London Taught Course on Spectral Theory
Final Test

February 2013

Your work on this test must be emailed to E.Brian.Davies@kcl.ac.uk by the end of February. Please make sure that it is readable before sending it. If you post it to Professor E.B. Davies, Department of Mathematics, King's College London, Strand, London WC2R 2LS, you *must* make and keep a copy first. You may use any books you like, or the web, to gather information, but must acknowledge sources in the text, not just once at the end, and must write out the solution in your own words.

Let $\mathcal{H} = L^2([0,\infty), \mathrm{e}^{-x}\,\mathrm{d}x)$ be the Hilbert space of all functions $f : [0,\infty) \to \mathbf{C}$ such that

$$\|f\| = \left\{ \int_0^\infty |f(x)|^2 \mathrm{e}^{-x}\,\mathrm{d}x \right\}^{1/2} < \infty,$$

with inner product

$$\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)}\mathrm{e}^{-x}\,\mathrm{d}x.$$

Let

$$(Hf)(x) = -xf''(x) + (x-1)f'(x)$$

on the domain $\mathrm{Dom}(H)$ of all twice continuously differentiable $f : [0,\infty) \to \mathbf{C}$ such that such that f, f', f'' are polynomially bounded.

Use integration by parts to prove that

$$\langle Hf, g\rangle = \langle f, Hg\rangle = \int_0^\infty x\mathrm{e}^{-x} f'(x)\,\overline{g'(x)}\,\mathrm{d}x$$

for all f, $g \in \mathrm{Dom}(H)$. This implies that H is symmetric. This differential operator is associated with Laguerre.

Prove that if $\mathcal{P}_n$ is the linear subspace of all polynomials that have degree at most n, then $H(\mathcal{P}_n) \subseteq \mathcal{P}_n$. Apply the Gram-Schmidt orthogonalization procedure to the sequence u_0, u_1, $u_2, \ldots$, where $u_n(x) = x^n$, to prove that there exists an orthonormal sequence e_0, e_1, $e_2, \ldots$ of polynomials in $\mathcal{H}$ such that e_n has degree n for every n.

Prove that

$$\mathrm{lin}\{e_r : 0 \le r \le n\} = \mathcal{P}_n$$

for all n. The e_n are known as Laguerre polynomials, of which there are many accounts.

By using the symmetry of the operator H and $H(\mathcal{P}_n) \subseteq \mathcal{P}_n$, prove that there exist $\lambda_n \in \mathbf{R}$ such that $He_n = \lambda_n e_n$ for all n.

Write down a formula for λ_n for all n. If you do not prove the formula yourself, state where you obtained the formula from.

Assuming without proof that the subspace $\mathcal{P}$ of all polynomials is norm dense in $\mathcal{H}$, prove the abstract Parseval identity, namely

$$\|f\|^2 = \sum_{n=0}^{\infty} |\langle f, e_n \rangle|^2$$

for all $f \in \mathcal{H}$, by proving the equivalence of the relevant definitions of completeness of an orthonormal sequence in $\mathcal{H}$.

Use the properties of H proved above, write down without proof the statement of a theorem that proves that H is essentially self-adjoint on $\mathcal{P}$. What is the spectrum of the closure $\overline{H}$ of H?

Write down without proof the statement of a variational theorem that enables you to obtain explicit upper and lower bounds on the eigenvalues of the self-adjoint operator K acting on $\mathcal{H} = L^2([0,\infty), \mathrm{e}^{-x}\,\mathrm{d}x)$ and defined by

$$(Kf)(x) = -xf''(x) + (x-1)f'(x) + \frac{1}{3}\sin(x)f(x),$$

where $\mathrm{Dom}(K) = \mathrm{Dom}(\overline{H})$. What are the bounds that this method yields?