LECTURE 2

We saw last time that, in principle,

$$\boxed{\mathrm{ind}\, D = \mathrm{Tr}(e^{-t\Delta}) - \mathrm{Tr}(e^{-t\hat{\Delta}}) \quad \forall t>0} \qquad (1)$$

order 1 elliptic differential

where $\Delta = D^*D$, $\Delta = DD^*$ and $D : C^\infty(X,E^+) \to C^\infty(X,E^-)$

Aim: show

$$\boxed{\lim_{t\to 0} \text{RHS of } (1) = \int_X \underbrace{\hat{A}(X)\,\mathrm{ch}(E)}_{\text{index density}}}$$

For this need to compute heat trace asymptotics

$$\mathrm{Tr}(e^{-t\Delta}) \sim h_{-\frac{n}{2}} t^{-\frac{n}{2}} + h_{\frac{-n+1}{2}} t^{\frac{-n+1}{2}} + \dots + h_0 t^0 + h_{\frac{1}{2}} t^{\frac{1}{2}} + \dots \quad \text{as } t \to 0 \qquad (2)$$

$$h_{\frac{-n+j}{2}} \in \mathbb{R}$$

$n = 2m = \dim X$, $\boxed{h_{\frac{-2m+2k+1}{2}} = 0 \quad \text{(we will see why!)}}$

hence only integer powers

$$(3) \qquad = h_{-m} t^{-m} + h_{-m+1} t^{-m+1} + \dots + h_0 t^0 + \dots$$

Likewise,

$$\mathrm{Tr}(e^{-t\tilde{\Delta}}) \sim \tilde{h}_{-\frac{n}{2}} t^{-\frac{n}{2}} + \tilde{h}_{-\frac{n+1}{2}} t^{-\frac{n+1}{2}} + \dots + \tilde{h}_0 t^0 + \tilde{h}_{\frac{1}{2}} t^{\frac{1}{2}} + \dots$$

as $t \to 0$

some $\tilde{h}_{-\frac{n+j}{2}} \in \mathbb{R}$

$n = 2m = \dim X$, $h_{\frac{-2m+(2k+1)}{2}} = 0$ (will see)

so

$$\textcircled{4} \quad = \tilde{h}_{-m} t^{-m} + \tilde{h}_{-m+1} t^{-m+1} + \dots + \tilde{h}_0 t^0 + \dots$$

Then will see: $\boxed{h_j = \tilde{h}_j}$ $\boxed{\text{for } j = -m, \dots, -1.}$

'Thus':

$$\textcircled{5} \quad \boxed{\mathrm{ind}\, D = h_0 - \tilde{h}_0} = \lim_{t \to 0} \text{RHS of } \textcircled{1}$$

∴ this limit exists

So how do we get these numbers

$$\textcircled{6} \quad \boxed{h_{\frac{-n+j}{2}} = \int_M h_{\frac{-n+j}{2}}(x)\, |dx|}\ ?$$

'Lebesgue measure'

First, recall that

$$⑦ \rightarrow \boxed{e^{-t\Delta} := \int_\Gamma e^{-\lambda} \underbrace{(t\Delta - \lambda)^{-1}}_{\text{'resolvent'}} \, d\lambda}$$

$$d\lambda = \frac{i}{2\pi} d\lambda$$

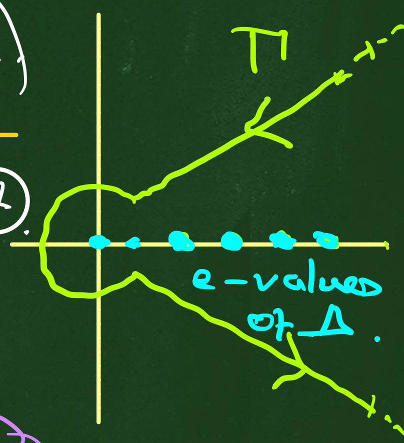
$$f(w) = \int_\gamma \frac{f(\lambda)}{w-\lambda} d\lambda$$

cf. Cauchy integral formula

or

$$\boxed{e^{-t\Delta} = \int_\Gamma e^{-t\lambda} (\Delta - \lambda)^{-1} \, d\lambda}$$

(by $\lambda \longrightarrow t\lambda$ change of variable)



Need to extract the heat kernel from ⑦

$$H_t(x,y) \longleftarrow C^\infty \text{ in } (x,y)$$

$$e^{-t\Delta} \psi(x) = \int_X H_t(x,y) \psi(y) \, dy$$

⑧ since

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) = \int_X \mathrm{tr}\big(\underbrace{H_t(x,x)}_{\text{generally } k\times k \text{ matrix for each } x.}\big) \, |dx|}$$

usual matrix trace

We do that via the kernel $R(x,y,\lambda)$ of the resolvent $(\Delta - \lambda I)^{-1}$ — then use the classical symbol calculus to compute ②/③.

The (resolvent) kernel of $(\Delta-\lambda)^{-1}$ given by

$$\boxed{(\Delta-\lambda)^{-1}\psi(x) \overset{\text{distributionally}}{=} \int_X \underbrace{R(x,y,\lambda)}_{\text{distributionally}}\psi(y)|dy| \qquad (8)}$$

From (7) and (8)

$$\left(e^{-t\Delta}\psi\right)(x) := \int_\Gamma e^{-t\lambda}(\Delta-\lambda)^{-1}\psi(x)\,d\lambda$$

$$= \int_\Gamma e^{-t\lambda}\int_X R(x,y,\lambda)\,\psi(y)|dy|\,d\lambda$$

$$= \int_X \underbrace{\boxed{\int_\Gamma e^{-t\lambda}R(x,y,\lambda)\,d\lambda}}_{=:H_t(x,y)}\psi(y)|dy|$$

Comment:

"GLOBAL"

Only way to compute exactly is via Selberg 'trace theory

- Π discrete group acting on (simple) M

$$\boxed{\begin{gathered} X = M/\Pi \\ \text{e.g. } S^1 = \mathbb{R}^1/2\pi\mathbb{Z} \\ \int_X = \sum_{\gamma\in\Pi}\int_M \end{gathered}}$$

"LOCAL"

But on any closed Riemannian manifold can compute 'pieces' of the heat trace using ψDO symbol calculus.

So looks like

$$\mathrm{Tr}(e^{-t\Delta}) = \int_X \mathrm{tr}\left(\int_\Gamma e^{-t\lambda} \underbrace{R(x,y,\lambda)}_{\substack{\text{singular along} \\ x=y}} d\lambda\right)\Bigg|_{y=x} |dx|$$

this is C^∞ along $x=y$.

Why? $\rightarrow \int e^{-t\lambda}(\Delta-\lambda)^{-1} d\lambda = \int_\Gamma t^k e^{-t\lambda} \underbrace{\partial_\lambda^k (\Delta-\lambda)^{-1}}_{k!(\Delta-\lambda)^{-k-1}} d\lambda$

$k!(\Delta-\lambda)^{-k-1}$: $\underbrace{H^s(X,E^+)}_{\text{sobolev.}} \to H^{s+2k+2}(X,E^+)$

independent of k

$\Rightarrow$ range in $\bigcap_t H^t(X,E^+) = C^\infty(X,E^+)$.

Key step:

Replace $R(x,y,\lambda)$ by approximation

$$R(x,y,\lambda) \sim \sum_{j\geq 0} \underbrace{R_{-2-j}(x,y,\lambda)}$$

these become smoother as $j \to \infty$

$n = \dim X$

Because Fourier Transform takes distributions homog deg $-k$ to ones homog deg $-n+k$

each $R_{-2-j}(x,y,\lambda=0)$ is homogeneous in $x-y$ of order $-n+2+j$, and so smoother with increasing j.

We get the components R_{-2-j} from the symbol calculus:

$$R(x,y,\lambda) = \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} \underbrace{r(x,y,\xi,\lambda)}_{\text{symbol of } (\Delta-\lambda)^{-1}} \, d\xi_{\mathbb{R}^n}$$

$$= \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} \sum_{j\geq 0} r_{-2-j}(x,\xi,\lambda)\, d\xi_{\mathbb{R}^n}$$

"Error" $\longrightarrow$ $+\; E_{-\infty}(x,y)$

$\leftarrow$ smoothing operator more with $\mathrm{Tr}(E_{-\infty})=0$

"Thus"

$$\boxed{R(x,y,\lambda) \sim \sum_{j\geq 0} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)}\, \underline{\underline{r_{-2-j}(x,\xi,\lambda)\, d\xi}}}$$

The $r_{-2-j}(x,\xi,\lambda)$ are computed recursively as the quasi-homogeneous components of the symbol of $(\Delta-\lambda)^{-1}$, as follows. Their computation depends on having explicit formulae for Δ in local coordinates on X:

Write locally (in local coordinates $(x_1,\dots,x_n)$)

$$\Delta = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum b_j(x)\frac{\partial}{\partial x_j} + c(x)$$

for us

$$= -\frac{1}{\sqrt{|g|}}\sum_{k,l} \partial_k \left(g^{kl}\sqrt{|g|}\right)\partial_l$$

+ first-order + 0-order terms

$$= \boxed{\sum_{k,l=1}^{n} - g^{kl}(x)\partial_k \partial_l} + \text{first-order} + \text{0-order}$$

$$\left(g^{kl}(x)\right)^{-1} = \left(g_{jl}(x)\right).$$

$|g| = \det(g_{ij}(x))$

$g(x) = (g_{ij}(x))$

Riemannian matrix

$\sum g_{ij}(x)\, dx_i \otimes dx_j$

Obtain $a(x,\xi) = \sigma(\Delta)(x,\xi)$ by

$$-i\partial_k \longleftrightarrow \xi_k$$

$a_1(x,\xi)$

Δ ⊙ 9

$$a(x,\xi) = \overset{a_2(x,\xi)}{|\xi|_g^2} + \sum_k \tilde{b}_k(x)\,\xi_k + c(x) \quad -\lambda \text{ if } \Delta - \lambda$$

$$|\xi|_g^2 = \sum_{i,j=1}^{n} g^{ij}(x)\,\xi_i \xi_j$$

$\tilde{b}_i = i b_j$

Then the $r_{-2-j}(x,\xi,\lambda)$ are given by the recursive formulae:

$$r_{-2} = \left(a_2(x,\xi) - \lambda\right)^{-1} \qquad 8-k-|\mu| + -8-2 \qquad (10)$$

$$r_{-2-j} = r_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!}\, \partial_\xi^\mu a_{2-k}\, D_x^\mu r_{-2-l}$$

$\mu = (\mu_1, \ldots, \mu_n) \qquad \mu_j \in \mathbb{N} = \{0, 1, 2, \ldots\}$

$\dfrac{1}{\mu!} = \dfrac{1}{\mu_1! \cdots \mu_n!} \qquad \partial_\xi^\mu = \partial_{\xi_1}^{\mu_1} \cdots \partial_{\xi_n}^{\mu_n}, \quad D_{x_j} = -i\partial_{x_j}$

If A and B and ΨDOs with symbols $a(x,\xi)$, $b(x,\xi)$, then the composition AB has symbol

$$a \circ b\,(x,\xi) = \sum_\mu \frac{1}{\mu!}\, \partial_\xi^\mu a(x,\xi)\, D_x^\mu b(x,\xi)$$

If $AB = I$ modulo smoothing operators (i.e. B is a 'parametrix' for A), then $a \circ b\,(x,\xi) = 1$ (identity)

Applying this to $a(x,\xi) = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi) - \lambda$

and $b(x,\xi) := r(x,\xi,\lambda) \sim \sum_{j\geq 0} \underbrace{r_{-2-j}(x,\xi,\lambda)}_{\text{quasi-homogeneous}}$

($a_j(x,\xi)$ – homog deg j in ξ)

Applying this easily yields the formulae (10)

– we will go through this next time!

At any rate substituting in the above we end up with formulae for the asymptotic expansion of the heat kernel: one has

$$H_t(x,y) = \int_{\Gamma} e^{-t\lambda} R(x,y,\lambda)\, d\lambda \qquad d\!\!{}^-\xi = \frac{d\xi}{(2\pi)^n}$$

$$\sim \int_{\Gamma} e^{-t\lambda} \sum_{j\geq 0} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)}\, r_{-2-j}(x,\xi,\lambda)\, d\lambda\, d\!\!{}^-\xi$$

So by change of variable and quasi-homogeneity of r_{-2-j} one has

$$\boxed{H_t(x,x) \sim \sum_{j\geq 0} \underbrace{\int_{\mathbb{R}^n}\int_{\Gamma} e^{-t\lambda}\, r_{-2-j}(x,\xi,\lambda)\, d\lambda\, d\!\!{}^-\xi}_{h_{-\frac{n+j}{2}}(x)}\; t^{-\frac{n+j}{2}}}$$

Giving us an explicit formula for computing

Next time we will get on with computing these terms, and hence the index formula!