LECTURE 1 Atiyah-Singer index theorem

M manifold, closed $\boxed{= \text{cpct with } \partial M = \emptyset}$

Riemannian $= (M, g)$, + spin structure.

Atiyah-Singer index formula states that:

$$\underbrace{\operatorname{ind}(\not{D}^{V})} \overset{①}{=} \left\langle \hat{A}(M)\, ch(V), [M] \right\rangle \overset{②}{=} \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_M \hat{A}(M, R)\, ch(V, F)$$

$n = \dim M$

index of coupled Dirac operator with $\dim M = 2n$

$$\boxed{\operatorname{ind}(\not{D}^{V}) = \dim \operatorname{Ker}(\not{D}^{V}) - \dim\left(\operatorname{Ker}(\not{D}^{V})^{*}\right)}$$

$$\not{D}^{V} : C^\infty(M, S^+ \otimes V) \longrightarrow C^\infty(M, S^- \otimes V)$$

$\sum e_i \cdot \nabla^{S^+\otimes V}_{e_i}$

spin connection

Clifford matrices

+ve (complex) spinor bundle

"coeff" vector bundle ("coupled to external gauge fields")

−ve spinor bundle

$\boxed{\text{Task of this course is to explain all these terms and components, and prove ①, ②.}}$

In ①, the $\hat{A}$-cohomology class

$$\boxed{\hat{A}(M) = 1 + \hat{A}_4(M) + \hat{A}_8(M) + \dots \in H^*(M,\mathbb{Q}),\ \hat{A}_{4k} \in H^{4k}(M,\mathbb{Q})}$$

is computed from the power series $Q(z) = \dfrac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$

as $$\hat{A}(M) = \prod_{j=1}^{N} \frac{x_j/2}{\sinh(x_j/2)} \qquad x_j = c_1(L_j)$$

where $TM \otimes_{\mathbb{R}} \mathbb{C} = L_1 \oplus \bar{L}_1 \oplus \dots \oplus L_N \oplus \bar{L}_N$,

$\leftarrow$ cx line bundle.

giving

$$\hat{A}_4 = -\frac{1}{24} p_1, \quad \hat{A}_8 = \frac{1}{5760}(7p_1^2 - 4p_2), \dots$$

where $p_j = p_j(M) \in H^{4j}(M,\mathbb{Z})$ is the j^{th} Pontryagin class. Also, $ch(V) = \sum_{k \geq 0} \frac{1}{k!}(x_1 + \dots + x_R)^k$ with $x_j = c_j(L_j)$, $V = L_1 \oplus \dots \oplus L_R$, is the Chern character of the complex vector bundle V.

The evaluation $\langle \hat{A} \cdot ch, [M] \rangle$ in ① is the linear map $H^n(M,\mathbb{Q}) \longrightarrow \mathbb{Q}$, $\alpha \mapsto \langle \alpha, [M] \rangle$ with $[M]$ the fundamental homology class and $\langle\ ,\ \rangle$ the pairing between cohomology and homology

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The equality (2) is the de Rham formulation, with

$$\boxed{\hat{A}(M,R) = \sqrt{\det\left(\frac{R/2}{\sinh(R/2)}\right)}}, \quad \boxed{ch(V,F^V) = \mathrm{Tr}\left(e^{F^V}\right)}, \quad F^V = \text{curvature of } V \text{ wrt connection}$$

R = Riemannian curvature on $M \in \Omega^2(M, \mathrm{End}(TM))$.

Comment: The cobordism ring of manifolds

$$\mathcal{N} = \{[X] \mid X \text{ closed}\}$$

is defined by

$X \sim X'$ if

$\Rightarrow [X]$

Ring structure:
- $+$ = disjoint union $X_0 \sqcup X_1$
- $\cdot$ = direct product $X_0 \times X_1$

The $\hat{A}$-hat class defines a ring homomorphism

$$\hat{A} : \mathcal{N} \longrightarrow \mathbb{Q}, \quad [M] \longmapsto \int_M \hat{A}(M)$$

which is integer-valued when M is a spin-manifold.

The index theorem says why - because then $\int_M \hat{A}(M)$ is the index $\in \mathbb{Z}$ of the Dirac operator.

Cobordant manifolds have the same index!

In fact, characteristic classes originally arose as cobordism invariants:

$X \sim X'$ iff $sw_J(X) = sw_J(X')$

← Stieffel-Whitney numbers

if also X, X' oriented cobordant — AND — $p_J(X) = p_J(X')$

← Pontryagin numbers.

if X, X' have stable complex structures and cobordant — iff $c_J(X) = c_J(X')$

← Chern numbers

In this course we prove the AS index thm using the heat equation method, via the McKean-Singer index formula:

$$③\quad \boxed{\operatorname{ind}(D) = \operatorname{Tr}\left(e^{-t\Delta}\right) - \operatorname{Tr}\left(e^{-t\tilde{\Delta}}\right)} \qquad t > 0.$$

"trace"

Here:

D = elliptic 1st order differential operator.

$D : C^\infty(M, E^+) \longrightarrow C^\infty(M, E^-)$ $\quad E^\pm$ vector bundles.

E.g. $D = i\frac{d}{dx}$ on $S^1 = \mathbb{R}^1/2\pi\mathbb{Z}$ — acting on $2\pi\mathbb{Z}$ periodic functions on $\mathbb{R}^1$.

$$\Delta = D^*D : C^\infty(M, E^+) \longrightarrow C^\infty(M, E^+)$$
$$\tilde{\Delta} = DD^* : C^\infty(M, E^-) \longrightarrow C^\infty(M, E^-)$$

adjoint operator w.r.t. metrics on E & M.

Why does ③ hold?

Let $A : H \to H$ be trace-class operator on a Hilbert space H.

$\Rightarrow$
- A has a discrete spectrum consisting of e-values
  $\mu_0, \mu_1, \mu_2, \ldots \longrightarrow 0$
  (with repetitions)
- $\sum_j |\mu_j| < \infty$.

Then, if

$$\boxed{A : C^\infty(M,E) \longrightarrow C^\infty(M,E)}$$

$E \to M$ a vector bundle.

is an integral operator, so that

$$\boxed{(A\psi)(x) = \int_M k_A(x,y)\,\psi(y)\,d_M y.}$$

vol form

$\psi \in C^\infty(M,E)$

with its $k_A : M \times M \to \mathrm{End}\,E$, $(x,x) \longmapsto K_A(x,x) \in \mathrm{End}(E_x)$

n×n matrix

v. space = fibre of E at x

Then

$$\boxed{\mathrm{Tr}(A) = \int_M \mathrm{tr}(k(x,x))\, d_M x}$$

tr = matrix trace.

(4)

$$\boxed{= \sum_{j \geq 0} \mu_j}$$

In (3), $e^{-t\Delta}$ is the heat operator, solving the heat equation (elliptic PDE on M)

(5)

$$\boxed{\partial_t \psi + \Delta \psi = 0}$$

HEAT EQUATION

$\psi = \psi(x,t)$

$\psi : M \times (0,\infty) \to \mathrm{End}\,E$

$\boxed{\psi(x,0) = \phi(x)}$

initial heat distribution

Precisely,

$$\boxed{\psi(x,t) = e^{-t\Delta}\phi(x)}$$

Heat kernel.

(6)

$$e^{-t\Delta}\phi(x) = \int_M H_t(x,y)\,\phi(y)\,d_M y$$

We shall see that $e^{-t\Delta}$ is trace-class and so from (4)

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) = \int_M \mathrm{tr}\,(H_t(x,x))\, d_M x.}$$

(7)

$$\boxed{= \sum e^{-t\lambda_j}}$$

λ_j an e-value of Δ (repeated with multiplicity)

But since D is elliptic then the positive self-adjoint Laplacian $\Delta = D^*D$ has a discrete spectrum of eigenvalues which are real and positive and accumulating at $+\infty$:

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \longrightarrow +\infty.$$

E.g. $D = i\frac{d}{dx}$ on S^1, then $\Delta = -\frac{d^2}{dx^2}$ with e-values

$$0, 0, 1, 1, 4, 4, \ldots, n^2, n^2, \ldots$$

and

$$\mathrm{Tr}(e^{-t\Delta}) = 2\sum_{n=0}^{\infty} e^{-n^2 t}$$

Similarly, $\tilde{\Delta} = DD^*$ has spectrum

$$0 \leq \tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \dots \longrightarrow \infty$$

and $$\mathrm{Tr}(e^{-t\tilde{\Delta}}) = \sum_{i\geq 0} e^{-t\tilde{\lambda}_i}.$$

LEMMA 1: Δ and $\tilde{\Delta}$ have the same non-zero eigenvalues (with same multiplicity).

PROOF: Let $\lambda \in \mathrm{spec}(\Delta)\setminus\{0\}$. Then $\Delta\phi_\lambda = \lambda\phi_\lambda$ some $\phi_\lambda \in C^\infty(M, E^+)$. Then set $\rho_\lambda = D\phi_\lambda \in C^\infty(M, S^-)$.

Then $$\tilde{\Delta}\rho_\lambda = (DD^*)D\phi_\lambda = D\Delta\phi_\lambda = \lambda D\phi_\lambda = \lambda\rho_\lambda.$$

Since this is so for a basis of e-sections $\{\rho_\lambda\}$ for $L^2(M,E)$, and since the argument is symmetric $\tilde{\Delta} \rightsquigarrow \Delta$, the result follows. ▨

COROLLARY 2:

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) - \mathrm{Tr}(e^{-t\tilde{\Delta}}) = \dim\operatorname{Ker}\Delta - \dim\operatorname{Ker}\tilde{\Delta}} \qquad (8)$$

PROOF: $$\text{LHS } (8) = \sum e^{-t\lambda} - \sum e^{-t\tilde{\lambda}}$$

$$\overset{\text{by LEMMA 1}}{=} \sum_{\lambda=0} e^{-t\lambda} - \sum_{\tilde{\lambda}=0} e^{-t\tilde{\lambda}}$$ } sum is over multiplicity of 0-evalue.

$$\overset{\text{defn}}{=} \dim\operatorname{Ker}\Delta - \dim\operatorname{Ker}\tilde{\Delta}.$$ ▨

LEMMA 3: $\boxed{\begin{array}{l} \ker \Delta = \ker D \\ \ker \tilde{\Delta} = \ker D^* \end{array}} \begin{array}{l} \subset C^\infty(M, E^+) \\ \subset C^\infty(M, E^-) \end{array}$

& know kernel consists of *smooth* sections.

PROOF: Clearly, $\ker D \subset \ker \Delta$.

Suppose $\psi \in \ker \Delta$, then $0 = \Delta\psi = D^*D\psi$.

So
$$0 = \langle \Delta\psi, \psi \rangle = \langle D^*D\psi, \psi \rangle = \langle D\psi, D\psi \rangle = \|D\psi\|^2.$$

L^2 inner product

$$\Rightarrow D\psi = 0.$$

So $\psi \in \ker D$. Hence $\ker \Delta = \ker D$. ∎

Thus, Corollary 2 + LEMMA 3 $\Rightarrow$ (3) (McKean-Singer)

But there is a more delicate analytic view point on the McKean-Singer formula which leads to a proof of the AS-index formula (2).

For this we use the "supertrace"

(4) $$\mathrm{Tr}_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{Tr}(a) - \mathrm{Tr}(d)$$

of a trace-class operator $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{H}^+ \oplus \mathcal{H}^- \to \mathcal{H}^+ \oplus \mathcal{H}^-$

Write

$$\mathbb{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} : C^\infty(M, E^+ \oplus E^-) \longrightarrow C^\infty(M, E^+ \oplus E^-)$$

and

$$\boxed{\Delta = \mathbb{D}^2 = \begin{pmatrix} \Delta^+ & 0 \\ 0 & \Delta^- \end{pmatrix}}$$

Since (super) traces vanish on (super) commutators

Then we shall see that

$$\boxed{\frac{d}{dt} \mathrm{Tr}_s\left(e^{-t\Delta}\right) = -\frac{1}{2} \mathrm{Tr}_s\left([\mathbb{D}, \mathbb{D}e^{-t\Delta}]\right) = 0.}$$

Assuming this exists!

and hence that

(10) $$\underbrace{\lim_{t\to\infty} \mathrm{Tr}_s\left(e^{-t\Delta}\right)}_{= \mathrm{ind}(D)} = \underbrace{\lim_{t\to 0} \mathrm{Tr}_s\left(e^{-t\Delta}\right)}_{= \int_M \underbrace{\text{index density}}_{= \hat{A}(M)\, ch(V)}}$$

for coupled Dirac.

which is the A-S index formula!

This side holds just as above

This side is where we have to do some computation

The key fact in computing the RHS of (10) is that there is an asymptotic expansion

$$\boxed{\operatorname{Tr}\left(e^{-t\Delta}\right) = h_{-\frac{n}{2}}\, t^{-\frac{n}{2}} + \dots + h_0 t^0 + h_{\frac{1}{2}}\, t^{\frac{1}{2}} + \dots}$$

($n = \dim M$)

as $t \to 0+$

and likewise

$$\boxed{\operatorname{Tr}\left(e^{-t\tilde{\Delta}}\right) = \tilde{h}_{-\frac{n}{2}}\, t^{-\frac{n}{2}} + \dots + \tilde{h}_0 t^0 + \tilde{h}_{\frac{1}{2}}\, t^{\frac{1}{2}} + \dots}$$

with each "heat trace coefficient" explicitly computable from the geometric-analytic data

— metrics, connections, curvature etc.

Moreover, we will see that

$$\boxed{h_{-n/2} = \tilde{h}_{-n/2}}, \quad \dots, \quad \boxed{h_{-1} = \tilde{h}_{-1}}, \quad \boxed{h_{-1/2} = \tilde{h}_{-1/2}}$$

So that

$$\boxed{\underbrace{\lim_{t \to 0} \operatorname{Tr}_s\left(e^{-t\Delta}\right)}_{\text{exists}} = h_0 - \tilde{h}_0}$$

For example, we'll show very shortly that

$$\boxed{h_{-n/2} = \operatorname{vol}_g(M) \cdot c} \qquad c \text{ a universal const}$$

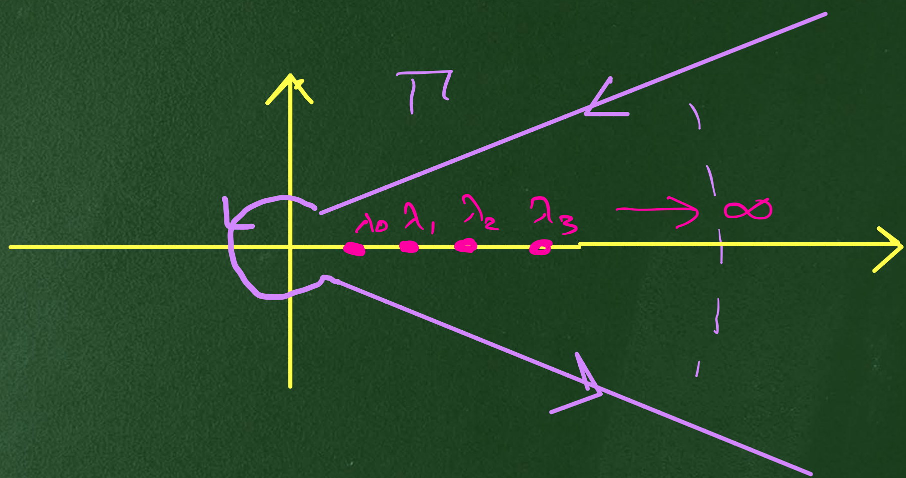
To do the computations, we use the construction

$$(11) \qquad e^{-t\Delta} = \int_\Gamma e^{-t\lambda}(\Delta-\lambda)^{-1}\, đ\lambda \qquad đ\lambda = \frac{i}{2\pi}\,d\lambda$$

of the heat operator as a contour integral

cf. Cauchy Thm $\quad f(z) = \int_\Gamma f(\lambda)(\lambda-z)^{-1}\, đ\lambda$

$$\left(e^{-t\Delta}\psi\right)(x) = \int_\Gamma e^{-t\lambda}\left((\Delta-\lambda)^{-1}\psi\right)(x)\, đ\lambda$$



The key, in view of (11), to all that follows is the resolvent operator

$$(\Delta-\lambda)^{-1} : H^s(M,E^+) \longrightarrow H^{s-2}(M,E^+)$$

Sobolev sections.

$$(\Delta-\lambda)^{-1}\psi(x) = \int_M \underbrace{K_\Delta(x,y,\lambda)}_{\text{Resolvent kernel.}}\, \psi(y)\, dy_M$$

ΨDO

$$\boxed{K_\Delta(x,y,\lambda) = \int_{\mathbb{R}^{n=\dim M}} e^{i\xi(x-y)} \underbrace{r(x,y,\xi,\lambda)}_{\text{symbol amplitude.}}\, d\xi.}$$

- smooth for $x \neq y$, singular along diag. $x = y$.

- Can replace $r(x,y,\xi,\lambda)$ by $r(x,\xi,\lambda)$ and

$$r(x,\xi,\lambda) \sim \sum_{j\geq 0} r_{-2-j}(x,\xi,\lambda)$$

with

$$r_{-2-j}(x,t\xi,t^2\lambda) = t^{-2-j}\, r_{-2-j}(x,\xi,\lambda)$$

What makes the index a "local invariant" is that we can compute it from just finitely many of the r_{-2-j}.

Next time we will see how this works with a proof of (*) on a Riemann surface — the Hirzebruch-Riemann-Roch theorem for a surface.