. Also, if (u_λ) is an approximate unit, then $\varphi(x^*) = \lim_\lambda \varphi(x^*u_\lambda) = \lim_\lambda \overline{\varphi(u_\lambda x)} = \overline{\varphi(x)}$. Hence

$$|\varphi(x)|^2 = \lim_\lambda |\varphi(u_\lambda x)|^2 \leq \limsup_\lambda \varphi(u_\lambda^2)\varphi(x^*x) \leq \|\varphi\|\,\varphi(x^*x).$$

□

Theorem 7.6. *Let A be a C*-algebra, $\varphi : A \to \mathbb{C}$ a continuous functional. The following are equivalent:*

(1) $\varphi \geq 0$,
(2) *For every approximate unit (u_λ) in A, we have $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$,*
(3) *There is an approximate unit (u_λ) in A such that $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$.*

Proof. (1) $\Rightarrow$ (2): Without loss of generality assume that $\|\varphi\| = 1$. $(\varphi(u_\lambda))$ is an increasing bounded net in $\mathbb{C}$, so $\lim_\lambda \varphi(u_\lambda) = \alpha \leq 1$. For $x \in A$ with $\|x\| \leq 1$, we have

$$|\varphi(x)|^2 = \lim_\lambda |\varphi(u_\lambda x)|^2 \leq \limsup_\lambda \varphi(u_\lambda^2)\varphi(x^*x) \leq \lim_\lambda \varphi(u_\lambda)\varphi(x^*x) \leq \alpha \leq 1.$$

As $\|\varphi\| = 1$, there must exist $x \in A$ with $|\varphi(x)|$ arbitrarily close to 1, so that $\alpha = 1$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): Assume that $1 = \|\varphi\| = \lim_\lambda \varphi(u_\lambda)$. We first show that given $x \in A_{\mathrm{sa}}$ with $\|x\| \leq 1$, we must have $\varphi(x) \in \mathbb{R}$. Write $\varphi(x) = \alpha + \mathrm{i}\cdot\beta$. We may assume $\beta \leq 0$ (otherwise replace x by $-x$). Suppose that $\beta < 0$. Then

$$\|x - \mathrm{i}\cdot nu_\lambda\|^2 = \|(x+\mathrm{i}\cdot nu_\lambda)(x-\mathrm{i}\cdot nu_\lambda)\| = \left\|x^2 + n^2u_\lambda^2 - \mathrm{i}\cdot n(xu_\lambda - u_\lambda x)\right\| \leq 1 + n^2 + n\|xu_\lambda - u_\lambda x\|,$$

so that

$$|\beta|^2 + 2|\beta|n + n^2 = (|\beta|+n)^2 \leq |\varphi(x) - \mathrm{i}\cdot n|^2 = \lim_\lambda |\varphi(x - \mathrm{i}\cdot nu_\lambda)|^2 \leq \lim_\lambda 1 + n^2 + n\|xu_\lambda - u_\lambda x\| = 1 + n^2.$$

But this would imply $|\beta|^2+2|\beta|n \leq 1$ for all $n \in \mathbb{N}$. $\lightning$

Now let $x \geq 0$ with $\|x\| \leq 1$. Then $-1 \leq u_\lambda - x \leq 1$, so that $\|u_\lambda - x\| \leq 1$, and hence $1-\varphi(x) = \lim_\lambda \varphi(u_\lambda - x) \leq 1$. This implies $0 \leq \varphi(x)$. ☐

Corollary 7.7. *Let A be a C*-algebra with unit 1 and φ a continuous functional on A. Then $\varphi \geq 0 \Leftrightarrow \varphi(1) = \|\varphi\|$.*

Proof. Just take $u_\lambda = 1$. ☐

Corollary 7.8. *Let φ and φ' be two positive functionals on a C*-algebra A. Then $\|\varphi+\varphi'\| = \|\varphi\| + \|\varphi'\|$.*

Proof. Given an approximate unit (u_λ), we have

$$\|\varphi\| + \left\|\varphi'\right\| = \lim_\lambda \varphi(u_\lambda) + \varphi'(u_\lambda) = \lim_\lambda (\varphi+\varphi')(u_\lambda) = \left\|\varphi+\varphi'\right\|.$$

☐

Definition 7.9. *A state on a C*-algebra A is a positive functional φ with $\|\varphi\| = 1$.*

Theorem 7.10. *Let A be a C*-algebra, $x \in A$ normal. Then there is a state φ on A with $|\varphi(x)| = \|x\|$.*

Proof. $C^*(x,1) \subseteq \tilde{A}$ is commutative, so by Theorem 4.19, there exists $\varphi_0 \in \operatorname{Spec} C^*(x,1)$ with $\varphi_0(1) = 1$ and $|\varphi_0(x)| = \|x\|$. Using the Hahn-Banach Theorem, we can extend φ_0 to a continuous functional $\tilde{\varphi}$ on $\tilde{A}$ with $\|\tilde{\varphi}\| = \|\varphi_0\| = 1 = \varphi_0(1) = \tilde{\varphi}(1)$. By Theorem 7.6, $\tilde{\varphi} \geq 0$. $\varphi := \tilde{\varphi}|_A$ is the desired functional on A. ☐