LTCC Environmental Flows

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Abstract

Short description Simplified mathematical models of some large-scale atmospheric and oceanic flow features will be presented, using geophysical fluid dynamics.

Syllabus This course is about aspects of geophysical fluid dynamics, relevant to the climate system. The flows to be considered will be large-scale oceanic and atmospheric features. With the aim of constructing simplified idealised models that illustrate basic mechanisms, mathematical models of several such features will be derived and analysed. After introducing the quasigeostrophic equations, applications to be covered will be baroclinic instability, multiple equilibria, ocean spinup.

Prerequisites: The course is suitable for students who have taken undergraduate fluid mechanics courses. Familiarity with geophysical fluid dynamics will be helpful, but not assumed.

Recommended reading The course will be self-contained. The following books provide much wider coverage of the theme of oceanic and atmospheric dynamics.

Gill, A.E. (1982) Atmosphere-Ocean Dynamics. Academic Press.

Pedlosky, J. (1987, 1992) Geophysical Fluid Dynamics. Springer-Verlag.

Vallis, G.K. (2006) Atmospheric and oceanic fluid dynamics. Cambridge University Press.

Content of notes The notes provided here are a summary of the course material presented in 2009 and 2011. The main sections are:

1. Governing equations for quasigeostrophic stratified flow
2. Ekman layers
3. Rossby waves
4. Mid-latitude wind-driven ocean spinup and circulation
5. Baroclinic instability
6. Multiple equilibria and blocking

The lectures also contained numerous diagrams sketched on the board, but these are not included here.

1 Governing equations for quasigeostrophic stratified flow

1.1 Equations of motion in a rotating frame of reference

For flow relative to a rotating frame of reference (such as the Earth) with rotation vector $\underline{\Omega}$ the Navier-Stokes equations have the form

$$\underline{u}_t + (\underline{u}.\nabla)\underline{u} - 2\underline{\Omega} \times \underline{u} = -\nabla p/\rho + \nu\nabla^2\underline{u} + \text{gravity} + \text{other forces} \quad . \tag{1.1}$$

Compared to the size of the planet, the ocean and atmosphere are thin layers on the Earth's surface, and we will be considering flows with large horizontal scale compared to the depth in each medium.

For convenience, we will use local Cartesian co-ordinates, with z vertically upwards, x in the zonal (west to east) direction, and y in the meridional (south to north) direction. The fluid velocity components will be denoted u (zonal), v (meridional) and w (upward).

At latitude θ, the vertically upwards component of the rotation vector is $\underline{\Omega}$ is $2\Omega\sin\theta$, denoted by f, and referred to as the Coriolis parameter. ($\Omega = |\underline{\Omega}|$ is the rate of rotation of the Earth, 2π radians per day.) Other components of the rotation vector have small influence, and will be ignored. Thus the horizontal momentum equations are

$$u_t + (\underline{u}.\nabla)u - fv = -(1/\rho)p_x + \nu\nabla^2 u + \text{other forces} \quad , \tag{1.2a}$$
$$v_t + (\underline{u}.\nabla)v + fu = -(1/\rho)p_y + \nu\nabla^2 v + \text{other forces} \quad . \tag{1.2b}$$

(Thus locally the Earth is regarded as flat, which is a useful approximation for many purposes.)

At some reference latitude θ_0, the meridional distance from the equator is $R_e\theta_0$, where R_e is the radius of the Earth. Relative to that latitude the meridional coordinate is $y = R_e(\theta - \theta_0)$, and the zonal distance around the Earth is $2\pi R_e \cos\theta_0$. At that latitude the Coriolis parameter is $f_0 = 2\Omega \sin\theta_0$.

1.2 The beta plane

For small ranges of latitudes, the Coriolis parameter can be approximated by

$$f = 2\Omega\sin\theta \approx 2\Omega\sin\theta_0 + 2\Omega(\theta - \theta_0)\cos\theta_0 \tag{1.3a}$$

$$= 2\Omega\sin\theta_0 + (2\Omega/R_e)\cos\theta_0\, y \tag{1.3b}$$

$$= f_0 + \beta y \tag{1.3c}$$

where $\beta = (2\Omega/R_e)\cos\theta_0$.

The is referred to as the 'beta-plane' approximation. (The situation in which variations of f with latitude are neglected, so $f = f_0$, is known as the 'f-plane'.)

1.3 Hydrostatic balance

For the large-scale ocean and atmosphere flows relevant to this course, the vertical equation of motion is dominated by the balance of vertical pressure gradient and gravitational force. To a very good approximation, the system is in 'hydrostatic balance', with

$$p_z = -\rho g \ . \tag{1.4}$$

(Effectively the pressure at any point is determined by the mass of overlying fluid, and is not influenced by the fluid motion.)

1.4 Continuity

For the flows of interest, to a good approximation the fluids can be regarded as incompressible. (Although the atmosphere is a compressible gas, the flow speeds are much less than the speed of sound and dynamically the flow can be regarded as incompressible.) Thus in our Cartesion representation,

$$u_x + v_y + w_z = 0 \ . \tag{1.5}$$

1.5 Geostrophic balance

Suppose the flow has length scale L, a (horizontal) velocity scale U, and an advective time scale L/U. The nondimensional Rossby number R is defined as

$$R = U/(f_0 L) \quad . \tag{1.6}$$

We will assume the Rossby number is small ($R \ll 1$), in which case the left hand side of equ 1.2 is dominated by the Coriolis terms fv and fu. Apart from thin layers near boundaries (see later), viscous and forcing effects are small. The dominant balance is between the Coriolis and pressure gradient terms:

$$-fv = -p_x/\rho \quad , \tag{1.7a}$$
$$fu = -p_y/\rho \quad . \tag{1.7b}$$

This is 'geostrophic balance'. For later use we define a geostrophic horizontal flow u_g, v_g by

$$-v_g = -p_x/(\rho_0 f_0) \quad , \tag{1.8a}$$
$$u_g = -p_y/(\rho_0 f_0) \quad , \tag{1.8b}$$

where ρ_0 is a typical density scale. Note that $\underline{u}_g.\nabla p = 0$: the geostrophic flow follows isobars. Flow is cyclonic around low pressure centres, and anti-cyclonic around high pressure centres. In the northern hemisphere, orientation is such that cyclonic flow is anti-clockwise. Note also that $u_{gx} + v_{gy} = 0$, so the geostrophic flow is horizontally non-divergent. Thus a streamfunction ψ can be defined: conventionally such that

$$u_g = -\psi_y \quad , \quad v_g = \psi_x \quad . \tag{1.9}$$

(Note: here and from here on we assume ∇ is the horizontal gradient operator, and $\underline{u} = (u, v)$, as should be obvious from the context: thus the continuity equation becomes $\nabla.\underline{u} + w_z = 0$.)

1.6 Representation of density distribution as layers

In practice density varies throughout the ocean and atmosphere. The ideal gas law applies to the atmosphere, and in the ocean density depends weakly on temperature, salinity and pressure. For this course, as is often done for theoretical investigations, we will consider a simple representation of density structure as layers in which each layer has a constant prescribed density. (See textbooks for alternative representations.) Thus we will not consider thermodynamic aspects, but concentrate on dynamic processes in this course.

Using hydrostatic balance, some useful relationships between pressures and layer distributions can be derived.

Suppose a system has N layers, with layer 1 at the top overlying layer 2 overlying layer 3 etc. Suppose layer n has density ρ_n, with $\rho_1 < \rho_2 < ....$ With the fluid at rest the interfaces are horizontal, and the layers have depths H_n which are constant, except for the lowest layer whose depth may vary with x and y to allow the possibility of varying 'topography'. Suppose the top of layer 1 is at $z = z_T$ when undisturbed, and suppose the perturbation of this surface is η_1, so the top of disturbed layer 1 is at $z_T + \eta_1$. Similarly, η_2 denotes the perturbation to the interface between layers 1 and 2, so the bottom of layer 1 (and the top of layer 2) is at $z_T - H_1 + \eta_2$, and so on. A fixed perturbation $\eta_{N+1}(x, y)$ at the base of layer N can be used to represent topography, so the bottom of layer N is at $z_T - (H_1 + ... + H_N) + \eta_{N+1}$.

Suppose the pressure at the top surface is p_T. From the hydrostatic relation, the pressure in layer 1 is

$$p_1 \;=\; p_T + \rho_1 g \eta_1 - \rho_1 g (z - z_T) \quad , \tag{1.10}$$

for $z_T - H_1 + \eta_2 < z < z_T + \eta_1$, and in layer 2

$$p_2 \;=\; p_T + \rho_1 g \eta_1 + (\rho_1 - \rho_2) g (H_1 - \eta_2) - \rho_2 g (z - z_T) \quad , \tag{1.11}$$

for $z_T - H_1 - H_2 + \eta_3 < z < z_T - H_1 + \eta_2$, etc.

Suppose p_T is constant. (For oceanic applications, p_T is the sea level atmospheric pressure, fluctuations in which are small compared to pressure fluctuations below the surface and negligible for most circumstances. For atmospheric applications p_T is effectively zero at the top of the atmosphere.) Then the horizontal pressure gradients are independent of depth within each layer. In layer 1,

$$\nabla p_1 \;= \rho_1 g \nabla \eta_1 \quad , \tag{1.12}$$

and in layer 2

$$\nabla p_2 \;=\; \rho_1 g \nabla \eta_1 + (\rho_2 - \rho_1) g \nabla \eta_2 \quad , \tag{1.13}$$

etc. Thus the geostrophic flow $\underline{u}_{g1}$ in layer 1 can be determined from η_1 and is independent of depth in layer 1; likewise $\underline{u}_{g2}$ is determined by η_1 and η_2, etc. (I.e. $\underline{u}_g$ is determined by the overlying density structure.) Note that

$$\nabla (p_2 - p_1) \;=\; (\rho_2 - \rho_1) g \nabla \eta_2 \quad , \tag{1.14}$$

so the difference $\underline{u}_{g1} - \underline{u}_{g2}$ is determined by η_2. (Thus the vertical geostrophic shear between two layers is determined by the horizontal gradient in the intervening density structure.)

1.7 Shallow water equations and potential vorticity equations

The above relations show how the density structure, pressure gradients, and geostrophic flows are diagnostically related. However more information is needed to find out how they evolve. The relevant equations can be derived rigorously through asymptotic expansions with e.g. the Rossby number as a small parameter (see textbooks for details). Here we take a more ad hoc approach, making a series of assumptions that can ultimately be justified more formally.

Within each layer (away from thin frictional layers at the upper and lower interfaces - see later) the horizontal flow is independent of depth and governed by the 'shallow water equations':

$$u_t \;+\; (\underline{u}.\nabla)u \;-\; fv \;=\; -\,p_x/\rho_0 \;+\; A\nabla^2 u \tag{1.15a}$$
$$v_t \;+\; (\underline{u}.\nabla)v \;+\; fu \;=\; -\,p_y/\rho_0 \;+\; A\nabla^2 v \;\;. \tag{1.15b}$$

Here A is a horizontal diffusivity coefficient, whose influence is negligible except in regions of strong gradients. (The situation is analogous to long waves on shallow (depth much less than a wavelength) water, hence the name.) Note that we have assumed a constant reference density in the pressure gradient terms, valid for all layers.

The vorticity (more correctly, the vertical component of the vorticity vector) is $\zeta = v_x - u_y$. We use the vector identity $(\underline{u}.\nabla)\underline{u} \;=\; \nabla\underline{u}^2/2 \;-\; \underline{u}\times\zeta\underline{k}$, where $\underline{k}$ is the unit vertical vector, to write 1.15 as

$$u_t \;+\; (\underline{u}^2)_x/2 \;-\; (f+\zeta)v \;=\; -\,p_x/\rho_0 \;+\; A\nabla^2 u \;\;, \tag{1.16a}$$
$$v_t \;+\; (\underline{u}^2)_y/2 \;+\; (f+\zeta)u \;=\; -\,p_y/\rho_0 \;+\; A\nabla^2 v \;\;. \tag{1.16b}$$

Eliminating p and $\underline{u}^2$ by cross-differentiating leads to the vorticity equation

$$\zeta_t \;+\; \underline{u}.\nabla(\zeta+f) \;+\; (\zeta+f)\nabla.\underline{u} \;=\; A\nabla^2\zeta \;\;. \tag{1.17}$$

From the continuity equation we have $\nabla.\underline{u} + w_z = 0$, so we obtain

$$\frac{D}{Dt}(\zeta+f) \;=\; (\zeta+f)w_z \;+\; A\nabla^2\zeta \;\;, \tag{1.18}$$

where the material derivative ('following the motion') is $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}.\nabla$. Thus changes in the total vorticity $\zeta + f$ are induced by vertical motion stretching or shrinking the fluid column within the layer and by dissipation.

As w_z is independent of depth within the layer (because $\underline{u}$ and hence $\nabla.\underline{u}$ is depth-independent), we have

$$w_z \;=\; (w_T - w_B)/h \quad , \tag{1.19}$$

where h denotes the layer depth, and w_T and w_B denote w near the top and bottom of the layer (i.e. outside any thin friction layers). Further, we have

$$w_T \;-\; w_B \;=\; \frac{D}{Dt}h + w_{ET} - w_{EB} \quad , \tag{1.20}$$

indicating a contribution from changes in the bounding interface positions and contributions from thin frictional Ekman layers (see later). Noting that

$$\frac{D}{Dt}\frac{(\zeta + f)}{h} \;=\; \frac{1}{h}\frac{D}{Dt}(\zeta + f) \;-\; \frac{(\zeta + f)}{h^2}\frac{D}{Dt}h \quad ,$$

we obtain

$$\frac{D}{Dt}\frac{(\zeta + f)}{h} \;=\; \frac{(\zeta + f)}{h}\frac{(w_{ET} - w_{EB})}{h} \;+\; \frac{1}{h}A\nabla^2\zeta \quad . \tag{1.21}$$

The expression $(\zeta + f)/h$ is known as the potential vorticity: a fundamental quantity in geophysical fluid dynamics. (There are equivalent expressions in continuously stratified systems.)

1.8 Quasigeostrophic potential vorticity

Suppose the layer displacements are small compared to the layer depth, so

$$\frac{1}{h_n} \;=\; \frac{1}{H_n + \eta_n - \eta_{n+1}} \;\approx\; \frac{1}{H_n}\Big(1 - \frac{\eta_n - \eta_{n+1}}{H_n}\Big) \quad . \tag{1.22}$$

With $\zeta_n + f = \zeta_n + f_0 + \beta y$, the potential vorticity is approximately

$$\frac{\zeta_n + f}{h_n} \;\approx\; \frac{f_0}{H_n}\Big(1 + \frac{\zeta_n + \beta y}{f_0}\Big)\,\Big(1 - \frac{\eta_n - \eta_{n+1}}{H_n}\Big) \quad .$$

Assuming $\zeta_n + \beta y$ is small compared to f_0, and neglecting the product of the small terms, we have

$$\frac{\zeta_n + f}{h_n} \;\approx\; \frac{f_0}{H_n}\Big(1 + \frac{\zeta_n + \beta y}{f_0} - \frac{\eta_n - \eta_{n+1}}{H_n}\Big) \quad . \tag{1.23}$$

Further, the flow in the layer is approximately geostrophic: $\underline{u} \approx \underline{u}_g$. Thus from equ 1.21 we obtain approximately

$$\frac{D_g}{Dt}\, q_n \;=\; \frac{f_0}{H_n}(w_{EnT} - w_{EnB}) \;+\; A\nabla^2\zeta_{gn} \quad , \tag{1.24}$$

where

$$q_n \;=\; \zeta_{gn} + \beta y - \frac{f_0}{H_n}(\eta_n - \eta_{n+1}) \quad , \tag{1.25}$$

with geostrophic vorticity $\zeta_{gn} = v_{gnx} - u_{gny}$ and $\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \underline{u}_g.\nabla$. Equ. 1.24 is a form of the quasigeostrophic potential vorticity equation. Note that in terms of the streamfunction introduced by equ 1.9, $\zeta_{gn} = \nabla^2\psi_n$.

The streamfunctions and interface displacements are related: e.g. using equs 1.8 and 1.12 etc.:

$$\psi_1 \;=\; \frac{g}{f_0}\,\eta_1 \quad , \tag{1.26a}$$

$$\psi_n \;=\; \psi_{n-1} \;+\; \frac{g'_n}{f_0}\,\eta_n \;\; \text{etc.} \quad . \tag{1.26b}$$

where

$$g'_n \;=\; \frac{(\rho_n - \rho_{n-1})}{\rho_0}\, g \tag{1.27}$$

is the 'reduced gravity'.

Thus the quasigeostrophic potential vorticities q_n can be written as:

$$q_1 \;=\; \nabla^2\psi_1 \;+\; \beta y \;-\; \frac{f_0^2}{H_1}\Big(\frac{\psi_1}{g} \;-\; \frac{(\psi_2 - \psi_1)}{g'_2}\Big) \quad , \tag{1.28a}$$

$$q_n \;=\; \nabla^2\psi_n \;+\; \beta y \;-\; \frac{f_0^2}{H_n}\Big(\frac{\psi_n - \psi_{n-1}}{g'_n} \;-\; \frac{(\psi_{n+1} - \psi_n)}{g'_{n+1}}\Big) \quad , \tag{1.28b}$$

$$q_N \;=\; \nabla^2\psi_N \;+\; \beta y \;-\; \frac{f_0^2}{H_N}\Big(\frac{\psi_N - \psi_{N-1}}{g'_N}\Big) \;+\; \frac{f_0}{H_N}\,\eta_{N+1} \quad . \tag{1.28c}$$

2 Ekman layers

There are boundary conditions to be applied at the top and/or base of the stratified flow system presented above, and at times also at the interfaces. Such boundary conditions involve thin boundary layers, called Ekman layers, within which viscous terms of the form $\nu \underline{u}_{zz}$ become important in the balance of forces. The main properties can be deduced by considering steady linear flow on an f-plane, ignoring horizontal dissipation terms, leaving the basic equations

$$-f_0 v \;=\; -\,p_x/\rho_0 \;+\; \nu u_{zz} \quad , \tag{2.1a}$$

$$f_0 u \;=\; -\,p_y/\rho_0 \;+\; \nu v_{zz} \quad . \tag{2.1b}$$

(Note: here ν should be regarded as some constant viscosity coefficient appropriate to large-scale flow.) In the flow interior, away from the boundary layer, the flow is geostrophic with $f_0 v_g = -p_x/\rho_0$ and $f_0 u_g = -p_y/\rho_0$. Define the departure from geostrophic flow by $\underline{u}' = \underline{u} - \underline{u}_g$. Then

$$-f_0 v' \;=\; \nu u'_{zz} \quad , \tag{2.2a}$$

$$f_0 u' \;=\; \nu v'_{zz} \quad . \tag{2.2b}$$

2.1 The Ekman layer near a solid boundary

Consider first flow over a flat bottom boundary at $z = z_B$, on which we require $\underline{u} = \underline{0}$. (This would be appropriate for oceanic flow over the sea floor, or atmospheric flow over land for example.) Then the boundary conditions for equs 2.2 are

$$\underline{u}' \;=\; -\,\underline{u}_g \;\text{ at } z = z_B \quad , \quad \underline{u}' \to 0 \;\text{ outside the boundary layer} \quad . \tag{2.3}$$

The solution can be written as

$$u' \;=\; -\,(u_g \cos Z \;+\; v_g \sin Z)e^{-Z} \quad , \tag{2.4a}$$

$$v' \;=\; -\,(v_g \cos Z \;-\; u_g \sin Z)e^{-Z} \quad , \tag{2.4b}$$

where

$$Z \;=\; (z - z_B)/H_E \;\text{ with }\; H_E \;=\; (2\nu/f_0)^{1/2} \quad . \tag{2.5}$$

(We assume here $f_0 > 0$, i.e. northern hemisphere conditions.) The velocity vector spirals with height (the 'Ekman spiral'), and the boundary layer depth scale is H_E . For large scale atmospheric and oceanic flows the boundary layer is thin, i.e. $H_E \ll H$ if H is the depth scale for the geostrophic flow (e.g. the thickness of one of our layers). The standard nondimensional parameter is the Ekman number E, defined as

$$E \;=\; 2\nu/(f_0 H^2) \quad . \tag{2.6}$$

Thus we have assumed $E \ll 1$. (Note: you may also see E defined without the factor of 2 in some descriptions.)

From the continuity equation $u_x + v_y + w_z = 0$ and the boundary condition $w = 0$ at $z = z_B$, we obtain

$$w \;=\; H_E \; \zeta_g \; [1 \;-\; e^{-Z}(\cos Z + \sin Z) \;] \; / \; 2 \quad . \tag{2.7}$$

In particular, approaching the upper (outer) side of the boundary layer (i.e. large Z) we find

$$w \;\rightarrow\; H_E \; \zeta_g \; /2 \quad , \tag{2.8}$$

at the base of the geostrophic flow. Thus the mechanism of adjustment of the interior geostrophic (to lowest order) flow to the boundary conditions via the frictional Ekman layer induces a vertical velocity (the 'Ekman pump' effect). This influences the evolution of the geostrophic flow: in this case equ 2.8 provides the term w_{EB} in the quasigeostrophic potential vorticity equation.

Similarly, at a motionless upper boundary at $z = z_T$ (such as might be found in a rotating tank experiment in a laboratory), and using for example $Z \;=\; (z_T - z)/H_E$, we find that near the base of the Ekman layer (the top of the geostrophic flow)

$$w \;\rightarrow\; - \; H_E \; \zeta_g \; /2 \quad , \tag{2.9}$$

which is the term w_{ET} in the quasigeostrophic potential vorticity equation.

More generally, if the upper boundary has a velocity $\underline{u}_T$ say (again, as might be found in a laboratory experiment) then

$$w \;\rightarrow\; H_E \; (\zeta_T - \zeta_g) \; /2 \quad , \tag{2.10}$$

where $\zeta_T = v_{Tx} - u_{Ty}$. Thus the 'pump' is proportional to the vorticity difference between the boundary and interior motion.

2.2 The Ekman layer driven by surface stress

Another important case is that of a boundary where a surface stress is given: in particular, the effect of surface wind stress on the ocean. Suppose the zonal and meridional components of the surface stress are

$$\tau^{(x)} \;=\; \rho\nu u_z \quad , \quad \tau^{(y)} \;=\; \rho\nu v_z \quad , \quad \text{at } z = z_T \quad . \tag{2.11}$$

With $Z \;=\; (z_T - z)/H_E$ as before, we find with these boundary conditions that

$$u' \;=\; (H_E/2\rho\nu)e^{-Z} \; [\; (\tau^{(x)} + \tau^{(y)}) \cos Z \;-\; (\tau^{(x)} - \tau^{(y)}) \sin Z \;] \;\;, \qquad (2.12a)$$
$$v' \;=\; -\; (H_E/2\rho\nu)e^{-Z} \; [\; (\tau^{(x)} - \tau^{(y)}) \cos Z \;+\; (\tau^{(x)} + \tau^{(y)}) \sin Z \;] \;\;. \qquad (2.12b)$$

Note: at the surface ($Z = 0$),

$$u' \;=\; (H_E/2\rho\nu) \; (\tau^{(x)} + \tau^{(y)}) \;\;, \qquad (2.13a)$$
$$v' \;=\; -\; (H_E/2\rho\nu) \; (\tau^{(x)} - \tau^{(y)}) \;\;, \qquad (2.13b)$$

which is directed 45^0 to the right of the wind (Northern hemisphere).

In this case the 'Ekman pump' at the base of the Ekman layer gives

$$w_{ET} \;=\; (\tau^{(y)}_x - \tau^{(x)}_y)/(\rho f_0) \;\;, \qquad (2.14)$$

i.e. proportional to the 'wind stress curl'. This is a very important mechanism for driving the ocean circulation in the layers below the ocean surface.

Ekman layers have other important properties that are not covered in this course: for example, the 'Ekman transport' which is is the mass transport in the Ekman layer.

3 Rossby waves in a two-layer stratified fluid

Variations in the Coriolis parameter f with latitude provide an important mechanism for large-scale waves in the ocean and atmosphere. We illustrate this by considering a two-layer system on a β-plane, ignoring the effects of dissipation, forcing and topography to focus on the Rossby waves. Further, we linearise the quasigeostrophic potential vorticity equations by omitting terms of the form $\underline{u}.\nabla\zeta$ and $\underline{u}.\nabla\eta$ (which involve multiples of derivatives of ψ), to obtain from equs 1.24 and 1.28

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 \;-\; \frac{f_0^2}{gH_1}\psi_1 \;+\; \frac{f_0^2}{g'H_1}(\psi_2-\psi_1)\right] + \;\beta\psi_{1x} \;=\; 0 \;\;, \tag{3.1a}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 \;-\; -\; \frac{f_0^2}{g'H_2}(\psi_2-\psi_1)\right] + \;\beta\psi_{2x} \;=\; 0 \;\;. \tag{3.1b}$$

These equations have wavelike solutions. It is convenient to look for the 'modes' which have coherent behaviour in each layer. While not strictly necessary, we also make some further assumptions to simplify the algebra.

3.1 Baroclinic mode equation

Taking the difference of the equations above, we find

$$\nabla^2(\psi_1-\psi_2)_t \;-\; \frac{f_0^2}{g'}\Big(\frac{1}{H_1}+\frac{1}{H_2}\Big)(\psi_1-\psi_2)_t \;-\; \frac{f_0^2}{gH_1}\psi_{1t} \;+\; \beta(\psi_1-\psi_2)_x \;=\; 0 \;\;. \tag{3.2}$$

For this mode, the surface displacement η_1 is much less than that of the interface displacement η_2, so we omit the term $(f_0^2/gH_1)\psi_1$, to obtain

$$\nabla^2\hat{\psi}_t \;-\; (1/a^2)\hat{\psi}_t \;+\; \beta\hat{\psi}_x \;=\; 0 \;\;, \tag{3.3}$$

where

$$\hat{\psi} \;=\; \psi_1 \;-\; \psi_2 \tag{3.4}$$

is a streamfunction for the baroclinic mode (and for the geostrophic shear $\underline{u}_{g1}-\underline{u}_{g2}$), and

$$a^2 \;=\; g'H_1H_2/f_0^2(H_1+H_2) \tag{3.5}$$

defines a length scale a known variously as the 'internal Rossby radius' or 'internal deformation scale' or 'baroclinic Rossby radius' etc. (This scale is the distance travelled by an internal gravity wave with speed $\sqrt{g'H_1H_2/(H_1+H_2)}$ in time $1/f_0$, which is a scale valid also for f-plane motion.) For the baroclinic mode, internal density variations (here, variations in the interface displacement) are essential.

3.2 Barotropic mode equation

Taking the depth-weighted sum of the equations 3.1 leads instead to

$$\nabla^2(H_1\psi_1 + H_2\psi_2)_t \;-\; \frac{f_0^2}{g}\psi_{1t} \;+\; \beta(H_1\psi_1 + H_2\psi_2)_x \;=\; 0 \;\;. \tag{3.6}$$

Define a 'barotropic streamfunction' by

$$\bar{\psi} \;=\; (H_1\psi_1 + H_2\psi_2)/(H_1 + H_2) \;\;, \tag{3.7}$$

which is a streamfunction for the depth-averaged geostrophic flow. For barotropic flow, which is the same in both layers to a good approximation so $\psi_1 = \psi_2 = \bar{\psi}$, we can write 3.6 as

$$\nabla^2\bar{\psi}_t \;-\; (1/\bar{a}^2)\bar{\psi}_t \;+\; \beta\bar{\psi}_x \;=\; 0 \;\;, \tag{3.8}$$

where

$$\bar{a}^2 \;=\; g(H_1 + H_2)/f_0^2 \tag{3.9}$$

defines an adjustment scale $\bar{a}$ analagous to a, but with reference to the 'external' gravity wave speed $\sqrt{g(H_1 + H_2)}$. The barotropic mode behaves as though the fluid were not stratified. The barotropic adjustment scale is much larger than the baroclinic scale, and in practice the term $(1/\bar{a}^2)\bar{\psi}_t$ can often be neglected in 3.8 .

3.3 Wavelike solutions

For the baroclinic equation 3.3, consider a solution of the form

$$\hat{\psi} \;=\; Ae^{i(kx+ly-\omega t)} \;\;, \tag{3.10}$$

where A is an arbitrary amplitude, k is zonal wavenumber, l is meridional wavenumber, and ω is frequency. Then $\nabla^2\hat{\psi} = -K^2\hat{\psi}$, where $K^2 = k^2 + l^2$, and $\hat{\psi}_x = -ik\hat{\psi}$ etc., so 3.3 leads to the dispersion relation

$$\omega \;=\; -\,\beta a^2 k/(K^2a^2 + 1) \;\;. \tag{3.11}$$

Thus there are wavelike solutions, known as Rossby waves, when β is non-zero. The zonal phase speed of these baroclinic Rossby waves is

$$\omega/k \;=\; -\,\beta a^2/(K^2a^2 + 1) \;\;, \tag{3.12}$$

which is negative. Zonal propagation of information by these waves is determined by the zonal group velocity, which is

$$\partial\omega/\partial k \;=\; \beta a^2 \;[\; (k^2 - l^2)a^2 - 1 \;]\; / \;(K^2a^2 + 1)^2 \;\;, \tag{3.13}$$

This may be positive or negative: waves with $k^2a^2 < l^2a^2+1$ have westward zonal group velocity while waves with $k^2a^2 > l^2a^2+1$ have eastward zonal group velocity. This is illustrated for the case $al = 0$ in Fig. 3.1 .

Note that for long waves (ka and $la \ll 1$; wavelength large compared to a) we have

$$\omega/k \approx -\beta a^2 \quad , \quad \partial\omega/\partial k \approx -\beta a^2 \quad . \tag{3.14}$$

Thus long Rossby waves are non-dispersive and propagate westward.

Equ 3.8 has the same form as equ 3.3, and has analogous barotropic Rossby wave solutions: just replace a by $\bar{a}$ in the above dispersion and wave speed results.

If the surface displacement is ignored (a 'rigid lid' approximation; effectively $\bar{a} \to \infty$), then equ 3.8 has Rossby wave solutions with

$$\omega/k = -\beta/K^2 \quad , \tag{3.15}$$

$$\partial\omega/\partial k = \beta\ (k^2 - l^2)\ /\ K^4 \quad . \tag{3.16}$$

3.4 Ocean scales

The contrast between baroclinic and barotropic scales and speeds is particularly marked for the ocean. As typical values for a mid-latitude region (say $40^0 N$), consider

$$H_1 = 1000 \text{ m} \quad , \quad H_2 = 4000 \text{ m} \quad , \tag{3.17a}$$

$$f_0 = 9.4 \times 10^{-5} \text{ sec}^{-1} \quad , \quad \beta = 1.75 \times 10^{-11} \text{ m}^{-1}\text{sec}^{-1} \quad , \tag{3.17b}$$

$$\rho = 1000 \text{ Kg m}^{-3} \quad , \quad \Delta\rho/\rho = 0.003 \quad . \tag{3.17c}$$

For the baroclinic mode,

$$a \approx 50 \text{ Km} \quad , \quad \beta a^2 \approx 5 \text{ cm sec}^{-1} \quad , \tag{3.18}$$

and for the barotropic mode

$$\bar{a} \approx 2300 \text{ Km} \quad , \quad \beta \bar{a}^2 \approx 100 \text{ m sec}^{-1} \quad . \tag{3.19}$$

4 Wind-driven mid-latitude ocean circulation

To illustrate some basic properties of mid-latitude wind-driven stratified ocean circulation, we consider a highly idealised 'textbook' scenario using a two-layer β-plane system. The ocean is a square basin, with southern boundary at $y = -L/2$, northern boundary at $y = L/2$, western boundary at $x = x_W = 0$ and eastern boundary at $x = x_E = L$. The system is forced by surface wind stress, and damped by bottom friction ν and lateral dissipation A. There are Ekman layers at the surface and bottom of the ocean.

4.1 Governing equations

For simplicity, we assume 'rigid lid' dynamics and linearise the quasigeostrophic vorticity equations about a state of rest, to obtain

$$\nabla^2\psi_{1t} \ + \ \beta\psi_{1x} \ + \ \frac{1}{a_1^2}(\psi_2 - \psi_1)_t \ = \ \frac{1}{\rho H_1}\underline{k}.\nabla \times \underline{\tau} \ + \ A\nabla^4\psi_1 \quad , \tag{4.1a}$$

$$\nabla^2\psi_{2t} \ + \ \beta\psi_{2x} \ - \ \frac{1}{a_2^2}(\psi_2 - \psi_1)_t \ = \ - f_0\frac{H_E}{H_2}\nabla^2\psi_2 \ + \ A\nabla^4\psi_2 \quad , \tag{4.1b}$$

where

$$a_1^2 \ = \ g'H_1/f_0^2 \quad , \quad a_2^2 \ = \ g'H_2/f_0^2 \quad , \quad g' \ = \ g(\rho_2 - \rho_1)/\rho_0 \quad , \tag{4.2}$$

For a northern hemisphere mid-latitude ocean (like the North Atlantic) the prevailing surface winds are westerly (toward the east) in the northern half and easterly in the southern half of the region: a simple representation of this is

$$\tau^{(x)} \ = \ \tau_0 \sin(\pi y/L) \quad , \quad \tau^{(y)} \ = \ 0 \quad , \tag{4.3}$$

with $\tau_0 > 0$. Then the wind stress curl is

$$\underline{k}.\nabla \times \underline{\tau} \ = \ \tau_x^{(y)} - \tau_y^{(x)} \ = \ - \tau_0(\pi/L)\cos(\pi y/L) \quad . \tag{4.4}$$

Thus the wind stress driven 'Ekman pump' is negative, with largest amplitude at $y = 0$. (Mass transport in the surface Ekman layer is directed to the right of the wind direction in the northern hemisphere: in this simple model the result is converging meridional Ekman transport requiring downward motion to compensate.)

With this choice, the streamfunctions in our linear model have the same dependence on y as the wind stress curl, so we can separate variables by defining

$$\psi_j \ = \ S_j(x,t)\cos(ly) \quad , \tag{4.5}$$

where $l = \pi/L$. Then for example the vorticity is

$$\nabla^2\psi_j \;=\; (S_{jxx} - l^2 S_j)\cos(ly) \quad . \tag{4.6}$$

At this point we opt to omit the bottom friction term in equ 4.1b, and retain the lateral diffusion. (The alternative, to retain the bottom friction and omit lateral diffusion, is also often used as a simple example - see textbooks.) Then equ 4.1 becomes

$$(S_{1xx} - l^2 S_1)_t \;+\; \beta S_{1x} \;+\; \frac{1}{a_1^2}(S_2 - S_1)_t \;=\; -\frac{1}{\rho_0 H_1}\tau_0 l \;+\; A(S_{1xxxx} - 2l^2 S_{1xx} + l^4 S_1) \quad , \tag{4.7a}$$

$$(S_{2xx} - l^2 S_2)_t \;+\; \beta S_{2x} \;-\; \frac{1}{a_2^2}(S_2 - S_1)_t \;=\; A(S_{2xxxx} - 2l^2 S_{2xx} + l^4 S_2) \quad . \tag{4.7b}$$

It is useful to write these in terms of barotropic and baroclinic modes. Define the barotropic (depth-average) and baroclinic parts by

$$\bar{S} \;=\; (H_1 S_1 + H_2 S_2)/(H_1 + H_2) \quad , \quad \hat{S} \;=\; S_1 \;-\; S_2 \quad . \tag{4.8}$$

Then we find from equ 4.7:

$$(\bar{S}_{xx} - l^2\bar{S})_t \;+\; \beta\bar{S}_x \;=\; -\frac{1}{\rho_0(H_1 + H_2)}\tau_0 l \;+\; A(\bar{S}_{xxxx} - 2l^2\bar{S}_{xx} + l^4\bar{S}) \quad , \tag{4.9a}$$

$$(\hat{S}_{xx} - l^2\hat{S})_t \;+\; \beta\hat{S}_x \;-\; \frac{1}{a^2}\hat{S}_t \;=\; -\frac{1}{\rho_0 H_1}\tau_0 l \;+\; A(\hat{S}_{xxxx} - 2l^2\hat{S}_{xx} + l^4\hat{S}) \quad , \tag{4.9b}$$

where

$$\frac{1}{a^2} \;=\; \frac{1}{a_1^2} \;+\; \frac{1}{a_2^2} \;=\; \frac{f_0^2(H_1 + H_2)}{g' H_1 H_2} \quad . \tag{4.10}$$

Thus we have the usual rigid-lid barotropic and baroclinic Rossby wave equations, now with forcing and dissipation.

4.2 Boundary conditions

There is no flow across the boundaries, so the perpendicular velocity component must be zero on each boundary in each layer, which is satisfied if the streamfunction is constant along each boundary: we choose $\psi_j = 0$ on the east and west boundaries.

With lateral dissipation, further conditions are required. The obvious condition is 'no slip': $v_j = 0$ on the east and west boundaries requires

$$\psi_{jx} \;=\; 0 \quad \text{on } x = x_E \;, x = x_W \quad . \tag{4.11}$$

However, our equations do not properly represent flow near a coastline, and in ocean models a common alternative is to use 'free slip' conditions with $v_{jx} = 0$ on the east and west boundaries, which requires

$$\psi_{jxx} \;=\; 0 \quad \text{on } x = x_E \;, x = x_W \quad . \tag{4.12}$$

4.3 Steady state

We first consider the steady state that is obtained when the ocean has adjusted to the steady wind stress applied in our scenario.

From equ 4.7b with no slip or free slip conditions we find

$$S_2 \;=\; 0 \quad , \tag{4.13}$$

so the lower layer is at rest.

From equ 4.7a we obtain

$$S_{1x} \;=\; -\,C \;+\; (A/\beta)(S_{1xxxx} - 2l^2 S_{1xx} + l^4 S_1) \quad , \tag{4.14}$$

where for convenience we have defined $C = \tau_0 l/(\beta\rho_0 H_1)$.

For reasonable choices of parameter, it turns out that the length scale defined by $(A/\beta)^{1/3}$ is small compared to the scale L of the ocean basin, and dissipation effects are confined to thin layers near the lateral boundaries. We define a small parameter by $\epsilon = (A/\beta)^{1/3}/L \ll 1$, and write equ 4.14 as

$$\epsilon^3 L^3 (S_{1xxxx} - 2l^2 S_{1xx} + l^4 S_1) \;-\; S_{1x} \;=\; C \quad . \tag{4.15}$$

This is a typical boundary layer problem that can be solved using matched asymptotic expansions. Away from the thin boundary layers, the 'outer' equation is

$$S_{1x} \;=\; -\,C \quad . \tag{4.16}$$

Note: S_{1x} is the meridional flow at $y = 0$, so in layer 1 in our example there is a southward flow (the same at all longitudes) indirectly driven by the wind. (This geostrophic current is in balance with a zonal pressure gradient that is associated with displacement of the upper surface, and generated by spinup - see below.) This is a particular example of 'Sverdrup balance' in which meridional geostrophic ocean flow is related to the surface wind stress curl.

The general solution of equ 4.16 is

$$S_1 \;=\; \alpha \;-\; Cx \quad , \tag{4.17}$$

where α is some constant to be determined by matching to the near-boundary behaviour determined below.

4.3.1 The western boundary layer

With hindsight, for the 'inner' expansion define the rescaled zonal co-ordinate X by

$$x \;=\; x_W \;+\; \epsilon X \quad , \tag{4.18}$$

so equ 4.15 gives

$$L^4(S_{1XXXX} - 2l^2\epsilon^2 S_{1XX} + l^4\epsilon^4 S_1) \;-\; LS_{1x} \;=\; \epsilon CL \quad . \tag{4.19}$$

To leading order

$$L^4 S_{1XXXX} \;-\; LS_{1x} \;=\; 0 \quad . \tag{4.20}$$

The general solution that does not grow exponentially eastward, and thus can be matched to the 'outer' solution, is

$$S_1 \;=\; \gamma_0 \;+\; e^{-X/2L}\Big[\gamma_1 \cos(\sqrt{3}X/2L) \;+\; \gamma_2 \sin(\sqrt{3}X/2L)\Big] \quad . \tag{4.21}$$

A simple match of inner and outer solutions is sufficient in this problem: for large X equ 4.21 must match 4.17 as $x \to x_W$, so

$$\gamma_0 \;=\; \alpha \;-\; Cx_W \quad . \tag{4.22}$$

With noslip conditions at the western boundary, we require $S_1 \;=\; S_{1X} \;=\; 0$ at $X = 0$, and find

$$S_1 \;=\; \gamma_0 \;-\; \gamma_0\, e^{-X/2L}\Big[\cos(\sqrt{3}X/2L) \;+\; \frac{1}{\sqrt{3}}\sin(\sqrt{3}X/2L)\Big] \quad . \tag{4.23}$$

With freeslip conditions at the western boundary, we require $S_1 = S_{1XX} = 0$ at $X = 0$, and find

$$S_1 \;=\; \gamma_0 \;-\; \gamma_0\, e^{-X/2L}\Big[\cos(\sqrt{3}X/2L) \;-\; \frac{1}{\sqrt{3}}\sin(\sqrt{3}X/2L)\Big] \quad . \tag{4.24}$$

4.3.2 The eastern boundary layer

For the 'inner' expansion near the eastern boundary define instead the rescaled zonal co-ordinate X by

$$x \; = \; x_E \; - \; \epsilon X \quad . \tag{4.25}$$

This time to leading order

$$L^4 S_{1XXXX} \; + \; L S_{1x} \; = \; 0 \quad , \tag{4.26}$$

and to satisfy the boundary conditions at $X = 0$ and have no exponential growth we require

$$S_1 = 0 \quad . \tag{4.27}$$

Then a simple match to the outer solution equ 4.17 as $x \to x_E$ requires

$$\alpha \; = \; C X_E \quad , \quad \text{and hence} \;\; \gamma \; = \; C(x_E - x_W) \; = \; CL \quad . \tag{4.28}$$

(Note: thus to leading order there is no eastern boundary layer. The adjustments to satisfy the noslip boundary conditions are small higher order terms. With freeslip conditions the 'outer' solution satisfies the eastern boundary conditions and no boundary layer is required.)

4.3.3 The composite solution

A leading order additive composite solution, valid in both the outer and boundary regions, can be found using the form 'inner + outer - overlap'. The western 'overlap' term is simply γ_0. Thus for noslip conditions the composite is

$$S_1 \; = \; C(x_E - x) - C(x_E - x_W) \, e^{-X/2L} \Big[\cos(\sqrt{3}X/2L) \, + \, \frac{1}{\sqrt{3}} \sin(\sqrt{3}X/2L) \Big] \;\; , \tag{4.29}$$

with $X = (x - x_W)/\epsilon$, and similarly for freeslip conditions

$$S_1 \; = \; C(x_E - x) - C(x_E - x_W) \, e^{-X/2L} \Big[\cos(\sqrt{3}X/2L) \, - \, \frac{1}{\sqrt{3}} \sin(\sqrt{3}X/2L) \Big] \;\; . \tag{4.30}$$

Apart from details very near the western boundary, the two solutions are quite similar. (See Figs. 4.1 and 4.2 .) In a narrow western boundary layer with width scale $(A/\beta)^{1/3}$ there is a relatively fast northward current (the western boundary current; the Gulf Stream in the North Atlantic), with speed scale $1/\epsilon$ times that of the slow southward flow in the rest of the ocean. Streamlines are illustrated in Fig. 4.3 .

(Note: in the real world the western boundary current separates from the coast at some point due to a combination of geography, nonlinearity, wind stress distribution etc.)

Note: the surface displacement is $\eta_1 = (f_0/g)S_1 \cos(ly)$, while the interface displacement is $\eta_2 = -(g/g')\eta_1$.

Typical scales: with $A = 10^4 \mathrm{m}^2\ \mathrm{s}^{-1}$, $(A/\beta)^{1/3}$ is about 80 km.

4.4 Spinup

It is useful to see how the above steady state is reached starting from rest. Suppose

$$S_1 \;=\; S_2 \;=\; \hat{S} \;=\; \bar{S} \;=\; 0 \;\text{ at time } t = 0 \quad . \tag{4.31}$$

4.4.1 Initial adjustment away from boundary influences

Away from the east and west boundaries, the ocean initially evolves with no boundary influence. With wind stress curl independent of x, the x derivative terms are zero in the governing equations 4.7 and 4.9. We can assume lateral dissipation is negligibly small in this region, for simplicity, to obtain

$$\bar{S} \;=\; \frac{\tau_0 l}{\rho_0 (H_1 + H_2) l^2}\, t \quad , \tag{4.32a}$$

$$\hat{S} \;=\; a^2 l^2 \bar{S} \;\ll \bar{S} \quad , \tag{4.32b}$$

The effect of the wind-induced Ekman transport is to raise the sea level: the meridional gradient and geostrophic balance give a corresponding zonal geostrophic flow within the ocean.

This acceleration continues until Rossby waves arrive from the east and west boundaries, with information travelling at the group velocity speed. The fastest Rossby waves are the long ($k \to 0$) westward-propagating waves. The fastest eastward-propagating waves are much slower, so most of the ocean is first influenced by the Rossby waves arriving from the eastern boundary.

4.4.2 Barotropic adjustment

Consider some location $x = x_0$ between x_W and x_E. With the rigid lid approximation, the fastest barotropic waves have group (and phase) velocity $-\beta/l^2$, and the time of arrival at x_0 from x_E is $t_0 = (x_E - x_0)l^2/\beta$. After this time in turns out that barotropic adjustment ceases, so

$$\bar{S} \;=\; \frac{\tau_0 l}{\rho_0(H_1 + H_2)} \frac{(x_E - x_0)}{\beta} \quad \text{at time } t > t_0 \quad . \tag{4.33}$$

This is consistent with the barotropic part of the steady state solution described above. (Recall that, away from the western boundary layer, the steady solution is $S_1 = C(x_E - x)$, and $S_2 = 0$, so steady $\bar{S} = C(x_E - x)H_1/(H_1 + H_2)$.)

In the western boundary region short Rossby waves and dissipation play a role in developing the western boundary layer. By the time the long waves have crossed the ocean from east to west (i.e. after $t = (x_E - x_W)l^2/\beta$) the barotropic adjustment has completed and the barotropic flow has 'spun up'. For typical scales, this barotropic adjustment takes just a few days. (Indeed, for applications that involve timescales longer than a few days the barotropic adjustment can be regarded as effectively instantaneous: i.e. the barotropic mode is in balance with the wind stress curl.) However, the ocean is far from a steady state at this time: the baroclinic mode takes much longer to 'spin up', and both the upper and lower layer flows are far from their eventual steady states.

4.4.3 Baroclinic adjustment

The baroclinic adjustment process is similar to barotropic adjustment: i.e. most of the ocean is adjusted by long baroclinic Rossby waves from the eastern boundary. However the spin up time is much longer: the fastest baroclinic Rossby waves have group (and phase) velocity $-\beta a^2$, and the spinup time $(x_E - x_W)/\beta a^2$ is several years for, say, the mid-North Atlantic ocean.

As the long baroclinic Rossby wave advances from the eastern boundary and reaches location x_0 at time $t_C = (x_E - x_0)/\beta a^2$, the baroclinic mode at that location equilibrates at

$$\hat{S} \;=\; \frac{\tau_0 l}{\rho_0 H_1} \frac{(x_E - x_0)}{\beta} \quad \text{at time } t > t_C \quad . \tag{4.34}$$

Again this is consistent with the steady state solution above. After time t_C the lower layer is indeed at rest, and the upper layer has reached its steady state. As with the barotropic mode, the baroclinic western boundary layer also develops during the spin up phase.

The whole process is best visualised in a few diagrams - see Figs.4.4, 4.5 and 4.6
.

Note: baroclinic Rossby waves provide the ocean with a long 'memory'. In the real ocean the density structure supports many baroclinic modes that are even slower than the one described above, so that memory can be very long!

5 Baroclinic instability in a two-layer channel

In this section some basic baroclinic instability theory is described, to demonstrate that in a flow with vertical shear small disturbances may grow. (E.g. small amplitude large scale waves in the atmosphere may grow into fully developed weather systems; small disturbances in a stratified current in the ocean may grow to form mesoscale eddies.) As usual, we choose a model that is highly simplified to make the calculations straightforward, while retaining the essential physical processes necessary for the phenomenon.

Consider a two-layer zonally periodic beta-plane channel, with a rigid lid and flat topography. In each layer the quasigeostrophic potential vorticity q_j is

$$q_1 = \zeta_1 + \beta y + (f_0/H_1)\eta \quad , \tag{5.1a}$$

$$q_2 = \zeta_2 + \beta y - (f_0/H_2)\eta \quad , \tag{5.1b}$$

where η is the displacement of the interface between the layers. In terms of the streamfunction ψ_j for the geostrophic flow, recall that

$$u_j = -\psi_{jy} \quad , \quad v_j = \psi_{jx} \quad , \quad \zeta_j = \nabla^2\psi_j \quad , \quad \eta = (\psi_2 - \psi_1)f_0/g' \; . \tag{5.2}$$

We assume no explicit forcing or dissipation, so

$$Dq_j/Dt = q_{jt} + u_j q_{jx} + v_j q_{jy} = 0 \quad . \tag{5.3}$$

Note that the advection part can be written as $J(\psi_j, q_j)$ where J is the Jacobian operator.

Suppose the basic (undisturbed) state has a uniform zonal flow in each layer, maintained by some unspecified means, with $(u_j, v_j) = (U_j, 0)$, with streamfunction $\psi_j = -U_j y$ and interface displacement $\eta = (f_0/g')(U_1 - U_2)y$. Thus a positive shear $U_1 > U_2$ is associated with a northward increase in η (assuming the northern hemisphere): we can imagine this as corresponding to a basic poleward decrease in temperature.

The basic state potential vorticity is denoted Q_j with

$$Q_1 = \beta y + (f_0^2/g'H_1)(U_1 - U_2)y \quad , \tag{5.4a}$$

$$Q_2 = \beta y - (f_0^2/g'H_2)(U_1 - U_2)y \quad . \tag{5.4b}$$

Note that the basic state meridional potential vorticity gradient is

$$Q_{1y} = \beta + (f_0^2/g'H_1)(U_1 - U_2) \quad , \tag{5.5a}$$

$$Q_{2y} = \beta - (f_0^2/g'H_2)(U_1 - U_2) \quad . \tag{5.5b}$$

For simplicity, suppose the layer depths are equal: $H_1 = H_2 = H_0$. From the earlier definition of the baroclinic adjustment length scale a, i.e. $a^2 = g' H_1 H_2 / f_0^2 (H_1 + H_2)$, we find in this situation

$$a^2 \;=\; g' H_0 / 2 f_0^2 \quad , \quad \text{so} \;\; \mathrm{f_0^2/g'H_1} \;=\; \mathrm{f_0^2/g'H_1} \;=\; \mathrm{1/2a^2} \quad . \tag{5.6}$$

For convenience, denote $\lambda = 1/2a^2$, and $S = U_1 - U_2$.

Consider perturbations to the above basic state, so $u_j = U_j + u_j'$ etc. and denote the perturbation streamfunction by ϕ so $\psi_j = -U_{jy} + \phi_j$, and the perturbation potential vorticity q_j' is

$$q_1' \;=\; \nabla^2 \phi_1 \;+\; \lambda(\phi_2 - \phi_1) \quad , \tag{5.7a}$$
$$q_2' \;=\; \nabla^2 \phi_2 \;-\; \lambda(\phi_2 - \phi_1) \quad . \tag{5.7b}$$

The potential vorticity equations can be expressed as

$$q_{jt}' \;+\; U_j q_{jx}' \;+\; \phi_{jx} Q_{jy} \;+\; J(\phi_j, q_j') \;=\; 0 \quad , \tag{5.8}$$

5.1 Linearised equations and wavelike solutions

For small perturbations, the nonlinear terms $J(\phi_j, q_j')$ can be neglected as a first approximation, leaving the linear equations

$$q_{jt}' \;+\; U_j q_{jx}' \;+\; \phi_{jx} Q_{jy} \;=\; 0 \quad . \tag{5.9}$$

Suppose the zonal channel has 'sidewalls' at latitudes $y = \pm \pi b/2$, where $\phi_j = 0$, and zonal extent $2\pi L$. We seek wavelike solutions of the form

$$\phi_j \;=\; A_j e^{ik(x-ct)} \cos(ly) \quad , \tag{5.10}$$

where $k = m/L$ for integers m, and $l = 1/b$. Note that c may be complex.

Substituting, we obtain coupled algebraic equations for the amplitudes A_j:

$$(c - U_1)[(K^2 + \lambda)A_1 - \lambda A_2] \;+\; (\beta + \lambda S)A_1 \;=\; 0 \quad , \tag{5.11a}$$
$$(c - U_2)[(K^2 + \lambda)A_2 - \lambda A_1] \;+\; (\beta - \lambda S)A_2 \;=\; 0 \quad , \tag{5.11b}$$

where $K^2 = k^2 + l^2$.

It is convenient to put

$$c \;=\; (U_1 + U_2)/2 \;+\; d \;\;,\;\; \text{so}\;\; \mathrm{c - U_1} \;=\; \mathrm{d} \;-\; \mathrm{S/2}\;,\; \mathrm{c - U_2} \;=\; \mathrm{d} \;+\; \mathrm{S/2} \;\;. \tag{5.12}$$

The coupled equations become

$$[(d - S/2)(K^2 + \lambda) + \beta + \lambda S]A_1 \;-\; (d - S/2)\lambda A_2 \;=\; 0 \;\;, \tag{5.13a}$$
$$-\;(d + S/2)\lambda A_1 \;+ [(d + S/2)(K^2 + \lambda) + \beta - \lambda S]A_2 \;=\; 0 \;\;. \tag{5.13b}$$

Non-trivial solutions for A_j require the determinant to vanish, which requires

$$K^2(K^2 + \lambda)d^2 \;+\; 2\beta(K^2 + \lambda)d \;+\; \beta^2 + K^2 S^2(2\lambda - K^2)/4 \;=\; 0 \;\;. \tag{5.14}$$

Using $\lambda = 1/2a^2$ we obtain

$$d \;=\; \frac{-\beta a^2(1 + 2a^2K^2) \pm [\beta^2 a^4 + S^2 a^4 K^4 (a^4K^4 - 1)]^{1/2}}{2a^2K^2(1 + a^2K^2)} \;\;. \tag{5.15}$$

Note: a^2K^2 is non-dimensional, and is much less than 1 for waves with wavelengths that are large compare to a.

Note: if the term [...] in 5.15 is negative then d has an imaginary part d_I that is non-zero. In particular, positive d_I gives a factor $e^{kd_I t}$ in the wavelike solution that is exponentially growing, so we have the possibility of unstable solutions.

Note: there is a particular vertical wave structure for each d: i.e. each d has an associated value of A_1/A_2. The two values of d from 5.15 give two different wavelike modes. If the system is unstable, then for arbitrary initial conditions one mode will grow and dominate while the other decays. Moreover, A_1/A_2 is complex in general, so there will be a phaseshift between the waves in the two layers.

5.2 Some particular cases

5.2.1 No shear

Suppose $U_1 = U_2$, so $S = 0$. From 5.15 we obtain two real values for d:

$$d \;=\; -\,\beta a^2/(1 + a^2K^2) \;\;,\;\; \text{or}\;\; \mathrm{d} \;=\; -\,\beta/\mathrm{K}^2 \;\;. \tag{5.16}$$

With no shear the initial wavelike disturbance is stable and does not grow. The values above are just the baroclinic and barotropic Rossby wave phase speeds in the rigid-lid system, relative to the basic uniform flow, so an initial disturbance simply evolves as a superposition of these Rossby waves.

5.2.2 f-plane

Suppose we omit variation of the Coriolis parameter with latitude, by setting $\beta = 0$ (an f-plane model). Then from 5.15 we obtain

$$d^2 \ = \ (S^2/4)\ (a^2K^2 - 1)/(a^2K^2 + 1) \quad . \tag{5.17}$$

Thus d is real for $a^2K^2 > 1$, and pure imaginary for $a^2K^2 < 1$. (See Fig. 5.1 .) Thus unstable modes only occur for K^2 sufficiently small, i.e. sufficiently long wavelengths. For instability in our zonal channel, the zonal wavenumber $k = m/L$ must be less than $\sqrt{1/a^2 - 1/b^2}$. Note that the growth rate increases as K^2 decreases, so longer waves grow faster, and that the growth rate is proportional to $|S|$.

5.2.3 More general conditions: minimum shear

Instability requires the term [...] in 5.15 to be negative, i.e.

$$S^2a^4K^4(a^4K^4 - 1) \ > \ \beta^2a^4 \quad . \tag{5.18}$$

Thus again we require $a^2K^2 < 1$ for instability. The β effect now also requires the shear to be sufficiently large:

$$S^2a^4K^4(a^4K^4 - 1) \ > \ \beta^2a^4 \ / \ a^4K^4(a^4K^4 - 1) \quad . \tag{5.19}$$

As a function of a^4K^4, the right hand side has a minimum value of $4\beta^2a^4$ at $a^4K^4 = 1/2$ (see Fig. 5.2): i.e. instability can only occur for

$$|S| > |S|_{min} \ = \ 2\beta a^2 \quad . \tag{5.20}$$

Note: from 5.5 we have

$$Q_{1y} \ = \ \beta \ + \ S/\beta a^2 \ \ , \ \ Q_{2y} \ = \ \beta \ - \ S/\beta a^2 \ \ , \tag{5.21}$$

so the minimum shear criterion for instability also requires Q_{1y} and Q_{2y} to have opposite sign. This requirement for reversal with height of the basic state quasigeostrophic potential vorticity gradient also holds for more general circumstances.

Note also that if instability does occur then the range of unstable wavenumbers is restricted to a band centred on $a^4K^4 = 1/2$, with the range increasing as $|S|$ increases beyond the minimum value. In contrast to the f-plane case, very long waves are stable and fastest growth occurs for an intermediate wavelength. (See Fig. 5.3 for an example.)

Analysis beyond the scope of these lectures (see textbooks) shows that the phase difference between the layers is an essential feature of a growing mode.

5.3 Energy conservation and finite amplitude effects

Ignoring a constant density factor, in a fluid column the kinetic energy is $KE_n = H_n(u_n^2 + v_n^2)/2$ in layer n. If $< ... >$ denotes a zonal average and [...] denotes a meridional average across the channel, it can be shown that $[< KE_1 + KE_2 + g'\eta^2/2 >]$ is constant in our system with no forcing or dissipation. Here $[< g'\eta^2/2 >]$ is the net available potential energy (APE) in the channel.

The linear stability analysis, valid for small amplitude disturbances, demonstrates the possibility of exponentially growing unstable waves. In a growing unstable mode the increase in perturbation energy is accompanied by a decrease in the mean meridional interface gradient through nonlinear effects: effectively the basic state APE provides the energy source, and wave growth is limited by depletion of this source.

This process can be examined analytically for waves that evolve slowly to moderate finite amplitude, as when $|S|$ is just above some critical value S_c for instability. An outline of the calculation is provided here. For convenience, suppose $S > 0$, fix U_2, and put

$$S \;=\; S_c \;+\; \Delta \quad , \quad U_1 = U_2 \;+\; S_c \;+\; \Delta \quad . \tag{5.22}$$

In this situation the growth rate turns out to be $O(\Delta^{1/2})$, and a two-time analysis is appropriate with a 'slow time' defined by $T = \Delta^{1/2} t$, so

$$\frac{\partial}{\partial t} \;\rightarrow\; \frac{\partial}{\partial t} \;+\; \Delta^{1/2}\frac{\partial}{\partial T} \quad . \tag{5.23}$$

The nonlinear quasigeostrophic potential vorticity equations 5.8 are now

$$(\frac{\partial}{\partial t} + \Delta^{1/2}\frac{\partial}{\partial T} + U_j\frac{\partial}{\partial x})q_j' \;+\; \phi_{jx}Q_{jy} \;+\; J(\phi_j, q_j') \;=\; 0 \quad . \tag{5.24}$$

An asymptotic expansion is used of the form

$$\phi_j \;=\; \phi_j(x,y,t,T) \;=\; \Delta^{1/2}\phi_j^{(1)} \;+\; \Delta\phi_j^{(2)} \;+\; \Delta^{3/2}\phi_j^{(3)} \;+\; ... \quad . \tag{5.25}$$

To $O(\Delta^{1/2})$ the linear equations with $S = S_c$ are obtained, which as before have wavelike solutions of the form

$$\phi_j^{(1)} \;=\; A_j(T)e^{ik(x-ct)}\cos(ly) \quad , \tag{5.26}$$

where the possibility of slow changes in amplitude is now allowed. For convenience, denote $A_1 = A$ and $A_2 = \gamma A$: thus $\phi_2^{(1)} = \gamma\phi_1^{(1)}$. (NB γ is real, as with $S = S_c$ the A_j are real, as is c.) Thus

$$q_1^{(1)} \;=\; \nabla^2\phi_1^{(1)} \;+\; \lambda(\phi_2^{(1)} - \phi_1^{(1)}) \;=\; [-K^2 + \lambda(\gamma - 1)]\phi_1^{(1)} \quad , \tag{5.27a}$$

$$q_2^{(1)} \;=\; \nabla^2\phi_2^{(1)} \;-\; \lambda(\phi_2^{(1)} - \phi_1^{(1)}) \;=\; [-K^2\gamma - \lambda(\gamma - 1)]\phi_1^{(1)} \quad . \tag{5.27b}$$

In particular, note that $J(\phi_j^{(1)}, q_j^{(1)}) = 0$.

To $O(\Delta)$ we obtain

$$(\frac{\partial}{\partial t} + (U_2 + S_c)\frac{\partial}{\partial x})q_1^{(2)} \;+\; \phi_{1x}^{(2)}(\beta + \lambda S_c) \;=\; -\,\frac{\partial A}{\partial T}\,[\lambda(\gamma - 1) - K^2]\; e^{ik(x-ct)}\cos(ly) \quad , \tag{5.28a}$$

$$(\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x})q_2^{(2)} \;+\; \phi_{2x}^{(2)}(\beta - \lambda S_c) \;=\; -\,\frac{\partial A}{\partial T}\,[\lambda(\gamma - 1) + \gamma K^2]\; e^{ik(x-ct)}\cos(ly) \quad . \tag{5.28b}$$

The left hand side has the same form as the equations for $\phi_j^{(1)}$, so there are homogeneous solutions of the form

$$\phi_j^{(2)} \;=\; A_j^{(2)}(T)e^{ik(x-ct)}\cos(ly) \quad . \tag{5.29}$$

There are also solutions of the form $\phi^{(2)}(y, T)$ to be determined, that represent slow adjustments to the zonal mean flow and the zonal mean interface shape. As an initial small wavelike perturbation grows, the zonal mean state also adjusts and influences the amplitude evolution. In turns out that calculations to $\Delta^{3/2}$ are needed to obtain an equation that can be solved for $A(T)$. To cut a long story short, a nonlinear equation of the nondimensional form

$$A_{TT} \;=\; A \;-\; A(A^2 - A_0^2) \quad , \tag{5.30}$$

can be found for particular initial conditions, which is useful for illustrative examples. The 'energy' combination

$$E \;=\; (A_T)^2/2 \;+\; A^4/4 \;-\; (1 + A_0^2)A^2/2 \tag{5.31}$$

is conserved ($E_T = 0$), with an associated 'potential' $A^4/4 - (1 + A_0^2)A^2/2$. Solutions are oscillations, which may be near-sinusoidal in shape or more episodic (relatively brief moderate amplitude spells separated by long small amplitude spells), depending on the initial conditions. (See Figs. 5.4, 5.5, 5.6 .)) Such solutions provide a simple model of the 'lifecycles' of baroclinic instabilities.

6 Blocking and multiple equilibria

The 'highs' and 'lows' of mid-latitude weather systems fluctuate from day to day, with the systems generally moving eastward, in part carried along by the prevailingly westerly mean zonal flow. In some circumstances the mean flow weakens and a 'block' develops with a near-stationary high pressure system, and the weather remains static for several days.

6.1 Formulation of a simple model with topography

One of several possible mechanisms for 'blocking' involves interaction between the mean flow, Rossby waves and topography. In this section this mechanism is described in a highly idealised context. Again we choose a periodic zonal beta-plane channel, this time with a single layer of mean depth H to represent barotropic flow, but with the inclusion of topography $h(x, y)$. The length of the channel is $2\pi L$, with northern and southern boundaries ('sidewalls') at $y{=}\pm\pi L/2$ for simplicity. We neglect horizontal diffusivity for simplicity.

Consider a basic state in which there is a uniform zonal flow $\underline{u} = (U_0, 0)$ when the topography is flat. In the atmosphere this geostrophic flow is driven by a meridional pressure gradient that in turn is associated with a meridional temperature gradient. In our model channel, we suppose the basic flow is driven via Ekman layers via a 'rigid lid' with $\underline{u}_T = (2U_0, 0)$: such as might be realised in a laboratory rotating-tank experiment.

The governing quasigeostrophic potential vorticity equation is

$$\frac{D_g}{Dt}(\nabla^2\psi + \beta y + (f_0/H)h) \;=\; f_0 E^{1/2}\zeta \quad . \tag{6.1}$$

We state a general circulation condition that applies to this situation:

$$\oint \; \underline{u}_t + f_0 E^{1/2}(\underline{u} - \frac{1}{2}\underline{u}_T) \; . \; \underline{dl} \;=\; 0 \quad , \tag{6.2}$$

where the path integral is taken around a closed streamline of the geostrophic flow. In the periodic channel, this condition includes zonally periodic streamlines: in particular, it applies to the streamlines along the north and south boundaries. (A proof is not given here, but can be derived by considering Ekman transports.)

It is convenient to define some terminology for zonal and meridional averages:

$$< ... > \; = \; \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} ... \, dx \quad \text{zonal average,} \tag{6.3a}$$

$$[...] \; = \; \frac{1}{\pi L}\int_{0}^{\pi L} ... \, dy \qquad \text{meridional average} \quad . \tag{6.3b}$$

We write the geostrophic streamfunction as

$$\psi \; = \; -U(t)y \; + \; \Phi(y,t) \; + \; \phi(x,y,t) \quad , \tag{6.4}$$

where $U = [<u>]$ is the mean zonal flow, Φ is the streamfunction for the zonal mean shear $< u > -U$, and ϕ is the streamfunction for wavelike 'eddy' flow. By construction, $< \phi >= 0$ and $[\Phi_y] = 0$.

The streamfunction ψ is constant along the sidewalls: we choose for convenience

$$\psi \; = \; -U\pi L/2 \;\; \text{on} \;\; y = \pi L/2 \;\; , \;\; U\pi L/2 \;\; \text{on} \;\; y = -\pi L/2 \;\; , \tag{6.5}$$

and by construction $\Phi = \phi = 0$ on the sidewalls.

For convenience we choose topography $h(x,y)$ with $<h>= 0$, and with $h = 0$ on the sidewalls. More specifically, we choose the functional form

$$h \; = \; h_0 \cos(ly) \sum_{m=1}^{\infty} F_m \cos(kx) \quad , \tag{6.6}$$

where $l = 1/L$ and $k = m/L$. Thus the topography has a height scale h_0, a simple meridional profile largest at the channel centre, and a zonal profile represented by a Fourier series symmetric about $x = 0$. (For later examples we choose a simple shape with a maximum at $x = 0$: see Fig. 6.1 .)

Equ 6.1 can be re-written as

$$\Phi_{yyt} \; + \; \nabla^2\phi_t \; + \; <u>\big(\nabla^2\phi + (f_0/H)h\big)_x \; + \; \phi_x(\beta + \Phi_{yyy}) \; +$$

$$J\big(\phi, \nabla^2\phi + (f_0/H)h \;\big) \; = \; f_0 E^{1/2}(\Phi_{yy} + \nabla^2\phi) \quad . \tag{6.7}$$

Taking the zonal average of 6.7 leads to

$$\Phi_{yyt} + < J(\phi, \nabla^2\phi + (f_0/H)h) \; + \; f_0 E^{1/2}\Phi_{yy} \; = \; 0 \quad . \tag{6.8}$$

Note that

$$< J(\phi, h) > \; = \; < \phi_x h_y - \phi_y h_x >$$

$$= \; - < \phi h_{xy} + \phi_y h_x > \;\; = \; - < (\phi h_x)_y > \;\; = \; - < \phi h_x >_y \quad . \tag{6.9}$$

A similar result holds for $< J(\phi, \nabla^2 \phi) >$.
Thus 6.8 can be expressed as

$$\Big(\; \Phi_{yt} - < \phi(\nabla^2\phi + (f_0/H)h)_x > \;\; + \;\; f_0 E^{1/2} \Phi_y \; \Big)_y \; = \; 0 \quad . \tag{6.10}$$

Integrating 6.10 with respect to y from the sidewall at $y = -\pi L/2$ leads to

$$-\Phi_{yt} + < \phi(\nabla^2\phi + (f_0/H)h)_x > \;\; - \;\; f_0 E^{1/2} \Phi_y \; =$$

$$(\; - \Phi_{yt} \;\; - \;\; f_0 E^{1/2} \Phi_y \;) \mid_{y=-\pi L/2} \quad . \tag{6.11}$$

On the sidewalls, from 6.2 we obtain

$$U_t - \Phi_{yt} \; + \; f_0 E^{1/2}(U - \Phi_y - U_0) \; = \; 0 \;\; \text{at } y = -\pi L/2 \quad . \tag{6.12}$$

Thus

$$U_t - \Phi_{yt} + < \phi(\nabla^2\phi + (f_0/H)h)_x > \; - \; f_0 E^{1/2} \Phi_y \; + \; f_0 E^{1/2}(U - U_0) \; = \; 0 \;\; . \tag{6.13}$$

Taking the meridional average of 6.13, using $\Phi = 0$ on the sidewalls, and noting $[< \phi \nabla^2 \phi_x >] = 0$ (use periodicity and $\phi = 0$ on sidewalls), we obtain

$$U_t \; + (f_0/H)[< \phi h_x >] \; + \; f_0 E^{1/2}(U - U_0) \; = \; 0 \quad . \tag{6.14}$$

This is the key equation, and it has a simple interpretation. Recalling that the streamfunction is proportional to pressure (with the hydrostatic pressure component removed), the second term is proportional to the drag on the flow exerted by the pressure field associated with the wavelike 'eddy' flow component. The drag depends on ϕ, which in turn is related to U.

So far, we have just reformulated the original quasigeostrophic flow problem, which remains nonlinear and must generally be solved numerically to determine U, Φ and ϕ.

6.2 Steady quasilinear flow

Analytic progress can be made by neglecting eddy-eddy interactions in the quasigeostrophic potential vorticity equation, and by considering steady flow. For simplicity, we also ignore the mean shear component Φ at this stage. Effectively we set $< u > = U$ and $J(\Phi + \phi, \Phi_{yy} + \nabla^2\phi + (f_0/H)h \;) = 0$ in 6.7, to obtain

$$U(\nabla^2\phi + (f_0/H)h)_x \; + \beta\phi_x + f_0 E^{1/2} \nabla^2 \phi \; = \; 0 \quad . \tag{6.15}$$

Given the form 6.6 for h, we can similarly express ϕ in the form

$$\phi \;=\; \cos(ly)\sum_{m=1}^{\infty} A_m\cos(kx) + B_m\sin(kx) \quad , \tag{6.16}$$

and obtain expressions for the Fourier coefficients A_m and B_m:

$$A_m \;=\; \frac{Uf_0k^2(UK^2-\beta)(h_0/H)F_m}{k^2(UK^2-\beta)^2 \;+\; f_0^2EK^4} \quad , \tag{6.17a}$$

$$B_m \;=\; \frac{-Uf_0E^{1/2}K^2(h_0/H)kF_m}{k^2(UK^2-\beta)^2 \;+\; f_0^2EK^4} \quad , \tag{6.17b}$$

where $K^2 = k^2 + l^2$. We can now use these expressions in 6.14 to obtain an equation for U. Noting that $[<\phi h_x>] = -(h_0/4)\sum_m kB_mF_m$, this leads to

$$U - U_0 \;=\; -\,(U/4)f_0^2(h_0/H)^2\sum_m \frac{k^2K^2F_m^2}{k^2(UK^2-\beta)^2 + f_0^2EK^4} \quad , \tag{6.18}$$

which is an odd-order polynomial equation for U. Using $k = m/L$ and $l = 1/L$ and some re-arrangement, this is equivalent to

$$U/U_0 \;-\; 1 \;=$$
$$-\,(U/4U_0)(h_0/H)^2\sum_m \frac{m^2(1+m^2)F_m^2}{m^2\big((U/U_0)(U_0/f_0L)(1+m^2)-\beta L/f_0\big)^2 + E(1+m^2)^2} \quad . \tag{6.19}$$

Thus U/U_0 can be determined given the non-dimensional parameters U_0/f_0L) (Rossby number), $\beta L/f_0$, E, h_0/H and the topography shape F_m.

Note: re-arranging further,

$$U/U_0 \;=$$
$$\left(1 \;+\; (h_0/2H)^2\sum_m \frac{m^2(1+m^2)F_m^2}{m^2((U/U_0)(U_0/f_0L)(1+m^2)-\beta L/f_0)^2 + E(1+m^2)^2}\right)^{-1} \quad . \tag{6.20}$$

The sum is positive, so it is clear that $0 < U/U_0 \leq 1$: i.e. the topographic drag slows down the flow.

Note: 6.18 is a nonlinear equation for U: but it can be regarded as a linear equation for U_0 given U (and the other parameters), which is convenient for calculating actual solutions.

Note: the combination $UK^2 - \beta$ has an important role in the solution. The physical interpretation is that $U - \beta/K^2$ is the Doppler-shifted phase speed of free barotropic rigid-lid Rossby waves with $k^2 + l^2 = K^2$ in a mean zonal flow of speed U.

6.3 Blocked and unblocked flows

To describe this simply, we consider topography with just one zonal wavenumber, so the wavelike flow has the same wavenumber and $\nabla^2\phi = -K^2\phi$. The quasilinear quasigeostrophic vorticity equation 6.15 becomes

$$(\beta - U/K^2)\phi_x - f_0 E^{1/2} K^2 \phi + U(f_0/H)h_x \ = \ 0 \ . \tag{6.21}$$

For small $E^{1/2}$ and $\beta - U/K^2$ not close to zero, the dominant balance is between vorticity generated by the flow over the topography and advection of vorticity by the mean flow (modified by the Rossby wave), giving

$$\phi \ \approx \ \frac{U f_0}{H(UK^2 - \beta)} h \ . \tag{6.22}$$

In this case ϕ is nearly in phase with h, the topographic drag is low, and $U/U_0 \approx 1$. The wavelike flow is small, and the mean flow is 'unblocked'. (Note that ϕ may have the same or opposite sign to h: with $U > \beta/K^2$ it has the same sign and turns out to have a peak slightly upstream of the topographic peak.)

If instead $\beta - U/K^2$ is close to zero, the dominant balance is between vorticity generated by the flow over the topography and weak dissipation via the Ekman layers, giving

$$\phi \ \approx \ \frac{U}{E^{1/2} K^2 H} h_x \ . \tag{6.23}$$

Now the wavelike flow is relatively large and out of phase with h, resulting in large drag and a substantial reduction in U: the flow is 'blocked'.

6.4 Multiple equilibria and regimes

Another feature of the simple model is that 6.18 may have more than one solution for the same parameter choices. This is best illustrated by an example.

Fig 6.1a shows the topography selected, constructed with 10 Fourier modes. Fig 6.2 shows U/U_0 as a function of $R_0 = U_0/f_0 L$, with $E = 10^{-4}$, $h/h_0 = 0.2$, and $\beta L/f_0 = 0.2$. The various curved spikes correspond to regions where various Rossby waves 'resonate'. In this example several such spikes are sufficiently curved for there to be 3 values of U for several ranges of values of U_0. More advanced analysis shows that the middle value is unstable, and in a time-dependent model the solution would settle to one of the other two values of U: the smaller value is a 'blocked' flow, while the other is not blocked.

The upper panel of Fig. 6.3 shows streamlines (contours of the streamfunction) for the unblocked option for $R_0 = 0.2$: the flow (dominated here by the $m = 1$ wave) is only slightly disturbed by the topography, and $U/U_0 = 0.97$.

By contrast the lower panel shows streamlines the blocked alternative: the zonal mean is reduced to $U/U_0 = 0.46$, and the flow is sufficiently distorted to produce closed streamlines within the channel.

As to flow regimes: consider starting at the blocked state with $R_0 = 0.2$. As the driving value U_0 is increased the flow remains blocked until $R_0 \approx 0.3$: beyond this point the system suddenly adjusts to an unblocked state as the only possibility, and remains unblocked if U_0 is increased further. But if U_0 is then decreased, the system remains in the unblocked regime until $R_0 \approx 0.16$, beyond which point the system switches to a moderately blocked state, then switches to a strongly blocked state dominated by the $m = 2$ wave.

6.5 An improvement of the quasilinear model

The reduction of the mean zonal flow is greatest around the centre of the channel for the form assumed for h. Instead of assuming $<u> = U$, a meridional profile with $<u> = U - (U_0 - U)\cos(2ly)$ (so $[<u>] = U$ as before) can be used. This leads to solutions of comparable simplicity that compare well with full numerical solutions of the problem.

3 Rossby waves

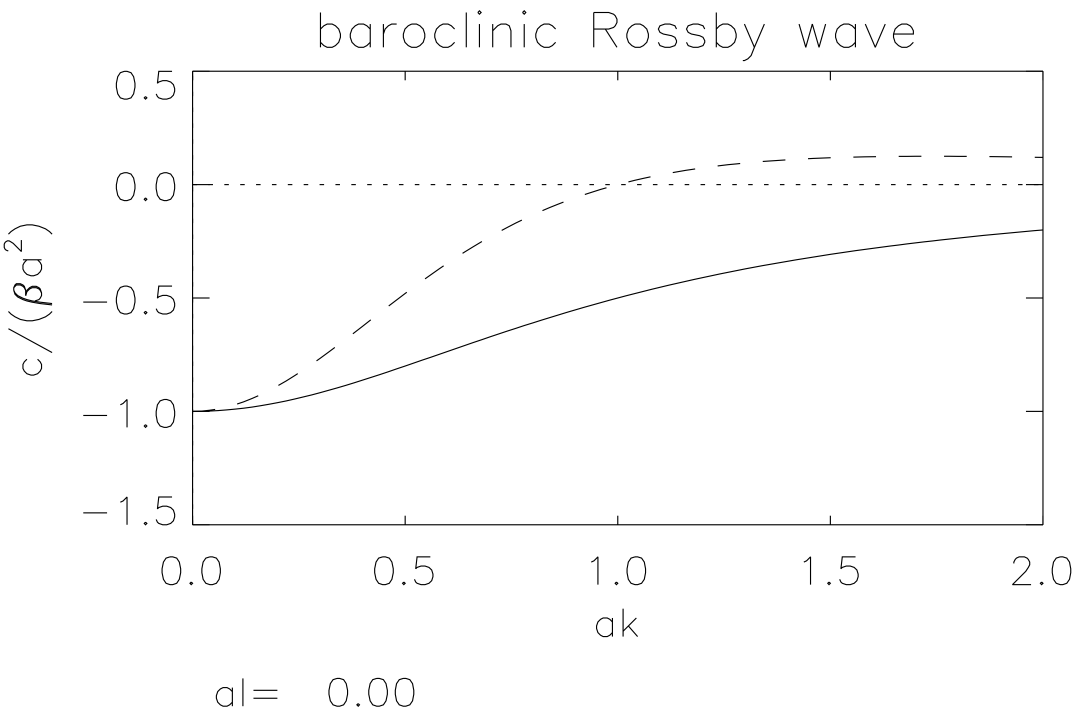

Figure 3.1: Baroclinic Rossby wave: zonal phase and group velocities

4 Mid-latitude 2-layer ocean

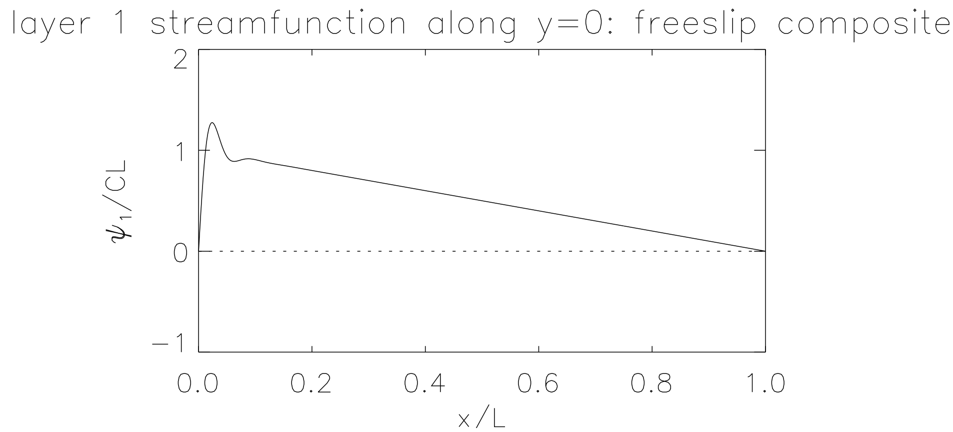

Figure 4.1:

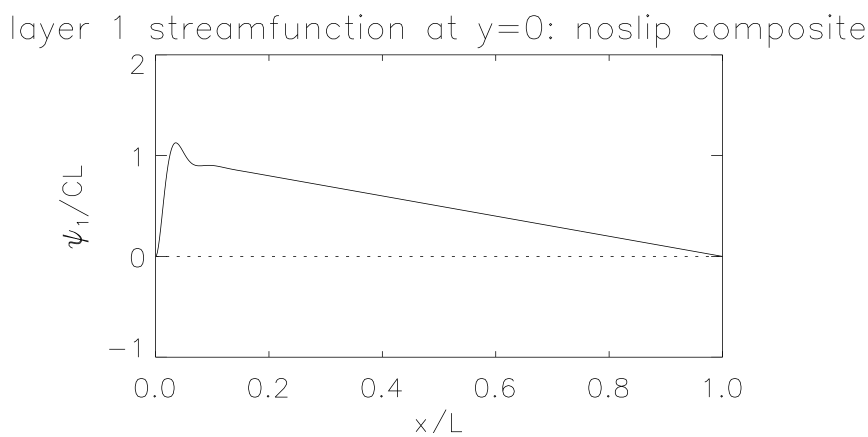

Figure 4.2:

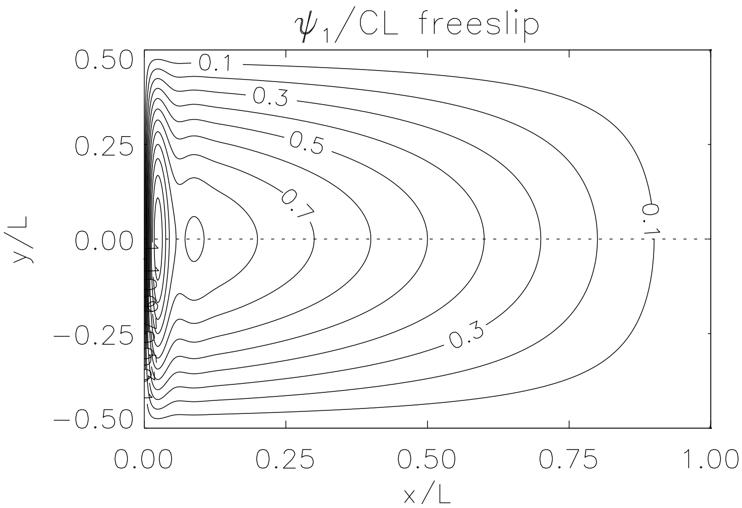

Figure 4.3: steady state

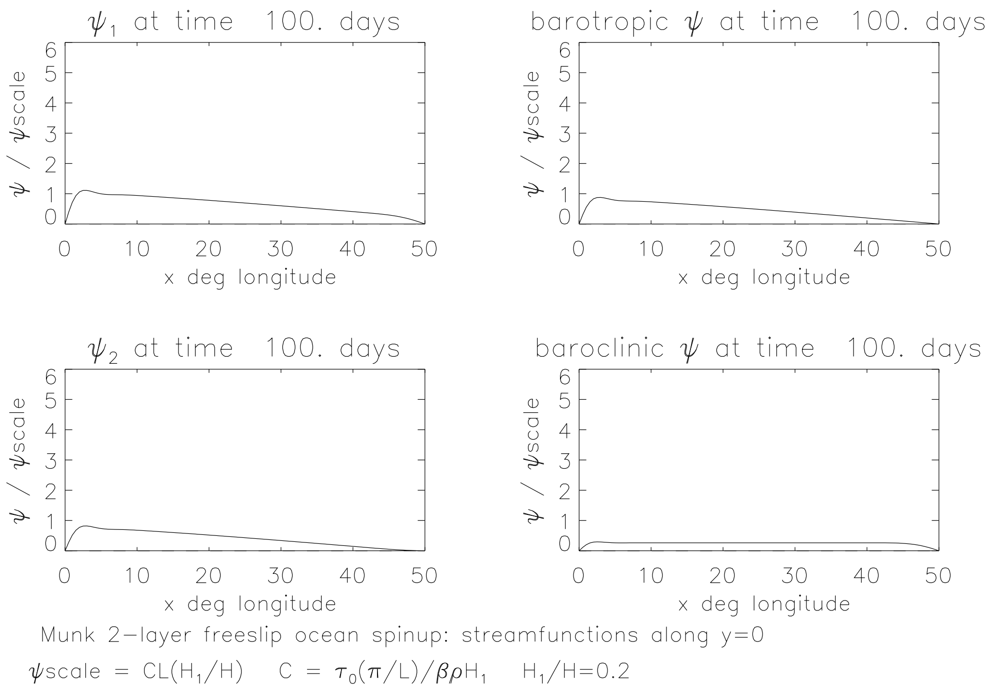

Figure 4.4: Ocean spinup - day 100

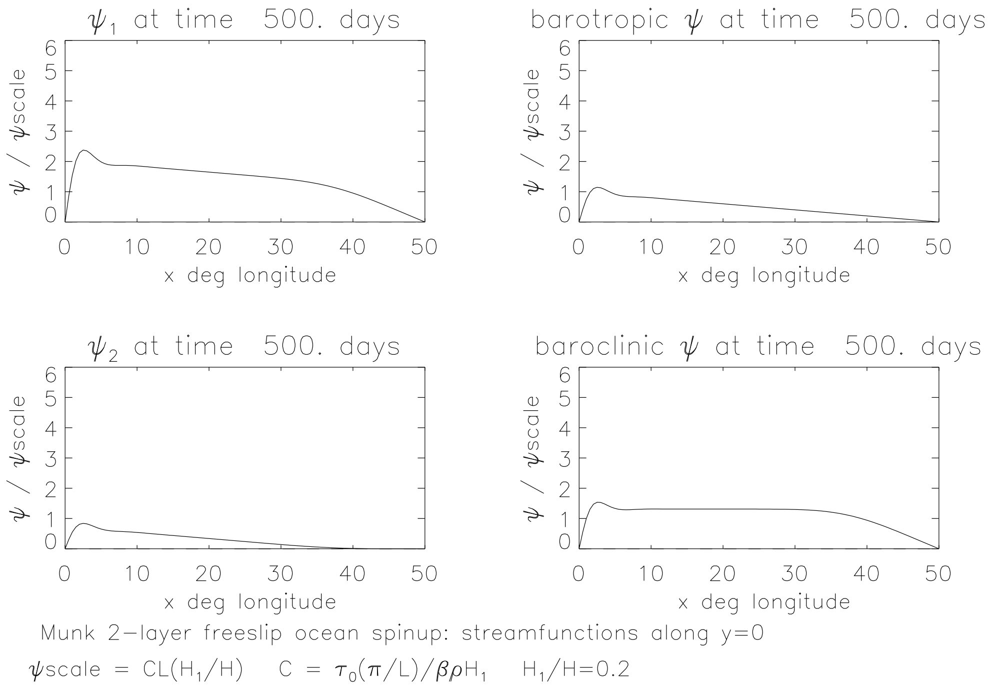

Figure 4.5: Ocean spinup - day 500

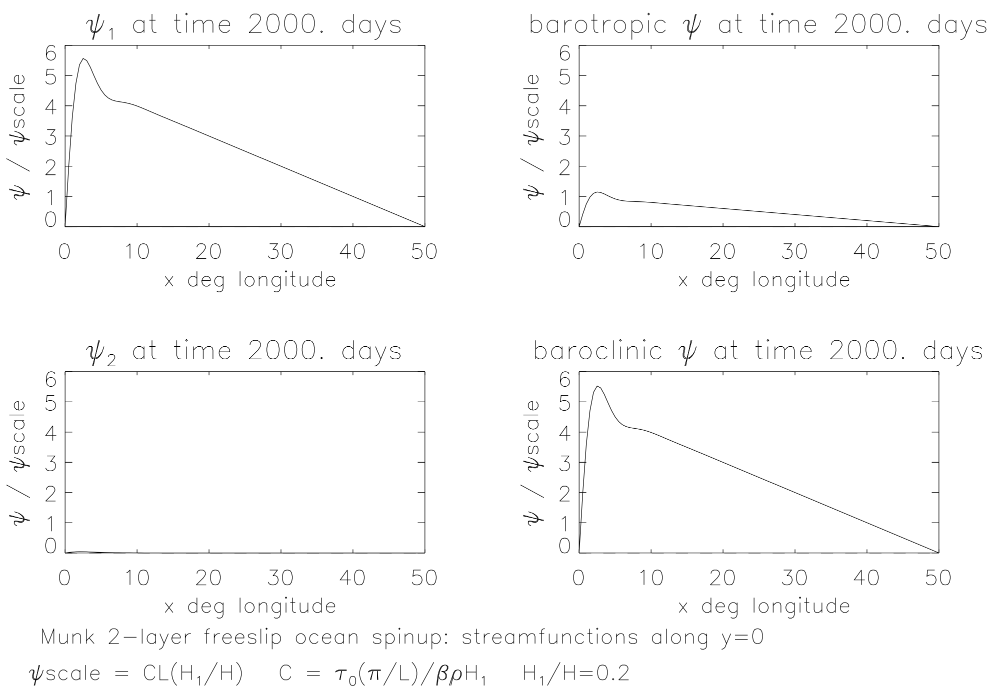

Figure 4.6: Ocean spinup - day 2000

5 Baroclinic instability

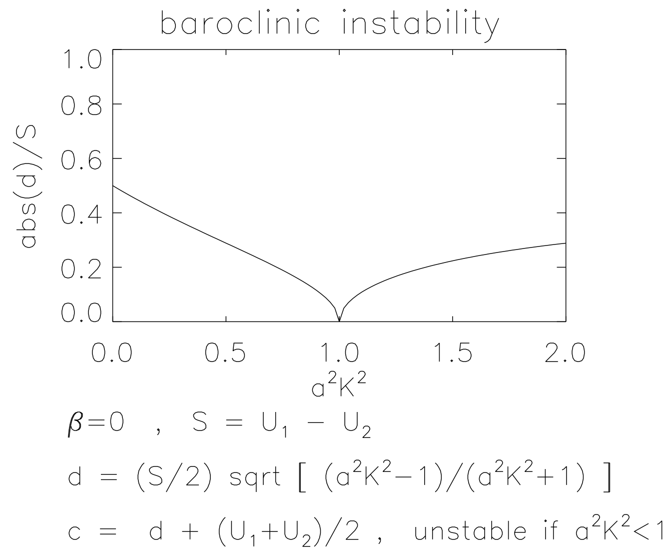

Figure 5.1:

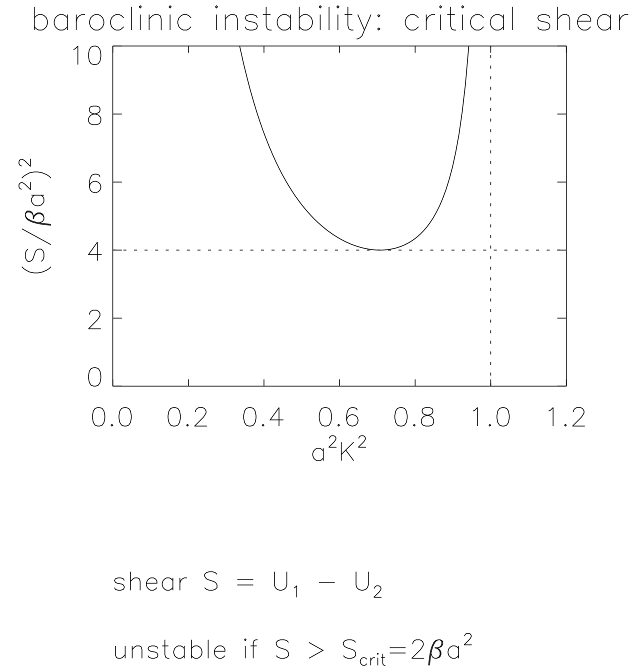
baroclinic instability: critical shear
(S/βa²)²
a²K²
0
2
4
6
8
10
0.0
0.2
0.4
0.6
0.8
1.0
1.2
shear S = U₁ − U₂
unstable if S > S_crit=2βa²

Figure 5.2:

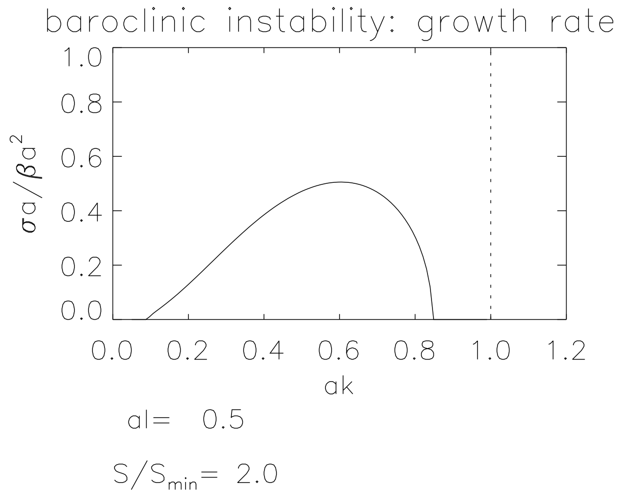
baroclinic instability: growth rate
1.0
0.8
0.6
0.4
0.2
0.0
$\sigma a/\beta a^2$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
ak
al= 0.5
S/S_{min}= 2.0

Figure 5.3:

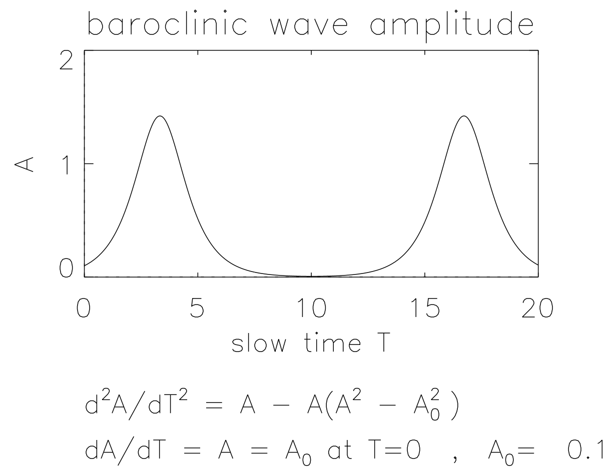
baroclinic wave amplitude
2
1
0
A
0
5
10
15
20
slow time T
d²A/dT² = A − A(A² − A₀²)
dA/dT = A = A₀ at T=0 , A₀= 0.1

Figure 5.4:

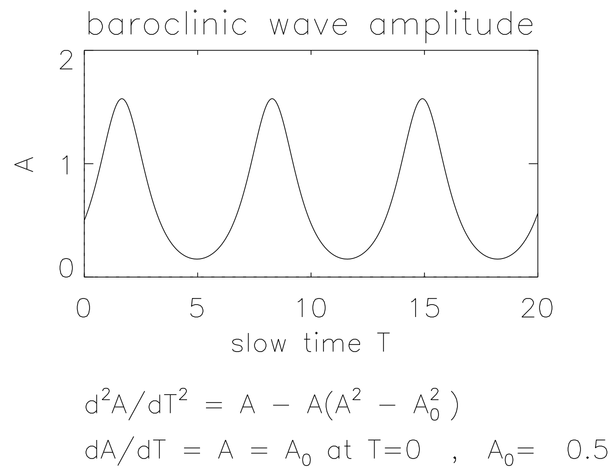
baroclinic wave amplitude
A
0
1
2
0
5
10
15
20
slow time T
d²A/dT² = A − A(A² − A₀²)
dA/dT = A = A₀ at T=0 , A₀= 0.5

Figure 5.5:

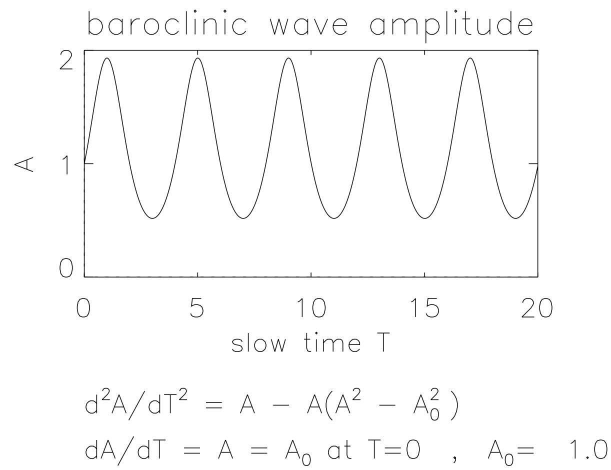
baroclinic wave amplitude
A
0
1
2
0
5
10
15
20
slow time T
d²A/dT² = A − A(A² − A₀²)
dA/dT = A = A₀ at T=0 , A₀= 1.0

Figure 5.6:

6 Multiple equilibia

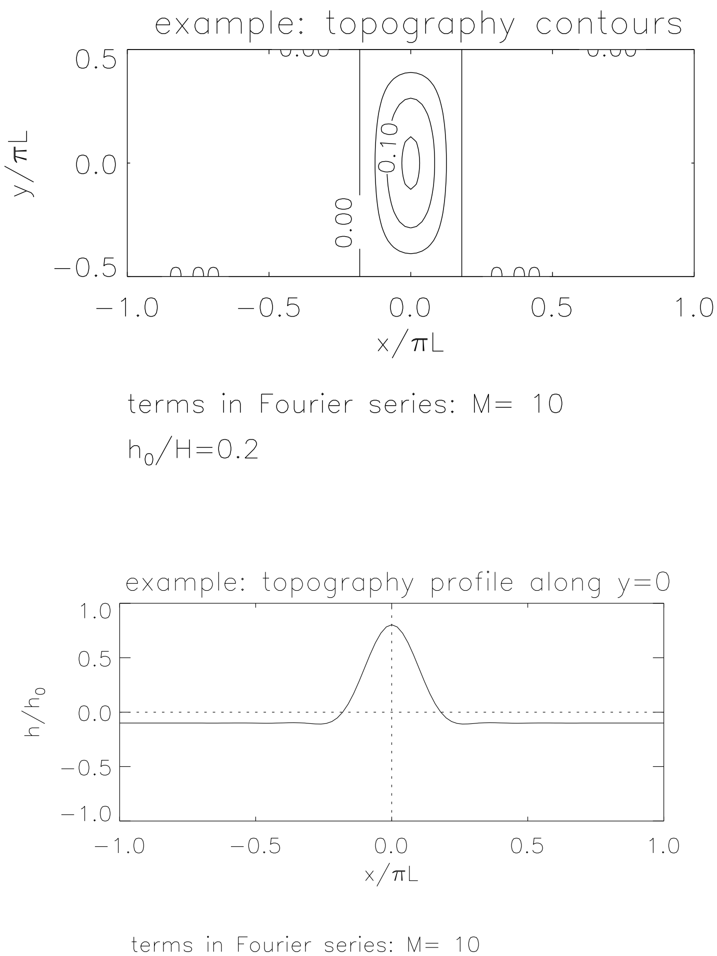

Figure 6.1:

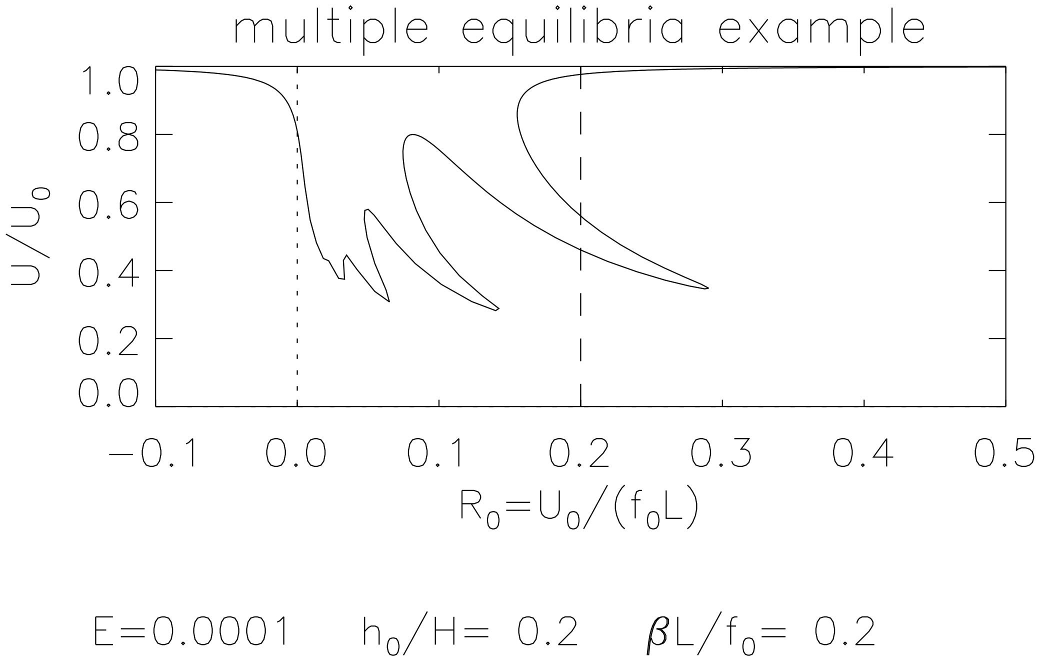
multiple equilibria example
1.0
0.8
0.6
0.4
0.2
0.0
U/U0
−0.1
0.0
0.1
0.2
0.3
0.4
0.5
R0=U0/(f0L)
E=0.0001 h0/H= 0.2 βL/f0= 0.2

Figure 6.2:

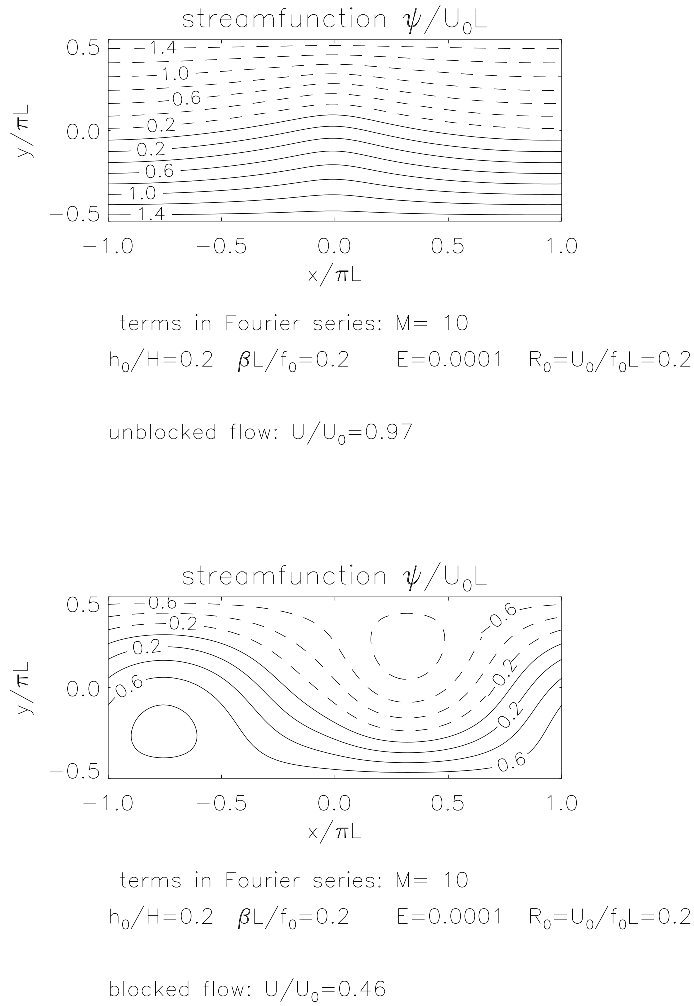

Figure 6.3: streamlines for multiple equilibria