LTCC LECTURE 4

From last time, in order to prove the Riemann Roch Hirzebruch theorem (RRH) on a closed Riemann surface Σ of genus g, endowed with a holomorphic vector bundle $V \to \Sigma$, we have to show that:

$$\int_{\mathbb{R}^n}\int_{\Gamma} e^{-\lambda} r_{-4}(x,\xi,\lambda)\, d\lambda\, d\xi\, |dx| \quad \leftarrow h_0 = \text{LHS}$$

$$(1) \qquad = \frac{1}{4\pi i}\bar{\partial}\partial \log \det(E) + \frac{N}{12\pi i}\bar{\partial}\partial \log h \; ,$$

We will assume here the formulae on pages 5, 6, 7 of lecture 3 for the quantities on the RHS of (1).

For the LHS of (1), r_{-4} is computed recursively via the formulae (from Lect 3):

$$(2) \qquad r_{-2}(x,\xi,\lambda) = \left(|\xi|_g^2 - \lambda\right)^{-1} \qquad \text{“}|\xi|_g^2\text{” = principal symbol of Laplacian } \Delta$$

$$r_{-2-j} = r_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!}\partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l}$$

Indeed, the numbers h_j in the heat trace expansion

$$\mathrm{Tr}(e^{-t\Delta}) \sim h_{-1}t^{-1} + h_{-\frac{1}{2}}t^{-\frac{1}{2}} + h_0 t^0 + \cdots$$

are given by

$$h_{\frac{-2+j}{2}} = \int_\Sigma h_{\frac{-2+j}{2}}(x)\,|dx|$$

with the function $h_{\frac{-2+j}{2}}(x)$ given by

③ $$\boxed{h_{\frac{-2+j}{2}}(x) = \int_{\mathbb{R}^2}\int_\Gamma e^{-\lambda}\, r_{-2-j}(x,\xi,\lambda)\,d\lambda\,d\xi}$$

To compute these integrals we need explicit formulae for $r_{-2-j}(x,\xi,\lambda)$ and for that we need explicit <u>local</u> formulae for our Dirac $\bar\partial^\nu$ operator and the associated Laplacians. For this, using the local coordinate chart formulae of lect 3 p 5,6,7 : over a chart $\mathcal{U}$

④ $$\bar\partial = \bar\partial^\nu|_{\mathcal{U}} = \partial_{\bar z}\,d\bar z \quad \text{acting on } C^\infty(\mathcal{U}, \mathcal{V}|_{\mathcal{U}} \cong \mathcal{U}\times\mathbb{C}^N)$$

⑤ $$\bar\partial^* = \bar\partial^{\nu *}|_{\mathcal{U}} = -h^{-1}\partial_z - (hE^T)^{-1}(\partial_z E^t)$$

Globally, recall, $\bar\partial : C^\infty(\Sigma, \mathcal{V}) \to C^\infty(\Sigma, \mathcal{V}\otimes T^{0,1}_{\mathbb{C}}\Sigma)$, while

$$\Delta = \bar\partial^* \bar\partial : \underset{\Omega^0(\Sigma,\mathcal{V})}{\overset{\|}{}} \Omega^0(\Sigma,\mathcal{V}) \longrightarrow \Omega^0(\Sigma,\mathcal{V})$$

is therefore given in local coordinates on the chart $\mathcal{U}$ by

$$\boxed{\Delta = -(hE^T)^{-1}\partial_z \cdot (E^t \bar\partial_z) = -h^{-1}\bar\partial_z\partial_z - (hE^T)^{-1}(\partial_z E^t)\bar\partial_z} \quad (6)$$

and $\tilde\Delta = \bar\partial\bar\partial^* : C^\infty(\Sigma, \mathcal{V}^*\otimes T^{1,0}_{\mathbb{C}}\Sigma) \circlearrowleft$ by

$$\boxed{\tilde\Delta = -E\,\partial_z \cdot (hE)^{-1}\bar\partial_z = -h\,\bar\partial_z\partial_z - E(\partial_z (hE)^{-1})\bar\partial_z.} \quad (7)$$

Since $z = x_1 + ix_2$, $\bar z = x_1 - ix_2$, and therefore

$$\partial_z = \tfrac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \bar\partial_z = \tfrac{1}{2}(\partial_{x_1} + i\partial_{x_2})$$

and writing (since at this point we do not need the exact formulae)

$$g = \frac{h(z,\bar z)^{-1}}{4} \quad (8)$$

"g" is just a letter here, not a Riemman metric

and

$$\alpha = \frac{1}{2i}(hE^t)^{-1}\frac{\partial E^t}{\partial z} \quad (9)$$

we can rewrite (6) as

$$\boxed{\Delta = -g(\partial_{x_1}^2 + \partial_{x_2}^2) - \alpha(i\partial_{x_1} - \partial_{x_2})} \quad (10)$$

To obtain the local symbol of Δ on $\mathcal{U}$ we just have to replace $-i\partial_{x_k}$ with ξ_k, by Fourier transforms, hence Δ has local symbol

(11) $$\boxed{a(x,\xi) = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi)}$$

with

(12) $$\boxed{a_2(x,\xi) = g(x_1,x_2)\,|\xi|^2} \qquad \left(|\xi|^2 = \xi_1^2 + \xi_2^2\right)$$

(13) $$\boxed{a_1(x,\xi) = \alpha(x_1,x_2)(\xi_1 + i\xi_2)}$$

(14) $$\boxed{a_0(x,\xi) = 0}$$

It follows from (2) that

$$r_{-2}(x,\xi,\lambda) = \left(g|\xi|^2 - \lambda\right)^{-1},$$

which is easy enough $\overset{\nabla}{\circ}$. For r_{-3}, which is used to compute $h_{-\frac{1}{2}}$, we have to work a little harder: from (2)

(15) $$r_{-3} = -r_{-2} \sum_{\substack{|\mu|+k+l=1 \\ l<1}} \frac{1}{\mu!}\,\partial_\xi^\mu a_{2-k}\, D_x^\mu r_{-2-l}.$$

Since $l<1$, then $l=0$, and the condition $|\mu|+k+l=1$ forces $\mu! = 1$ in all summands. Thus (15) simplifies to

$$r_{-3} = -r_{-2} \sum_{|\mu|+k=1} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2},$$

which can then be expanded into two summands

16 $$r_{-3} = -r_{-2}\underbrace{\left(\sum_{j=0}^{2} \partial_{\xi_j} a_2 D_{x_j} r_{-2}\right)}_{|\mu|=1,\ k=0} - r_{-2}\underbrace{\left(a_1 r_{-2}\right)}_{|\mu|=0,\ k=1}$$

For the first term, note that

17 $$\partial_{\xi_j} a_2 = 2g\,\xi_j$$

and also

18 $$D_{x_j} r_{-2}^{m} = -m\, r_2^{m+1} (D_{x_j} g)\, |\xi|^2$$

for any $m \in \mathbb{N}$. This and $a_1 = \alpha(\xi_1 + i\xi_2)$ reduces 16 to

14 $$\boxed{r_{-3} = 2r_{-2}^{3} \sum_j g\,(D_{x_j} g)\, \xi_j\, |\xi|^2 - r_{-2}^{2}\,\alpha(\xi_1 + i\,\xi_2)}$$

The computation of r_{-4} and subsequent terms, proceeds along similar elementary lines. However, though no more complicated than the above, each step is increasingly arduous. Fortunately, for the RRH theorem we only need to know r_{-4}: it starts off as follows.

We have

$$r_{-4} = -r_{-2} \sum_{\substack{|\mu|+k+l=2 \\ l<2}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l} \qquad (20)$$

The condition $l<2$ implies

$$l = 0 \text{ or } 1,$$

so we can break-up the above expression into two terms:

$$r_{-4} = -r_{-3} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l}$$

$$(21) \qquad -r_{-3} \sum_{|\mu|+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} D_x^\mu r_{-2-l}.$$

and repeat the process for each of the terms on the RHS of (21), letting $k=0,1,2$ in the first summand and $k=0$ or 1 in t -- sa e or two

second. After a computation one arrives at:

LTCC STUDENTS ARE INVITED TO COMPLETE THE r_{-4} COMPUTATION:

PROPOSITION 1:

$$r_{-2} = \left(g(x)\,|\xi|^2 - \lambda\right)^{-1} \tag{22}$$

$$r_{-3} = 2r_{-2}^3 \sum_l g\left(D_{x_l} g\right) \xi_l |\xi|^2 - r_{-2}\,\alpha\left(\xi_1 + i\xi_2\right) \tag{23}$$

DONE ABOVE.

$$\boxed{\begin{aligned} r_{-4} = {} & 12\, r_{-2}^5 \sum_{k,l} g^2 \left(D_{x_k} g\right)\left(D_{x_l} g\right) \xi_k \xi_l |\xi|^4 \\ & - 2 r_{-2}^4 \sum_k g\left(D_{x_k} g\right)^2 |\xi|^4 \\ & - 4 r_{-2}^4 \sum_{k,l} g\left(D_{x_k} g\right)\left(D_{x_l} g\right) \xi_k \xi_l |\xi|^2 \\ & - 4 r_{-2}^4 \sum_{k,l} g^2 \left(D^2_{x_k, x_l} g\right) \xi_k \xi_l |\xi|^2 \\ & - 6 r_{-2}^4 \sum_k \alpha g \left(D_{x_k} g\right)\left(\xi_1 + i\xi_2\right) |\xi|^2 \\ & + r_{-2}^3 \sum_k g\left(D^2_{x_k} g\right) |\xi|^2 \\ & + 2 r_{-2}^3 \sum_k \left(D_{x_k} \alpha\right) g\left(\xi_1 + i\xi_2\right) \xi_k \\ & + r_{-2}^3\, \alpha\left(D_{x_1} g\right) |\xi|^2 \\ & + i r_{-2}^3\, \alpha\left(D_{x_2} g\right) |\xi|^2 + r_{-2}^3\, \alpha^2 \left(\xi_1 + i\xi_2\right)^2 \end{aligned}} \tag{24}$$

Using Proposition 1 and (3), one has:

Proposition 2:

identity $N\times N$ matrix – since we are computing on a trivialization of rank N bundle $\mathcal{V}\to\Sigma$

(25) $$h_{-1}(x) = \frac{1}{4\pi g(x)} I_N \quad \text{and} \quad h_{-1} = \frac{rk(\mathcal{V})}{\pi} \mathrm{Vol}_h(\Sigma)$$

(26) $$h_{-\frac{1}{2}}(x) = 0 \quad \text{and} \quad h_{-\frac{1}{2}} = 0$$

(27) $$h_0(x) = \frac{1}{4\pi}\Big(-\frac{1}{6} D_{x_1}\big(g^{-1} D_{x_1} g\big) - \frac{1}{6} D_{x_2}\big(g^{-1} D_{x_2} g\big) + \frac{1}{2} D_{x_1}\big(g^{-1}\alpha\big) + \frac{1}{2} D_{x_2}\big(g^{-1}\alpha\big)\Big)$$

For the calculations let us note the identities:

(28) $$\boxed{\int_{\Gamma} e^{-\lambda}\big(\beta|\xi|^2 - \lambda I\big)^{-k} d\lambda = \frac{e^{-\beta|\xi|^2}}{(k-1)!}} \quad k\in\mathbb{N}$$

(29) $$\boxed{n_k\in\mathbb{N},\ \int_{\mathbb{R}^2} \xi^{2n_1}\xi^{2n_2} e^{-\beta|\xi|^2} d\xi = \frac{\Gamma(n_1+\frac{1}{2})\Gamma(n_2+\frac{1}{2})}{\beta^{n_1+\frac{1}{2}}\beta^{n_2+\frac{1}{2}}}}$$

Gamma function

(30) $$\boxed{m_1 \text{ or } m_2 \in\mathbb{N} \text{ is odd} \Longrightarrow \int_{\mathbb{R}^2} \xi^{m_1}\xi^{m_2} e^{-\beta|\xi|^2} d\xi = 0}$$

I have, regrettably, used the same letter "h" for the heat coeffts $h_{-k} = \int h_{-k}(x)dx$ and the hermitian metric h on Σ – please note the difference

PROOF OF (25): We have

$$h_{-1}(x) = \int_{\mathbb{R}^2}\int_{\Gamma} e^{-\lambda} r_{-2}\, d\lambda\, d\xi$$

$$= \int_{\mathbb{R}^2} \frac{i}{2\pi}\int_{\Gamma} e^{-\lambda}(g|\xi|^2-\lambda)^{-1} d\lambda\, d\xi$$

$$\overset{(28)}{=} \int_{\mathbb{R}^2} e^{-g|\xi|^2} d\xi$$

$$\overset{(29)}{=} \frac{1}{4\pi^2 g}\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{1}{4\pi g}, \qquad \text{since } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

From eqn (8) we have $(4g)^{-1} = h$, while $\frac{i}{2} dz\wedge d\bar{z} = dx_1 dx_2$, so

$$\operatorname{tr}(h_{-1}(x))|dx| = \operatorname{tr}\left((4\pi g)^{-1} I_N\right) dx_1 dx_2$$

$$= \frac{\operatorname{tr}(I_N)}{\pi} h \cdot \frac{i}{2} dz\wedge d\bar{z}$$

$$= \frac{N}{\pi} d_h\Sigma \qquad \text{by eqn (6) of lect 3.}$$

Thus,

$$h_{-1} := \int_{\Sigma} \operatorname{tr}(h_{-1}(x))\,|dx| = \frac{N}{\pi}\int_{\Sigma} 1 . d_h\Sigma = \frac{N}{\pi}\operatorname{Vol}_h(\Sigma) \qquad \blacksquare$$

PROOF OF (26): We have

$$h_{-\frac{1}{2}}(x) = \int_{\mathbb{R}^2}\int_{\Gamma} e^{-\lambda}\, r_{-3}\, d\lambda\, d\xi$$

$$\overset{(23)}{=} \int_{\mathbb{R}^2} 2\left(\int_{\Gamma} e^{-\lambda} r_{-2}^{3}\, d\lambda\right) \sum_{l} g\left(D_{x_l} g\right) \xi_l |\xi|^2\, d\xi$$

$$- \int_{\mathbb{R}^2}\left(\int_{\Gamma} e^{-\lambda} r_{-2}\, d\lambda\right)(\xi_1 + i\xi_2)\, d\xi\, \alpha(x)$$

$$\overset{(28)}{=} \sum_{l} g\left(D_{x} g\right) \int_{\mathbb{R}^2} e^{-g|\xi|^2} \xi_l |\xi|^2\, d\xi$$

$$- \int_{\mathbb{R}^2} e^{-g|\xi|^2}(\xi_1 + i\xi_2)\, d\xi\, \alpha(x)$$

$$\overset{(30)}{=} 0. \qquad \blacksquare$$

In fact, it's not hard to show, similarly, that:

PROPOSITION 3:

$$j \in \mathbb{N} \text{ odd} \Rightarrow h_{\frac{-2+j}{2}}(x) = 0 \qquad \left(\text{and } h_{\frac{-2+j}{2}} = 0\right).$$

EXERCISE — PARTICULARLY FOR LTCC STUDENTS!

Thus, the heat trace expansion only has integer powers of t.

OUTLINE PROOF OF 27.

LTCC — EXERCISE: PROVIDE FULL PROOF.

We have

$$h_0(\alpha) = \int_{\mathbb{R}^2}\int_{\Gamma} e^{-\lambda}\, r_{-4}\, d\lambda\, d\xi \, ;$$

with r_{-4} as stated in 24:

$$\begin{aligned} r_{-4} = {} & 12\, r_{-2}^5 \sum_{k,l} g^2 (D_{x_k} g)(D_{x_l} g)\, \xi_k \xi_l |\xi|^4 \\ & - 2\, r_{-2}^4 \sum_{k} g (D_{x_k} g)^2 |\xi|^4 \\ & - 4\, r_{-2}^4 \sum_{k,l} g (D_{x_k} g)(D_{x_l} g)\, \xi_k \xi_l |\xi|^2 \\ & - 4\, r_{-2}^4 \sum_{k,l} g^2 (D^2_{x_k, x_l} g)\, \xi_k \xi_l |\xi|^2 \\ & \boxed{- 6\, r_{-2}^4 \sum_{k} \alpha g (D_{x_k} g)(\xi_1 + i\xi_2) |\xi|^2} \\ & + r_{-2}^3 \sum_{k} g (D^2_{x_k} g) |\xi|^2 \\ & \boxed{+ 2\, r_{-2}^3 \sum_{k} (D_{x_k} \alpha)\, g (\xi_1 + i\xi_2)\, \xi_k} \\ & + r_{-2}^3\, \alpha (D_{x_1} g) |\xi|^2 \\ & + i\, r_{-2}^3\, \alpha (D_{x_2} g) |\xi|^2 + r_{-2}^3\, \alpha^2 (\xi_1 + i\xi_2)^2 \end{aligned} \qquad (31)$$

Can ignore these terms since by 30 they integrate in ξ to zero.

The rest of the computation is left as an exercise. ∎

As far as computing the index $\operatorname{ind}(\bar\partial^{\nu})$ is concerned (i.e. proving the RRH Thm) we may also ignore all the terms <u>not</u> containing an α — since these will be the same, will also occur in $\tilde h_0(x)$, i.e. in $\operatorname{Tr}(e^{-t\tilde\Delta}) \sim \underset{\substack{\| \\ h_{-1}}}{\tilde h_{-1}}\, t^{-1} + \underset{\substack{\| \\ 0}}{\tilde h_{-\frac12}}\, t^{-\frac12} + \tilde h_0 + \cdots$

& h_0

Thus, looking at (31), in fact we need only compute

$$
\begin{aligned}
r_{-4} = {} & \cancel{12\, r_{-2}^5 \sum_{k,l} g^2 (D_{x_k} g)(D_{x_l} g)\, \xi_k \xi_l |\xi|^4} \\
& \cancel{2\, r_{-2}^4 \sum_k g (D_{x_k} g)^2 |\xi|^4} \\
& \cancel{-4\, r_{-2}^4 \sum_{k,l} g (D_{x_k} g)(D_{x_l} g)\, \xi_k \xi_l |\xi|^2} \\
& \cancel{4\, r_{-2}^4 \sum_{k,l} g^2 (D^2_{x_k,x_l} g)\, \xi_k \xi_l |\xi|^2} \\
& \cancel{-6\, r_{-2}^4 \sum_k \alpha g (D_{x_k} g)(\xi_1 + i\xi_2) |\xi|^2} \\
& \cancel{+ r_{-2}^3 \sum_k g (D^2_{x_k} g) |\xi|^2} \\
& \cancel{+ 2\, r_{-2}^3 \sum_k (D_{x_k} \alpha)\, g (\xi_1 + i\xi_2)\, \xi_k} \\
& + r_{-2}^3\, \alpha (D_{x_1} g) |\xi|^2 \\
& + i r_{-2}^3\, \alpha (D_{x_2} g) |\xi|^2 + r_{-2}^3\, \alpha^2 (\xi_1 + i\xi_2)^2
\end{aligned}
$$

for the terms <u>not</u> crossed-out to compute

$$\operatorname{ind}(\bar\partial^{\nu}) = h_0 - \tilde h_0$$

This shows quite drammatically the often commented 'remarkable cancellations' that occur in the super ($\mathbb{Z}_2$) trace of the heat kernel formula for the index — and demonstrates again the 'locality' and computability of the index.

To complete the proof of the RRH formula, we may use $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, $D_{x_1} = -i(\partial_z + \partial_{\bar{z}})$, $D_{x_2} = \partial_z - \partial_{\bar{z}}$, and $g = (4h)^{-1}$, $\alpha = \frac{1}{2i}(hE^t)^{-1}\frac{\partial E^t}{\partial z}$ to show:

PROPOSITION 4: One has

$$\boxed{h_0(x) = -\frac{1}{6\pi}\partial_z\partial_{\bar{z}}\log(h).I_n - \frac{1}{2\pi}\partial_{\bar{z}}\left((E^t)^{-1}\frac{\partial E^t}{\partial z}\right)} \quad (32)$$

PROOF: This is just a rearrangement of formula (27) using the above identities.

LTCC EXERCISE! ▨

Taking the trace of (32), simplifying and using

$$|dx| = \frac{i}{2}dz\wedge d\bar{z}$$

we have

$$\mathrm{tr}\big(h_0(x)\big) = -\frac{1}{6\pi}\,\mathrm{tr}\big(\partial_{\bar z}\partial_z \log h(x) \cdot I_N\big)$$

$$-\frac{1}{2\pi}\,\mathrm{tr}\left(\partial_{\bar z}\left((E^t)^{-1}\frac{\partial E^t}{\partial z}\right)\right)$$

$$= -\frac{N}{6\pi}\,\partial_z\partial_{\bar z}\log\big(h(z)\big) - \frac{1}{2\pi}\,\partial_z\partial_{\bar z}\log\det\big(E(z)\big)$$

$$= \frac{N}{12\pi i}\,\partial_z\partial_{\bar z}\log h \cdot dz\wedge d\bar z$$

$$+\frac{1}{4\pi i}\,\partial_z\partial_{\bar z}\log\det(E)\,dz\wedge d\bar z$$

Thus

$$h_0 = \frac{1}{4\pi i}\int_\Sigma \partial\bar\partial \log\det E + \frac{N}{12\pi i}\int_\Sigma \partial\bar\partial\log h.$$

The 'same' computation for $\tilde\Delta$ using (7) and

$$\tilde\alpha = \frac{1}{2i}\,E\,\frac{\partial (hE)^{-1}}{\partial z}$$

gives

$$\tilde h_0 = \frac{-1}{4\pi i}\int_\Sigma \partial\bar\partial\log\det E - \frac{N}{6\pi i}\int_\Sigma \partial\bar\partial\log h.$$

and hence that

$$\boxed{h_0 - \hat{h}_0 = \frac{1}{2\pi i}\int_{\Sigma} \partial\bar{\partial}\log\det E + \frac{N}{4\pi i}\int_{\Sigma}\partial\bar{\partial}\log h \,.}$$

This proves the RRH theorem on a closed Riemann surface.