Modular Group and Modular Forms: Exercises 4, 15 Dec. 2008

1. Use the mapping properties of the $\wp$-function to show that every complex torus $X(l_1, l_2)$ has a biholomorphic automorphism of order 2 with 4 fixed points corresponding to the 4 points of order 2 on the torus, $\wp-$images (denoted $e_0, e_1, \ldots$) of the points

$$0, \quad \omega_1 = l_1/2); \quad \omega_2 = l_2/2, \quad \omega_3 = (l_1 + l_2)/2.$$

Normalising so that $\tau = l_2/l_1$, define the λ-function as the cross-ratio

$$l(\tau) = \{e_1, e_2; e_3, e_0\} = \frac{e_3 - e_1}{e_3 - e_2}.$$

Show that under modular transformations $\tau \mapsto \gamma(\tau) = \frac{a\tau+b}{c\tau+d} \in \Gamma(1)$, the four functions $e_j(\gamma(\tau))$ are a permutation of the $e_j(\tau)$. By evaluating the permutation for the generators T, and U of $\Gamma(1)$, show that the λ-function is invariant under the subgroup consisting of $\gamma \cong \mathrm{Id} \bmod 2$ in the congruence subgroup $\Gamma(2)$.

2. Look this function up in the book by Jones & Singerman, pp. 293-296.

3. Show that the congruence subgroup $\Gamma(2)$ is distinct from the commutator subgroup $H(1)$ of $\Gamma(1)$ although they have the same index in $\Gamma(1)$.

What can you discover about the quotient surface $\mathcal{U}/H(1)$?

Weekly course summary:

Modular group/ automorphic forms (Bill Harvey, KCL).

Week 1. The upper half plane, Moebius maps and hyperbolic plane geometry.

Week 2. Action of SL(2,Z) as hyperbolic isometries.

Week 3. Lattices, elliptic functions and Eisenstein series.

Summary notes for these are posted on the LTCC website.

Week 4. More on Eisenstein series and the discriminant form. Automorphic forms in general; projective embedding of tori and the modular quotient surface.

Notes on this are still in preparation.

Week 5. q-expansions and related topics; time permitting, mention quadratic forms & theta functions, modular varieties and the Grothendieck-Belyi theory of arithmetically defined algebraic curves.