Our goal is to show that every C*-algebra A is isometrically isomorphic to a closed sub-*-algebra of $\mathcal{L}(H)$ for some Hilbert space H. It suffices to construct an injective *-homomorphism $\pi: A \to \mathcal{L}(H)$ for some H. In the following, let A be a C*-algebra.

Definition 8.1. *A representation of A is a pair (π, H), where H is a Hilbert space and $\pi: A \to \mathcal{L}(H)$ a *-homomorphism.*

Two representations (π_1, H_1) and (π_2, H_2) are called equivalent (written $(\pi_1, H_1) \sim (\pi_2, H_2)$) if there exists a unitary $U: H_1 \to H_2$ such that $\pi_2(x) = U\pi_1(x)U^* \ \forall x \in A$.

Given a family $((\pi_i, H_i)_{i\in I}$ of representations, we can form the representation $(\bigoplus_i \pi_i, \bigoplus_i H_i)$ on the Hilbert space $\bigoplus_i H_i = \left\{(\xi_i)\colon \xi_i \in H_i, \sum_i \|\xi_i\|^2 < \infty\right\}$, where $(\bigoplus_i \pi_i)(x)(\xi_i) = (\pi_i \xi_i)$.

Remark 8.2. Let H be a Hilbert space, $K \subseteq H$ a closed subspace. Let (π, H) be a representation of A. K is called π-invariant if $\pi(A)K \subseteq K$. Then the orthogonal complement $K^\perp$ is also π-invariant. This is because given $\xi \in K$ and $\eta \in K^\perp$, we have $\langle \xi, \pi(x)\eta \rangle = \langle \pi(x^*)\xi, \eta \rangle = 0 \ \forall x \in A$. Moreover, $H = K \oplus K^\perp$, and if we define $\pi_1(x) := \pi(x)|_K$, $\pi_2(x) := \pi(x)|_{K^\perp}$, then $(\pi, H) \sim (\pi_1 \oplus \pi_2, K \oplus K^\perp)$.

Definition 8.3. *A representation (π, H) is called non-degenerate if $\overline{\pi(A)H} = H$.*

A representation (π, H) is called cyclic if there exists $\xi \in H$ with $\overline{\pi(A)\xi} = H$. In that case, ξ is called a cyclic vector.

Remark 8.4. (π, H) cyclic $\Rightarrow$ (π, H) non-degenerate, but "$\Leftarrow$" is not true.

Remark 8.5. Let (π, H) be a non-degenerate representation, and let (u_λ) be an approximate unit in A. Then $\lim_\lambda \pi(u_\lambda)\eta = \eta \ \forall \eta \in H$. This is because given arbitrary $y \in A$ and $\xi \in H$, we have for $\eta := \pi(y)\xi$ that $\lim_\lambda \pi(u_\lambda)\eta = \lim_\lambda \pi(u_\lambda y)\xi = \pi(y)\xi = \eta$. Now use that $\pi(A)H$ is dense in H.

Proposition 8.6. *(a) Every representation is the direct sum of a non-degenerate representation and the zero representation.*

(b) Every non-degenerate representation is a direct sum of cyclic representations.

Proof. Let (π, H) be a representation.

(a) Let $K := \overline{\pi(A)H}$. Then K is a closed, π-invariant subspace. $\pi|_K$ is non-degenerate. Moreover, Remark 8.2 shows that $K^\perp$ is π-invariant. It is clear that $\pi|_{K^\perp} = 0$.

(b) Assume that (π, H) is non-degenerate. By Zorn's Lemma, there is a maximal family $(K_i, \xi_i)_{i\in I}$ of closed subspaces K_i which are pairwise orthogonal, and vectors $\xi_i \in K_i$ such that $K_i = \overline{\pi(A)\xi_i}$. It remains to show that $\bigoplus_i K_i = H$, or equivalently, $(\bigcup_i K_i)^\perp = \{0\}$. So choose $\eta \in H$, $\eta \perp K_i \ \forall i \in I$, and assume $\eta \neq 0$. Set $K := \overline{\pi(A)\eta}$. Then $K \perp K_i$, $K \neq \{0\}$ because given an approximate unit (u_λ) in A, we have $\lim_\lambda \pi(u_\lambda)\eta = \eta \neq 0$ by Remark 8.5. But then we could add (K, η) to our family $(K_i, \xi_i)_{i\in I}$, contradicting maximality. $\square$

Proposition 8.7. *Let (π_1, H_1) and (π_2, H_2) be cyclic representations with cyclic vectors ξ_1 and ξ_2. Define $f_i(x) := \langle \pi(x)\xi_i, \xi_i \rangle$ for $i = 1, 2$. Then f_i are positive functionals on A. If $f_1 = f_2$, then there is a unitary $U: H_1 \to H_2$ such that $U\xi_1 = \xi_2$ and $\pi_1(x) = U^*\pi_2(x)U \ \forall x \in A$.*

Proof. First, we show that $V: \pi_1(A)\xi_1 \to \pi_2(A)\xi_2$ is isometric (in particular well-defined). We have

$$\langle \pi_2(x)\xi_2, \pi_2(x)\xi_2 \rangle = \langle \pi_2(x^*x)\xi_2, \xi_2 \rangle = f_2(x^*x) = f_1(x^*x) = \langle \pi_1(x)\xi_1, \pi_1(x)\xi_1 \rangle.$$

So in particular, V is well-defined, because $\pi_1(x)\xi_1 = \pi_1(y)\xi_1 \Rightarrow \pi_1(x-y)\xi_1 = 0 \Rightarrow \pi_2(x-y)\xi_2 = 0$.

Now V extends to a unitary $U : \overline{\pi_1(A)\xi_1} \to \overline{\pi_2(A)\xi_2}$. We have for all $x, y \in A$

$$U\pi_1(x)U^*\pi_2(y)\xi_2 = U\pi_1(xy)\xi_1 = \pi_2(xy)\xi_2 = \pi_2(x)\pi_2(y)\xi_2,$$

hence $U\pi_1(x)U^* = \pi_2(x)$ $\forall x \in A$ because ξ_2 is a cyclic vector. Now let (u_λ) be an approximate unit in A. Then, by Remark 8.5, $U\xi_1 = U\lim_\lambda \pi_1(u_\lambda)\xi_1 = \lim_\lambda \pi_2(u_\lambda)\xi_2 = \xi_2$. □

Theorem 8.8. *Let f be a state on A. Then there exists a cyclic representation (π_f, H_f) of A with cyclic vector ξ_f such that $f(x) = \left\langle \pi_f(x)\xi_f, \xi_f \right\rangle$ $\forall x \in A$.*

Note that (π_f, H_f) and ξ_f are unique up to (unitary) equivalence by Proposition 8.7.

Proof. $\langle x, y\rangle := f(y^*x)$ defines a positive sesquilinear form on A. Let $N_f := \left\{x \in A\colon \langle x, x\rangle_f = 0\right\}$. $K_f := A/N_f$ is a pre-Hilbert space, and the $\|\cdot\|_f$-closure $H_f := \overline{K_f}$ is a Hilbert space. Let $\gamma\colon A \to K_f \subseteq H_f$ be the canonical quotient map. Then $\langle \gamma(x), \gamma(y)\rangle = f(y^*x)$. It follows that $\|\gamma(x)\|^2 = \langle \gamma(x), \gamma(x)\rangle = f(x^*x) \le \|x\|^2$, so that γ is continuous. Define $\pi_f^0(x)\gamma(y) := \gamma(xy)$. Then

$$\|\gamma(xy)\|^2 = f(y^*x^*xy) \le \|x^*x\|\, f(y^*y) = \|x\|^2\,\|\gamma(y)\|^2,$$

so that $\pi_f^0(x) : K_f \to K_f$ is well-defined and continuous with $\left\|\pi_f^0(x)\right\| \le \|x\|$. Now extend $\pi_f^0(x)$ continuously to $\pi_f(x) : H_f \to H_f$. We still have $\left\|\pi_f(x)\right\| \le \|x\|$. We have $\pi_f^0(x)\pi_f^0(y) = \pi_f^0(xy)$, so that $\pi_f(x)\pi_f(y) = \pi_f(xy)$. Moreover, we have $\pi_f(x^*) = \pi_f(x)^*$ because $\left\langle \pi_f^0(x)\gamma(y), \gamma(z)\right\rangle = f(z^*xy) = f((x^*z)^*y) = \left\langle \gamma(y), \pi_f^0(x^*)\gamma(z)\right\rangle$.

Now let (u_λ) be an approximate unit in A. We claim that $(\gamma(u_\lambda))$ is a Cauchy net. Given $\varepsilon > 0$, choose λ_0 such that for all $\lambda \ge \lambda_0$, $|f(u_\lambda) - 1| < \varepsilon$. Then for all $\lambda \ge \lambda_0$, we have

$$\left\|\gamma(u_\lambda) - \gamma(u_{\lambda_0})\right\|^2 = f((u_\lambda - u_{\lambda_0})^2) \le f(u_\lambda - u_{\lambda_0}) \le |f(u_\lambda) - 1| + |f(u_{\lambda_0}) - 1| < 2\varepsilon.$$

Hence we may define $\xi_f := \lim_\lambda \gamma(u_\lambda)$, and then we have

$$\left\langle \pi(x)\xi_f, \xi_f\right\rangle = \lim_\lambda f(u_\lambda x u_\lambda) = f(x).$$

Finally, to show that ξ_f is cyclic, take $x \in A$ arbitrary. Then $\lim_\lambda x u_\lambda = x$, so that $\pi_f(x)\xi_f = \lim_\lambda \pi_f(x)\gamma(u_\lambda) = \lim_\lambda \gamma(x u_\lambda) = \gamma(x)$. Hence $\gamma(x) \in \pi_f(A)\xi_f$ $\forall x \in A$. □

So in conclusion, we obtain a one-to-one correspondence between states on A and unitary equivalence classes of cyclic representations of A.

Theorem 8.9. *Every C*-algebra has an injective representation (π, H), i.e., every C*-algebra is isomorphic to a sub-C*-algebra of $\mathcal{L}(H)$.*

Proof. Let $x \in A$ with $x \neq 0$. By Theorem 7.10, there exists a state f on A with $f(x^*x) = \|x\|^2$. Thus $\left\|\pi_f(x)\xi_f\right\|^2 = f(x^*x) = \|x\|^2$. Define

$$\pi := \bigoplus_{f \text{ state}} \pi_f.$$

Then $\pi(x) \neq 0$ for every $x \neq 0$. □

Remark 8.10. If A is separable, say $\{x_n\}_n$ is a countable dense subset of A, then we may choose states f_n on A with $f_n(x_n^* x_n) = \|x_n\|^2$. Then $\pi := \bigoplus_n \pi_{f_n}$ is an injective representation, and the Hilbert space H_{f_n} is separable for every n.

So as a consequence, every separable C*-algebra is isomorphic to a sub-C*-algebra of $\mathcal{L}(\ell^2\mathbb{N})$.

Example 8.11. *Let A be a C*-algebra. Consider the algebra $M_n(A) := \{(a_{ij})_{1\leq i,j\leq n} \colon a_{ij} \in A\}$. $M_n(A)$ is a *-algebra with respect to the usual operations. To define a norm on $M_n(A)$ such that it becomes a C*-algebra, we may assume that $A \subseteq \mathcal{L}(H)$. Then we have an embedding $M_n(A) \subseteq \mathcal{L}(H^n)$, where $(a_{ij})(\xi_j) = (\sum_j a_{ij}\xi_j)_i$. Therefore, the norm on $\mathcal{L}(H^n)$ induces a norm on $M_n(A)$ which satisfies the C*-identity.*