LTCC Atiyah–Singer Index – Lecture 3

Recall,

$$\mathrm{ind}(D) = \underset{<\infty}{\dim \mathrm{Ker}(D)} - \underset{<\infty}{\dim(\mathrm{Ker}\, D^*)} \in \mathbb{Z}$$

$< \infty \leftarrow$ FREDHOLM.

$\dim V = \mathrm{Tr}(I_V)$

$$= \mathrm{Tr}(I_{\mathrm{Ker}\, D}) - \mathrm{Tr}(I_{\mathrm{Ker}\, D^*})$$

Identity operator on v. space Ker D.

$\Delta = D^* D$

$\tilde{\Delta} = D D^*$

See Lect 1

$$= \mathrm{Tr}(I_{\mathrm{Ker}\,\Delta}) - \mathrm{Tr}(I_{\mathrm{Ker}\,\tilde{\Delta}})$$

$D : C^\infty(X, E^+) \to C^\infty(X, E^-)$

$\Delta : C^\infty(X, E^+) \circlearrowleft$

$\tilde{\Delta} : C^\infty(X, E^-) \circlearrowleft$

since $\mathrm{spec}\{\Delta\} \setminus 0 = \mathrm{spec}(\tilde{\Delta}) \setminus \{0\}$.

$$= \sum_{\lambda \in \mathrm{spec}(\Delta)} e^{-t\lambda} - \sum_{\mu \in \mathrm{spec}(\hat{\Delta})} e^{-t\mu}$$

Lidskii's thm

$$= \mathrm{Tr}(e^{-t\Delta}) - \mathrm{Tr}(e^{-t\hat{\Delta}})$$

$$= \int_X \mathrm{tr}\big(\underbrace{H_t(x,x)}_{\text{heat kernel for } \Delta}\big)\,|dx| - \int_X \mathrm{tr}(\tilde{H}_t(x,x))\,|dx|$$

heat for $\hat{\Delta}$

$$\rightsquigarrow \quad \text{"} = \int_X \hat{A}(X)\, ch(E^D) \text{"} \leftarrow \text{topological index}$$

e.g. Euler numbers, Signature, $\hat{A}$-genus, etc all arise this way.

First task today is to explain how the asymptotic expansion as $t \to 0+$

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) \sim h_{-\frac{n}{2}} t^{-\frac{n}{2}} + h_{-\frac{n+1}{2}} t^{-\frac{n+1}{2}} + \dots + h_0 t^0 + h_{\frac{1}{2}} t^{\frac{1}{2}} + \dots} \qquad (*)$$

mentioned in previous lectures arises, where

$$\boxed{h_{\frac{-n+j}{2}} = \int_M h_{\frac{-n+j}{2}}(x)\,|dx| \in \mathbb{R}} \qquad (*)'$$

To derive $(*)$ recall that

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) = \int_M H_t(x,x)\,|dx|} \qquad (**)$$

where $H_t(x,y)$ is the heat kernel — a smooth (in (x,y)) section in $C^\infty(M \times M, \mathrm{End}(E^+))$, where $\Delta = D^*D : C^\infty(M, E^+) \to C^\infty(M, E^+)$ and

$$\boxed{(e^{-t\Delta}\psi)(x) = \int_M \underbrace{H_t(x,y)}_{\text{Schwartz kernel of } e^{-t\Delta}} \psi(y)\,|dy|}$$

Next, recall from Lect 2 that:

$$= \int_{\mathbb{R}^n} e^{i\xi.(x-y)} \underbrace{r(x,\xi,\lambda)}_{\sim \sum r_{-2-j}(x,\xi,\lambda)} d\xi$$

$$H_t(x,y) = \int_{\Gamma} e^{-t\lambda} \overbrace{R(x,y,\lambda)} d\lambda\, d\xi$$

$$① \sim \int_{\Gamma} e^{-t\lambda} \sum_{j\geq 0} \int_{\mathbb{R}^n} e^{i\xi.(x-y)} r_{-2-j}(x,\xi,\lambda)\, d\lambda\, d\xi$$

Crucially the r_{-2-j} are quasi-homogeneous:

$$② \quad \boxed{r_{-2-j}(x, s\xi, s^2\lambda) = s^{-2-j} r_{-2-j}(x,\xi,\lambda)}$$

See inductively that ② holds from the recursive formulae:

$$r_{-2}(x,\xi,\lambda) = (|\xi|_g^2 - \lambda)^{-1}$$

$$r_{-2-j} = r_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l}$$

So ① $\Rightarrow$ $\qquad n = \dim X$

$$H_t(x,x) \sim \sum_{j\geq 0} \int_{\mathbb{R}^n} \int_{\Gamma} e^{-t\lambda} r_{-2-j}(x,\xi,\lambda)\, d\lambda\, d\xi.$$

Change variable $\mu = t\lambda$ to obtain

$$H_t(x,x) \sim \sum_{j\geq 0} \int_{\mathbb{R}^n} \int_{\Gamma} e^{-\mu} r_{-2-j}\left(x,\xi,\frac{\mu}{t}\right) \frac{d\mu}{t}\, d\xi.$$

To use (2), take $s = t^{-\frac{1}{2}}$ and so change variable $t^{-\frac{1}{2}}\eta = \xi$ to give

$$t^{-\frac{n}{2}}\,d\eta = d\xi \qquad \xi_i = t^{-\frac{1}{2}}\eta_i$$

(3)
$$H_t(x,x) \sim \sum_{j\geq 0} \int_{\mathbb{R}^n}\int_{\Gamma} e^{-\mu}\, \underbrace{r_{-2-j}\left(x, \frac{\eta}{t^{\frac{1}{2}}}, \frac{\mu}{t}\right)}_{\overset{(2)}{=}\; t^{+1+\frac{j}{2}}\, r_{-2-j}(x,\eta,\mu)} \frac{\text{đ}\mu}{t}\, \frac{\text{đ}\eta}{t^{\frac{n}{2}}}$$

$$= \sum_{j\geq 0} \underbrace{\left(\int_{\mathbb{R}^n}\int_{\Gamma} e^{-\mu}\, r_{-2-j}(x,\eta,\mu)\, \text{đ}\mu\, \text{đ}\eta\right)}_{h_{\frac{-n+j}{2}}(x)} t^{\frac{-n+j}{2}}$$

Thus, the task at hand is to compute (in some sense - in terms of geometric invariants, such as Riemmanian curvature tensors (Ricci, scalar, Riemman, etc), or, perhaps, topological invariants in the form of characteristic classes in $H^*(M,\mathbb{Z})$.

Specifically, the Atiyah-Singer index formula is now

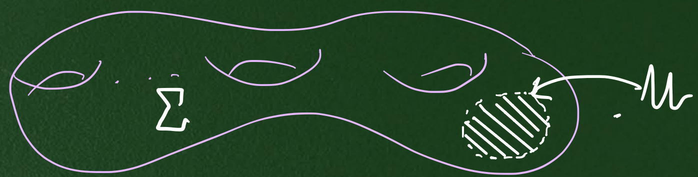
(4)
$$\boxed{\begin{gathered}\overbrace{\int_M \int_{\mathbb{R}^n}\int_{\Gamma} e^{-\lambda}\, r_{-2-n}(x,\xi,\lambda)\, \text{đ}\lambda\, \text{đ}\lambda\, |dx|}^{h_0 = \int_M h_0(x)\,|dx|} - \overbrace{\int_M \int_{\mathbb{R}^n}\int_{\Gamma} e^{-\lambda}\, \tilde{r}_{-2-n}(x,\xi,\lambda)\, \text{đ}\lambda\, \text{đ}\lambda\, |dx|}^{\tilde{h}_0 = \int_M \tilde{h}_0(x)\,|dx|} \\ = \int_M \omega_{ind} = \text{index}(D)\end{gathered}}$$

index density

$\hat{A}(M)\, ch(E)$

some v. bundle $E \to$

So we are going to carry out these computations on a Riemann surface, proving the Riemann-Roch-Hirzebruch theorem (RRH) on a surface Σ (complex curve) of genus g:



Assume an Hermitian metric h on Σ, given on a coordinate patch $U \subset \Sigma$ by

(5) $$\boxed{h(z,\bar{z})\, dz \otimes d\bar{z}}$$

IN THE FOLLOWING, THE FORMULAE IN YELLOW BOXES ARE EXERCISES FOR LTCC STUDENTS.

and induced volume element

(6) $$\boxed{d_h\Sigma = \frac{i}{2}\, h(z,\bar{z})\, dz \wedge d\bar{z}.}$$

Let $V \to \Sigma$ be a holomorphic vector bundle of rank n, given by holomorphic transition functions

(7) $$\boxed{g_{ij} : U_i \cap U_j \longrightarrow GL(n;\mathbb{C})}.$$

We shall assume that V is endowed with an Hermitian metric E, specified relative to the covering $\{U_j\}$ of Σ by a system $E_j : U_j \longrightarrow GL(n;\mathbb{C})$ of Hermitian matrices satisfying the compatibility condition (to define a global metric on $\mathcal{V}$)

(8) $$\boxed{g_{ij}^{*} E_i \, g_{ij} = E_j \quad \text{on} \quad \mathcal{U}_i \cap \mathcal{U}_j}.$$

This induces a Hermitian structure on the 'determinant line bundle' $\operatorname{Det} E := \Lambda^N E$, with $N = \operatorname{rank}(E)$, with correspondition transition rule

(9) $$\boxed{\det E_i \cdot |\det g_{ij}|^2 = \det E_j \quad \text{on} \quad \mathcal{U}_i \cap \mathcal{U}_j}.$$

Since the first Chern class $c_1(E)$ of E coincides with the first Chern class $c_1(\operatorname{Det} E)$ of $\operatorname{Det}(E)$, we have that $c_1(E)$ is represented as a differential form by

(10) $$\boxed{\underbrace{\bar\partial\partial \log \det E}_{:= \bar\partial\partial \log \det E_i \text{ on } \mathcal{U}_i} \in \Lambda^{1,1} T^{*}_{\mathbb{C}} \Sigma}$$

The first Chern number of $\mathcal{V}$ is then

(11) $$\boxed{c_1(\mathcal{V}) := \frac{1}{2\pi i} \int_{\Sigma} \bar\partial\partial \log \det E \in \mathbb{Z}}.$$

More classical terminology

(12)
$$\boxed{c_1(\mathcal{V}) = \deg(\mathcal{V})}$$

is the degree of $\mathcal{V}$ (or $\operatorname{Det}\mathcal{V}$).

Likewise, the differential form

(13)
$$\boxed{\bar\partial\partial \log h \in \Lambda^{1,1} T^*_{\mathbb{C}}\mathbb{C}}$$

computes the Euler characteristic

(14)
$$\boxed{c_1(\Sigma) = \frac{1}{2\pi i}\int_\Sigma \partial\bar\partial \log h = 2-2g = \chi(\Sigma)}$$

as the degree of the holomorphic tangent bundle.

The classical RRH formula is the identity

(15)
$$\boxed{h(\Sigma,\mathcal{V}) - h(\Sigma,\tilde{\mathcal{V}}) = \deg(\mathcal{V}) + \frac{\operatorname{rk}(\mathcal{V})\,\chi(\Sigma)}{2}}$$

where

$$h(\Sigma,\mathcal{E}) = \dim_{\mathbb{C}}\left(H^0(\Sigma,\mathcal{E})\right),$$

$H^0(\Sigma,\mathcal{E}) = 0^{th}$ degree Dolbeault cohomology group, for $\mathcal{E}\to\Sigma$ a holomorphic vector bundle

$$\operatorname{rk}(\mathcal{V}) = \text{rank of } \mathcal{V},$$

$$\tilde{\mathcal{V}} := \mathcal{V}^* \otimes T^{1,0}_{\mathbb{C}}\mathcal{E}$$

which we prove by observing that the left-hand side of (15) is the index of the "$\bar{\partial}$-operator"

(16)
$$\boxed{\begin{array}{ccc} \bar{\partial} = \bar{\partial}^{\mathcal{V}} : \Omega^{0}(\Sigma, \mathcal{V}) & \longrightarrow & \Omega^{0,1}(\Sigma, \mathcal{V}) \\ \| & & \| \\ \underset{\text{smooth sections of } \mathcal{V}.}{C^{\infty}(\Sigma, \mathcal{V})} & & C^{\infty}(\Sigma, \mathcal{V} \otimes T^{1,0}_{\mathbb{C}}\mathbb{C}.) \end{array}}$$

associated to the Hermitian structure on Σ and $\mathcal{V}$.

LEMMA 1 :

(17)
$$\boxed{\begin{array}{l} \operatorname{Ker} \bar{\partial} \cong H^{0}(\Sigma, \mathcal{V}) \\ \operatorname{Ker}(\bar{\partial}^{*}) \cong H^{0}(\Sigma, \tilde{\mathcal{V}}) \end{array}}$$

Note, by Serre duality
$\Omega^{0,1}(\Sigma, \mathcal{V}) \cong C^{\infty}(\Sigma, \mathcal{V}^{*} \otimes T^{1,0}_{\mathbb{C}}\Sigma)$

And hence

(18)
$$\boxed{\operatorname{index} \bar{\partial} = h^{0}(\Sigma, \mathcal{V}) - h^{0}(\Sigma, \tilde{\mathcal{V}})}$$

The general RRH formula for a holomorphic vector bundle $W \to M$ over a Kahler manifold M states, as an index formula:

(19)
$$\underbrace{\operatorname{index}\left(\bar{\partial}^{W} + (\bar{\partial}^{W})^{*}\right)}_{= \chi(M, W)} = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{M} Td(M)\, ch(W)$$

$n = \dim_{\mathbb{R}} M$

hol. Euler characteristic $= \sum_{0}^{n/2} (-1)^{r} \dim H^{0,r}(M, W)$.

for

(20)
$$\bar{\partial}^{W} + (\bar{\partial}^{W})^{*} : \bigoplus_{p \geq 0} \Omega^{0,2p}(M, W) \longrightarrow \bigoplus_{q \geq 0} \Omega^{0,2q+1}(M, W),$$

and

$$\mathrm{Td}(M,R) = \det\left(\frac{R}{e^{R}-1}\right) \qquad R \in \Omega^{1,1}(M, \mathrm{End}(T^{1,0}M))$$

curvature of $T^{1,0}_{\mathbb{C}}M$.

$$= 1 + \mathrm{Td}_1(M) + \cdots + \mathrm{Td}_{\frac{n}{2}}(M) \qquad \mathrm{Td}_k(M) \in \Omega^{k,k}(M).$$

is the Todd class, and

(21) $$\mathrm{ch}(W,F) = \mathrm{tr}(e^{F}) = \underbrace{\overset{\mathrm{rank}(W)}{N}}_{\mathrm{ch}_0(W,F)} + \underbrace{\mathrm{tr}(F)}_{\mathrm{ch}_1(W,F)} + \frac{1}{2!}\underbrace{\mathrm{tr}(F^2)}_{\mathrm{ch}_2(W,F)},$$

$\mathrm{ch}_1(W,F) = c_1(W,F)$.

The general RRH then states:

(22) $$\mathrm{index}(\bar\partial + \bar\partial^*) = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_M \mathrm{Td}(M,R)\,\mathrm{ch}(W,F).$$

For a surface $N=2$, the Dirac operator

(23) $$\bar\partial + \bar\partial^* = \bar\partial \quad \text{on } \Omega^0(\Sigma, \mathcal{E})$$

(24) $$\mathrm{Td}(\Sigma,R) = 1 - \frac{R}{2}, \qquad \mathrm{ch}(V,F) = \underset{\mathrm{rank}(V)}{N} - \mathrm{tr}(F)$$

and so (22) reduces to

(25) $$\mathrm{ind}(\bar\partial) = \int_\Sigma \left(-\frac{1}{2\pi i}\mathrm{tr}(F) - \frac{N}{2}\left(\frac{1}{2\pi i} R\right)\right)$$

That is, since from 10 & 13

$$R = \bar{\partial}\partial \log h \qquad \text{and} \qquad F = \bar{\partial}\partial \log \det(E),$$

that,

$$\boxed{\operatorname{ind}\bar{\partial} = \frac{1}{2\pi i}\int_{\Sigma} \bar{\partial}\partial \log \det(E) + \frac{N}{2\pi i}\int_{\Sigma} \bar{\partial}\partial \log h} \tag{26}$$

This is the formula we shall prove here, using the identity (4) for the index, i.e. that

$$\int_{\Sigma}\int_{\mathbb{R}^n}\int_{\Gamma} e^{-\lambda}\, r_{-4}(x,\xi,\lambda)\, d\lambda\, d\lambda\, |dx| - \int_{\Sigma}\int_{\mathbb{R}^n}\int_{\Gamma} e^{-\lambda}\, \tilde{r}_{-4}(x,\xi,\lambda)\, d\lambda\, d\lambda\, |dx|$$

$$= \frac{1}{2\pi i}\int_{\Sigma} \bar{\partial}\partial \log \det(E) + \frac{N}{2\pi i}\int_{\Sigma} \bar{\partial}\partial \log h. \tag{27}$$

Note in this case that the heat trace expansions (*) $\nearrow$ p.2
'reduce' to

$$\operatorname{Tr}(e^{-t\Delta}) \sim h_{-1}t^{-1} + h_{-\frac{1}{2}}t^{-\frac{1}{2}} + h_0 t^0 + \dots$$

and

$$\operatorname{Tr}(e^{-t\tilde{\Delta}}) \sim \tilde{h}_{-1}t^{-1} + \tilde{h}_{-\frac{1}{2}}t^{-\frac{1}{2}} + \tilde{h}_0 t^0 + \dots$$

where

$$\Delta = \bar{\partial}^* \bar{\partial} : \Omega^0(\Sigma, \mathcal{V}) \to \Omega^0(\Sigma, \mathcal{V})$$

$$\tilde{\Delta} = \bar{\partial} \bar{\partial}^* : \Omega^0(\Sigma, \mathcal{V}^* \otimes T^{1,0} \mathcal{V}) \circlearrowleft ,$$

and – we will show –

(28)
$$\boxed{\begin{aligned} h_{-1} &= \tilde{h}_{-1} = \frac{N}{\pi} \mathrm{Vol}_h(M) \\ N = \mathrm{rk}(\mathcal{V}), \quad &\mathrm{Vol}_h(M) := \int_\Sigma 1 \, d_h\Sigma \end{aligned}}$$

see (6)

= surface area of Σ wrt h

(29)
$$\boxed{h_{-1/2} = \tilde{h}_{-1/2} = 0}$$

(30)
$$\boxed{h_0 = \frac{1}{4\pi i} \int_\Sigma \bar{\partial} \partial \log \det(E) + \frac{N}{12\pi i} \int_\Sigma \bar{\partial} \partial \log h}$$

(31)
$$\boxed{\tilde{h}_0 = \frac{-1}{4\pi i} \int_\Sigma \bar{\partial} \partial \log \det(E) - \frac{N}{6\pi i} \int_\Sigma \bar{\partial} \partial \log h}$$

and hence, from (4), (30), (31), that

$$\boxed{\mathrm{index}(\bar\partial) = h_0 - \tilde h_0 = \text{RHS of } (27).} \qquad (32)$$

So, concretely, to prove the RRH formula on a surface Σ we are going to compute (we hope!) that (via (27)):

$$\int_{\mathbb{R}^n}\int_{\Gamma} e^{-\lambda}\, r_{-4}(x,\xi,\lambda)\, d\lambda\, d\xi\, |dx| \quad \leftarrow h_0 = \text{LHS}$$

$$= \frac{1}{4\pi i}\bar\partial\partial \log \det(E) + \frac{N}{12\pi i}\bar\partial\partial \log h, \qquad (33)$$

and a similar formula for $\tilde h_0$.

So that's what we need to do, and we will do that in the next lecture.