Graph Minors

An intensive course by Steven Noble
De Morgan House, London, 13-14 May 2010

A graph H is a minor of a graph G, if H can be obtained from G by deleting and contracting edges, and deleting isolated vertices.

Many important classes of graphs are closed under minors, in the sense that if G belongs to the class then so does any minor of G. Examples are the class of planar graphs, or more generally the graphs embeddable in any fixed surface, series-parallel graphs and outerplanar graphs.

Any minor closed class of graphs can be characterized by its excluded minors, i.e. the minor-minimal graphs that do not belong to the class. For example, a version of Kuratowski's famous theorem on planarity states that a graph is planar if and only if it does not contain either K_5 or $K_{3,3}$ as a minor.

Wagner conjectured that the set of excluded minors is always finite, or equivalently that in any infinite set of graphs, one can always find two non-isomorphic graphs G and H with G a minor of H. In a series of 21 papers, between 1984 and 2009, Robertson and Seymour proved Wagner's conjecture and, amongst many other important results on graph minors, showed that for each graph H there is a polynomial time algorithm to determine whether a graph contains H as a minor. This implies the existence of a polynomial time algorithm to determine membership in any class of graphs that is closed under minors.

In this course we will investigate some of the tools and techniques that are used in the proof, particularly those which have seen applications outside the graph minor project, and look at the algorithmic consequences of graph minors.

Examples of minor closed families of graphs: planar graphs, graphs embeddable on any fixed surface, series-parallel graphs, outerplanar graphs, linklessly embeddable graphs, delta-wye graphs.

Overview of the graph minor project: well-quasi-ordering and Wagner's conjecture, algorithmic consequences, well-quasi-ordering for trees.

Tree-decompositions: tree-width, partial k-trees, chordal graphs, min–max theorems for tree-width, linear time algorithms for NP-hard problems such as MAX-CLIQUE, CHROMATIC NUMBER and counting problems such as RELIABILITY.

Proof of the graph minor theorem: some key steps, such as well-quasi-ordering for graphs of bounded tree-width and bounds for tree-width when a planar graph is excluded.

Algorithmic consequences: polynomial time algorithm for minor testing, bidimensional parameters, parameter–tree-width bounds, applications to domination number, vertex cover and orienting a planar graph.