An Introduction to p-adic L-functions – Exercises I

Exercise 1. — (a) Let L be an extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$. Show that if $u \in L$ with $|u-1| < 1$, then there is a unique continuous character $\kappa_u : \mathbf{Z}_p \longrightarrow L^\times$ sending 1 to u.

(b) Show that there are no non-trivial characters $\mathbf{Z}_p \longrightarrow \overline{\mathbb{F}}_p^\times$.

(c) Deduce that if $\kappa : \mathbf{Z}_p \longrightarrow L^\times$ is a continuous character, then $\kappa = \kappa_u$ for some $u \in L$ with $|u-1| < 1$.

Exercise 2. — Let $\mu, \lambda \in \mathscr{M}(\mathbf{Z}_p, \mathscr{O}_L)$ be two measures on $\mathbf{Z}_p$, and define their *convolution* $\mu * \lambda \in \mathscr{M}(\mathbf{Z}_p, \mathscr{O}_L)$ by

$$\int_{\mathbf{Z}_p} f \cdot (\mu * \lambda) := \int_{\mathbf{Z}_p} \left(\int_{\mathbf{Z}_p} f(x+y) \cdot \lambda(y) \right) \cdot \mu(x).$$

(a) Show that convolution defines an $\mathscr{O}_L$-algebra structure on $\mathscr{M}(\mathbf{Z}_p, \mathscr{O}_L)$.

(b) Show that $\mathscr{A}_{\mu * \lambda} = \mathscr{A}_\mu \mathscr{A}_\lambda$. Deduce that the Mahler transform is an isomorphism of $\mathscr{O}_L$-algebras.

Exercise 3. — For $a \in \mathbf{Z}_p$, define the *Dirac measure* δ_a by

$$\int_{\mathbf{Z}_p} \phi \cdot \delta_a = \phi(a).$$

Show that the $\mathscr{O}_L$-module generated by the δ_a for $a \in \mathbf{N}$ is dense in $\Lambda(\mathbf{Z}_p)$.

Exercise 4. — (a) If ζ is any p-power root of unity, verify that $x \mapsto \zeta^x$ is a locally constant (hence continuous) function $\mathbf{Z}_p \mapsto \mathbf{C}_p$.

(b) Prove that the $\mathbf{C}_p$-submodule of $\mathscr{C}(\mathbf{Z}_p, \mathbf{C}_p)$ generated by the functions ζ^x, as ζ runs over all p-power roots of unity, is dense.

(c) Let μ be a measure on $\mathbf{Z}_p$ and let $\chi : \mathbf{Z}_p^\times \to L$ be a character of conductor p^n with $n \geq 1$. Show that

$$\int_{\mathbf{Z}_p} \chi(x) \cdot \mu = \frac{1}{G(\chi^{-1})} \sum_{b \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \chi^{-1}(b)\, \mathscr{A}_\mu(\zeta_{p^n}^b - 1).$$

(c) Recall that a power series $F \in \mathscr{O}_L[[T]]$ can be seen a (bounded analytic) function on the open unit ball. Interpret the above result in this language.

Exercise 5. — (a) Let Γ denote a group that is isomorphic to $\mathbf{Z}_p$. Show that, for any topological generator γ of Γ, there is a unique isomorphism

$$r_\gamma^n : \mathscr{O}_L[\Gamma/\gamma^{p^n}] \longrightarrow \mathscr{O}_L[T]/\varphi^n(T)$$

sending $[\gamma]$ to $1+T$.

(b) Using the fact that

$$\mathscr{O}_L[[T]] \cong \varprojlim \mathscr{O}_L[T]/\varphi^n(T),$$

deduce that there is a unique isomorphism

$$r_\gamma : \Lambda(\mathbf{Z}_p) \cong \mathscr{O}_L[[T]]$$

of $\mathscr{O}_L$-algebras sending the Dirac measure δ_γ to $1+T$.

(c) Fix an isomorphism $\theta : \mathbf{Z}_p \xrightarrow{\sim} \Gamma$, and let $\gamma = \theta(1)$. Show that the isomorphism $\Lambda(\mathbf{Z}_p) \cong \mathscr{O}_L[[T]]$ induced by θ and r_γ is the Mahler transform.

Exercise 6. — We can equip the space $\mathscr{M}(\mathbf{Z}_p, \mathscr{O}_L)$ with two natural topologies; the *strong topology*, which is the topology induced by the valuation

$$v_{\mathscr{M}}(\mu) = \inf_{\phi \in \mathscr{C}(\mathbf{Z}_p, \mathscr{O}_L)} \big(v_p\big(\mu(\phi)\big) - v_{\mathscr{C}}(\phi)\big),$$

and the *weak topology*, in which a sequence μ_n converges if and only if the limit $\mu_n(\phi)$ exists for all $\phi \in \mathscr{C}(\mathbf{Z}_p, \mathscr{O}_L)$. Show that, under the Mahler transform, the strong topology corresponds to the p-adic topology on $\mathbf{Z}_p[[T]]$, whilst the weak topology corresponds to the (p,T)-adic topology.

Exercise 7. — Recall that $\mathbf{Z}_p^\times$ acts on $\mathbf{Z}_p[[T]]$ by $\sigma_a(T) = (1+T)^a - 1$, $a \in \mathbf{Z}_p^\times$. Show that, for every $a \in \mathbf{Z}_p^\times$, σ_a defines an isometry on $\mathbf{Z}_p[[T]]$ (with the p-adic topology).

Exercise 8. — Define the *augmentation ideal* $I(\mathcal{G})$ to be the kernel of the degree map

$$\begin{aligned} \deg : \Lambda(\mathcal{G}) &\longrightarrow \mathbf{Z}_p, \\ \sum_{a\in\mathcal{G}} c_a[a] &\longmapsto \sum_{a\in\mathcal{G}} c_a. \end{aligned}$$

Show that if e is a topological generator of $\mathbf{Z}_p^\times$, then $I(\mathcal{G}) = (\sigma_e - 1)\Lambda(\mathcal{G})$. Hence show that $\zeta_p \in Q(\mathcal{G})$ is a pseudo-measure.

Exercise 9. — Let η be a Dirichlet character of conductor D prime to p, and recall the definition of μ_η from the lecture notes. Prove that, for any Dirichlet character χ of conductor p^n, we have

$$\int_{\mathbf{Z}_p^\times} \chi(x)x^k \cdot \mu_\eta = \big(1 - \chi\eta(p)p^k\big)L(\chi\eta, -k).$$

Exercise 10. — We define the *weight space* to be

$$\mathcal{W}(\mathbf{C}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times).$$

Show that:

(a) Topologically, $\mathcal{W}(\mathbf{C}_p)$ is the disjoint union of $p-1$ open unit balls in $\mathbf{C}_p$;

(b) We have $\mathbf{Z} \subset \mathcal{W}(\mathbf{C}_p)$, and two integers k, k' lie in the same open unit ball if and only if $k \equiv k' \pmod{p-1}$.

(c) A function $f : \mathcal{W}(\mathbf{C}_p) \to \mathbf{C}_p$ is defined to be a *rigid analytic function* if the restriction of f to any of the $p-1$ open unit balls $\{s \in \mathbf{C}_p : |s| < 1\}$ admits a power series expansion in the variable s.

(d) Let $\mu \in \mathscr{M}(\mathbf{Z}_p^\times, \mathscr{O}_L)$. Define the Fourier transform of μ to be

$$\begin{aligned} f_\mu : \mathcal{W}(\mathbf{C}_p) &\longrightarrow \mathbf{C}_p, \\ \eta &\longmapsto \int_{\mathbf{Z}_p^\times} \eta\cdot\mu. \end{aligned}$$

Show that f is a bounded rigid analytic function.

(e) Show that any bounded rigid analytic function on the weight space is the Fourier transform of a measure.

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