An Introduction to p-adic L-functions – Exercises II

Exercise 1. — Use the Weierstrass preparation theorem, as stated in the appendix of the lecture notes, to prove that any non-zero power series over $\mathbf{Z}_p$ has finitely many zeros in the maximal ideal of $\mathbf{C}_p$. (We used this fact to prove the uniqueness of the Coleman power series attached to a system of local units).

Exercise 2. — Let φ and $\mathcal{N}$ denote the Frobenius and norm operators on power series. Prove that:

(i) If $\varphi(f)(T) \equiv 1 \mod p^k$, then $f(T) \equiv 1 \mod p^k$.
(ii) If $f \in \mathbf{Z}_p[[T]]^\times$, then

$$\mathcal{N}(f) \equiv f \mod p.$$

Exercise 3. — We define the *Coates–Wiles homomorphisms* $\delta_k : \mathscr{U}_\infty \longrightarrow \mathbf{Z}_p$ by

$$\delta_k(u) = \left[\partial^k \log(f_u)\right](0).$$

Show that

$$\int_{\mathcal{G}} x^k \cdot \mathrm{Col}(u) = (1 - p^{k-1})\delta_k(u).$$

In other words, the Coleman map is a p-adic interpolation of the Coates–Wiles homomorphisms.

Exercise 4. — Recall we defined $\mathcal{G} = \mathrm{Gal}(F_\infty/\mathbf{Q})$ and $\mathscr{X}_\infty = \mathrm{Gal}(\mathscr{M}_\infty/F_\infty)$. We defined an action of $\mathcal{G}$ on $\mathscr{X}_\infty$ by

$$\sigma \cdot x = \widetilde{\sigma} x \widetilde{\sigma}^{-1}, \quad \sigma \in \mathcal{G}, x \in \mathscr{X}_\infty,$$

where $\widetilde{\sigma}$ is any lift of σ to $\mathrm{Gal}(\mathscr{M}_\infty/\mathbf{Q})$. Show that this action is well-defined and extends to an action of $\Lambda(\mathcal{G})$.

Exercise 5. — Recall that we defined the L-function of a p-adic Galois representation in the introduction of the lecture notes.

(a) Show that

$$L^{(p)}(\mathbf{Q}_p(1), s) = (1 - p^{-s-1})\zeta(s+1),$$

where the superscript (p) means that we delete the Euler factor at p. This explains the ‘twist by 1’ we introduced in the lecture notes when studying the Iwasawa main conjecture.

(b) More generally, if $V \in \mathrm{Rep}_L(\mathscr{G}_{\mathbf{Q}})$ is a p-adic representation of $\mathscr{G}_{\mathbf{Q}}$ that is unramified outside p, define $V(k)$ to be the representation of $\mathscr{G}_{\mathbf{Q}}$ whose underlying vector space is V, with Galois action defined by

$$\sigma \cdot_k v = \chi(\sigma)^k (\sigma \cdot v),$$

where $\chi : \mathscr{G}_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ is the cyclotomic character. Show that

$$L^{(p)}(V(k), s) = L^{(p)}(V, s+k).$$

(The same is true of the full L-functions, but this involves p-adic Hodge theoretic arguments; specifically, we need the characteristic polynomial of Frobenius at p acting on $\mathbf{D}_{\mathrm{cris}}(V)$).

Exercise 6. — (a) Let L/K be an extension of number fields such that there is no non-trivial subextension that is unramified everywhere (including at infinite places). Show that the class number of K divides the class number of L.

(b) Deduce that if $m|n$, then the class number of $\mathbf{Q}(\mu_m)$ divides the class number of $\mathbf{Q}(\mu_n)$.

Exercise 7. — Show that the cyclotomic units form an Euler system. Explicitly, recall that we defined

$$\mathbf{c}_m := \frac{\zeta_m^{-1}-1}{\zeta_m - 1},$$

where (ζ_m) is a compatible system of roots of unity (so that $\zeta_{\ell m}^{\ell} = \zeta_m$ for all integers ℓ). Show that

$$\mathrm{Norm}_{\mathbf{Q}(\mu_{\ell m})/\mathbf{Q}(\mu_m)}(\mathbf{c}_{\ell m}) = \begin{cases} \mathbf{c}_m & : \ell | m, \\ (1-\ell^{-1})\mathbf{c}_m & : \ell \nmid m. \end{cases}$$

Exercise 8. — Recall that

$$\mathscr{V}_{n,1}^{+} = \left\{u \in \mathscr{O}_{F_n^+}^{\times} : u \equiv 1 \pmod{\mathfrak{p}_n}\right\},$$

where $\mathfrak{p}_n$ is the unique prime of F_n^+ above p, and recall that $\mathscr{D}_{n,1}^+$ is the intersection of the cyclotomic units in F_n with $\mathscr{V}_{n,1}^+$.

(a) Suppose that $p \nmid h_1^+ := \#\mathrm{Cl}(\mathbf{Q}(\mu_p)^+)$. Show that the index $[\mathscr{V}_{n,1}^+ : \mathscr{D}_{n,1}^+]$ is finite and prime to p.

(b) Deduce that in this case, there is an isomorphism

$$\mathscr{D}_{n,1}^+ \otimes_{\mathbf{Z}} \mathbf{Z}_p \cong \mathscr{V}_{n,1}^+ \otimes_{\mathbf{Z}} \mathbf{Z}_p.$$

Exercise 9. — Let $\mathscr{A}$ be a finitely generated $\Lambda(\mathbf{Z}_p^\times)$-module. Let $\omega : \mathbf{Z}_p^\times \to \mu_{p-1} \subset \mathbf{Z}_p^\times$ be the usual Teichmüller character (that is, projection to the first factor in the decomposition $\mathbf{Z}_p^\times = \mu_{p-1} \times (1+p\mathbf{Z}_p)$).

(a) Let

$$e_i := \frac{1}{p-1}\sum_{a \in (\mathbf{Z}/p\mathbf{Z})^\times} \omega^{-i}(a)\cdot [a].$$

Show that there is a decomposition

$$\mathscr{A} \cong \bigoplus_{i=1}^{p-1} \mathscr{A}^{(i)},$$

where $\mathscr{A}^{(i)} = e_i\mathscr{A}$.

(b) Show that $a \in \mu_{p-1}$ acts on $\mathscr{A}^{(i)}$ by multiplication by $\omega^i(a)$.

(c) We can apply this to $\Lambda(\mathbf{Z}_p^\times)$ itself. Show that

$$\Lambda(\mathbf{Z}_p^\times)^{(i)} \cong \Lambda(\mathbf{Z}_p).$$

(d) Recall the definitions of $\mathscr{A}^+$ and $\mathscr{A}^-$ from the notes as the spaces where $-1 \in \mathbf{Z}_p^\times$ acts as $+1$ and -1 respectively. Show that

$$\mathscr{A}^+ = \bigoplus_{i \text{ even}} \mathscr{A}^{(i)}$$

and

$$\mathscr{A}^- = \bigoplus_{i \text{ odd}} \mathscr{A}^{(i)}.$$

(e) Let $\mu \in \Lambda(\mathbf{Z}_p^\times)$, and write $\mu^{(i)} = e_i\mu$ for its image in $\Lambda(\mathbf{Z}_p^\times)^{(i)}$. Show that $\mu^{(i)} = 0$ if and only if

$$\int_{\mathbf{Z}_p^\times} x^k \cdot \mu = 0$$

for all integers $k \equiv i \pmod{p-1}$.

The course will be accompanied by two exercise sheets. If you have questions about the exercises, or would like to submit solutions for feedback, please email us at the addresses below.

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