CHAPTER 2

Resolutions via Topology

In this section we shall see that we can construct resolutions once we have constructed models for classifying spaces. We shall introduce very quickly the basic topological notions used later. We shall, however introduce classifying spaces in a more general way than initially used. We will see how to construct classifying spaces for families of subgroups.

1. CW-complexes

In this section we only briefly introduce the concept of a CW-complex. The interested reader can find all detail in most Algebraic Topology textbooks, such as for example Hatcher's book [**9**], appendix.

A CW-complex can be thought of as built by the following proceedure:

(1) Start with a discrete set X^0, whose points are regarded as 0-cells. (This is the 0-skeleton).
(2) Inductively, from the $(n-1)$-skeleton X^{n-1} build the n-skeleton X^n by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. (This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identification $x \sim \varphi_\alpha x$ for $x \in D^n_\alpha$. Thus, as a set $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$, where each e^n_α is an open n-disk.)
(3) Put $X = \bigcup_n X^n$ where X is given the weak topology: A set $A \subset X$ is open if and only if $A \cap X^n$ is open for all n.

Example 2.1. A 1-dimensional CW-comples is just a graph with vertices the 0-cells and edges the 1-cells.

Example 2.2. $X = \mathbb{R}^2$ is a 2-dimensional CW-complex with $\mathbb{Z} \times \mathbb{Z}$ as the 0-cells, the open intervalls as the 1-cells and the interior of the unit squares as 2-cells.

Example 2.3. The sphere S^n has the structure of a CW-complex with one 0-cell and one n-cell.

Example 2.4. The real projective plane, $\mathbb{R}P^2$ can be seen as D^2 with antipodal points of $S^1 = \delta D^2$ identified. Hence $\mathbb{R}P^2 = e^0 \cup e^1 \cup e^n$.

Exercise 6. How can we see that $\mathbb{R}P^n$ has a CW-structure, $e^0 \cup e^1 \cup ... \cup e^n$?

Exercise 7. How can we see that a closed orientable surface M_g of genus g ($M_1 = T$, the torus) has a CW-structure given by: $e^0 \cup e^1_1 \cup e^1_2 \cup ... \cup e^1_{2g} \cup e^2$, i.e. has one 0-cell, $2g$ 1-cells and one 2-cell? (Identify edges on a regular $4g$-gon.)

2. G-spaces

In this course, all our groups are discrete groups. One can, however, define classifying spaces for families for arbitrary topological groups. For detail see tomDieck's book on transformation groups [**6**].

Definition 2.5. A G-space is a topological space X with a (continuous) left G-action

$$G \times X \to X, \qquad (g, x) \mapsto gx$$

satisfying

(1) $ex = x$ for all $x \in X$ and $e = e_g$ the identity of G.
(2) (gh)x = g(hx) for all $x \in X$ and all $g, h \in G$.

Example 2.6. (a) Let G be the infinite cyclic group with generator g, i.e. $G = \langle g \rangle$ and $X = \mathbb{R}$. X is a G-space with G acting by translation $g^i x = x + i$.

(b) Let $G = \mathbb{Z} \times \mathbb{Z}$. $X = \mathbb{R}^2$ is a G-space with G acting by translation.
(c) Let $\mathbb{H}$ be the upper half plane model of the hyperbolic plane,

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \,|\, y > o\}.$$

$Sl_2(\mathbb{Z}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a, b, c, d \in \mathbb{Z}, det(A) = 1\}$ acts on $\mathbb{H}$ by Möbius-transformations, i.e $Az = \frac{az+b}{cz+d}$.
(Check this really makes $\mathbb{H}$ into a $Sl_2(\mathbb{Z})$-space)
The kernel of this action consists of scalar multiples in $Sl_2(\mathbb{Z})$ of the identity matrix I. Hence $\mathbb{H}$ is a G space for $G = PSl_2(\mathbb{Z}) = Sl_2(\mathbb{Z})/\{\pm I\}$.

Definition 2.7. The stabilizer $G_x \leq G$ of a point $x \in X$ is the subgroup $\{g \in G \,|\, gx = x\}$.

Let us note that the Cartesian product $X \times Y$ of two G-spaces X and Y is again a G-space via the diagonal action $g(x, y) = (gx, gy)$ for all $x \in X, y \in Y$ and $g \in G$.

Definition 2.8. Let $H \subseteq G$ be a subgroup of G. Write X^H for the subspace of H-fixed points

$$X^H = \{x \in X \,|\, hx = x, \,\forall h \in H\}$$

and X/H for the space of H-orbits,

$$X/H = \{Hx \,|\, x \in X\}.$$

Let $N_G(H)$ denote the normalizer of H in G:

$$N_G(H) = \{g \in G \,|\, gH = Hg\}.$$

Then the G-action on X restricts to an $N_G(H)$-action on X^H with H acting trivally. Hence X^H is a $N_G(H)/H$-space.

Example 2.9. The space of left cosets G/H is a G-space via $(g, kH) \mapsto gkH$ for all $g, k \in G$.
(Fact: Every discrete G-space is a disjoint union of such G-spaces.)
Let $K \leq G$ a subgroup. Then $(G/H)^K$ consists of all cosets gH such that $KgH = gH \iff g^{-1}Kg \leq H$.

DEFINITION 2.10. A G-CW-complex consists of a G-space X together with a filtration

$$X^0 \subset X^1 \subset X^2 \subset ... \subset X$$

by G-subcomplexes such that

(1) Each X^n is closed in X.
(2) $\bigcup_{n\in\mathbb{N}} X^n = X$.
(3) X^0 is a discrete subspace of X.
(4) For each $n \geq 1$ there exists a discrete G-space Δ_n together with G-maps $F : S^{n-1} \times \Delta_n \to X^{n-1}$ and $\hat{f} : D^n \times \Delta_n \to X^n$ such that the following diagramme is a push-out:

$$\begin{array}{ccc} S^{n-1} \times \Delta_n & \to & X^{n-1} \\ \downarrow & & \downarrow \\ D^n \times \Delta_n & \to & X^n \end{array}$$

(5) A subspace Y of X is open if and only if $Y \cap X^n$ is open for all $n \geq 0$.

A map $f : X \to Y$ of G-CW-complexes is a G-map if $f(gx) = gf(x)$ for all $g \in G$, $x \in X$. If $G = \{e\}$ the trivial group, then a G-CW-complex is just a CW-complex as in Chapter 1. All our examples in 2.6 are G-CW-complexes.

EXAMPLE 2.11. Let $G = C_2$ be the cyclic group of order 2. Then the sphere, S^2 is a G-CW-complex with G acting by the antipodal map.

DEFINITION 2.12. (1) A G-CW-complex is called finite dimensional if $X^n = X$ for some $n \geq 0$. The least such n is called the dimension of X. (In case $dim(X) < \infty$, Axiom 5 above is redundant.)
(2) A G-CW-complex is said to be of finite type, if there are finitely many G-orbits in each dimension. (Equivalently, as X/G is a CW-complex, X/G only has finitely many cells in each dimension.)
(3) A G-CW-complex is called cocompact if X is finite dimensional and of finite type. (Equivalently, X/G is a finite CW-complex.)

All the examples we've seen so far, are cocompact. Before we can move on to defining classifying spaces, we need to have a quick look at an important construction, the join construction:

DEFINITION 2.13. Let $I = [0, 1]$ and let X, Y be G-CW-complexes. We define the join of X and XY to be:

$$X * Y = (I \times X \times Y)/\sim,$$

where $\sim$ is the equivalence relation generated by $(0, x, y_1) = (0, x, y_2)$ and $(1, x_1, y) = (1, x_2, y)$.

Hence the dimension of$X * Y$ is equal to $1 + dim(X) + dim(Y)$. Furthermore, the join of two G-spaces is again a G-space with diagonal G-action. One can also show that the join of two G-CW-complexes is again a G-CW-comples.

EXAMPLE 2.14. (1) $X * \{pt\} = CX$ the cone on X.
(2) $X * S^0 = \Sigma X$ the suspension on X.
(3) The n-fold join $\{pt\} * ... * \{pt\}$ is a $n - 1$-simplex

LEMMA 2.15. [**13**] *Let X be a non-empty and Y be a n-connected space. Then $X * Y$ is $n + 1$-connected. In particular, the infinite join of non-empty G-CW-complexes is contractible.*

A space X is called 0-connected if it is non-empty and path-connected; it is callen n-connected if X is 0-connected and for each $1 \leq i \leq n$, the homotopy group $\pi_i(X)$ is trivial. For detail on connectedness and higher homotopy groups see [**16**, Chapter 11].

3. Classifying spaces

Let $\mathfrak{F}$ denote a family of subgroups of a group G. This is a collection of subgroups closed under conjugation and finite intersection. The following are examples of such families:

- $\mathfrak{F} = \mathfrak{All}$, the family of all subgroups of G
- $\mathfrak{F} = \mathfrak{Fin}$, the family of all finite subgroups of G
- $\mathfrak{F} = \mathfrak{VC}$, the family of all virtually cyclic subgroups of G. (A group is virtually cyclic if it has a cyclic subgroup of finite index)
- $\mathfrak{F} = \{e\}$, the family consisting only of the trivial subgroup.

Later on, we will mainly be concerned with $\mathfrak{F} = \{e\}$, but will also talk about $\mathfrak{F} = \mathfrak{F}in$.

Definition 2.16. A G-CW-complex X is called a classifying space for the family $\mathfrak{F}$, or a model for $E_{\mathfrak{F}}G$, if for each subgroup $H \leq G$, the following holds:

$$X^H \simeq \begin{cases} * & \text{if} \quad H \in \mathfrak{F} \\ \emptyset & \text{otherwise} \end{cases}$$

Theorem 2.17. *For each group G there exists a model for $E_{\mathfrak{F}}G$.*

Proof To prove existence one could follows either Milnor's [**12**] or Segal's [**17**] construction of EG, the classifying space for free actions. We shall follow Milnor's model here: Let

$$\Delta = \bigsqcup_{H \in \mathfrak{F}} G/H$$

be the discrete G-CW-complex as in example 2.9. Now form the n-fold join

$$\Delta_n = \underbrace{\Delta * \ldots * \Delta}_{n}$$

and put

$$X = \bigcup_{n \in \mathbb{N}} \Delta_n.$$

Example 2.9 now implies that $\Delta^H = \emptyset \iff H \notin \mathfrak{F}$. Furthermore, since

$$\Delta^H * \ldots * \Delta^H = (\Delta * \ldots * \Delta)^H,$$

Lemma 2.15 implies that $X^H \simeq *$ for $H \in \mathfrak{F}$ and $X^H = \emptyset$ otherwise and X is therefore a model for $E_{\mathfrak{F}}G$. $\square$

This construction, however gives us an infinite dimensional model, which is not of finite type. In this course we will try to find "nice" models.

REMARK 2.18. when considering the family $\mathfrak{F} = \mathfrak{Fin}$, the we denote the classifyibg space $E_{\mathfrak{F}}G$ by $\underline{E}G$. This is the classifying space for proper action.

Let G be torsion-free and X be a model for $\underline{E}G$. Then X is contractible and G acts freely ($X^{\{e\}} \simeq *$ and $X^H = \emptyset$ for all $\{e\} \neq H \leq G$). Hence X is a model for EG, the classifying space for free actions, or equivalently the universal cover of a $K(G,1)$, an Eilenberg-Mac Lane space.

EXAMPLE 2.19. (Examples for torsion-free groups)

(a) $G = \mathbb{Z}$. Then $\mathbb{R}$ is a model for EG by Example 2.6 (a)
(b) $G = \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{R}^2$ is a model for EG by Example 2.6 (b).
(c) Let G be the free group on 2 generators, $G = \langle x, y \rangle$. Then the Cayley-graph is a tree, which is a model for EG .

EXAMPLE 2.20. (Examples for groups with torsion)

(a) If G is a finite group, then $\{*\}$ is a model for $\underline{E}G$.
(b) Let $G = D_\infty$ be the infinite dihedral group. Then $\mathbb{R}$ is a model for $\underline{E}G$, where the generator for the infinite cyclic group acts by translation and the generator of order two acts by reflection.
(c) Let G be a wallpaper group, i.e. an extension of $\mathbb{Z} \times \mathbb{Z}$ with a finite subgroup of O_2, the group of 2×2 orthogonal matrices. Then $\mathbb{R}^2$ is a model for $\underline{E}G$.
(d) Let $G = PSL_2(\mathbb{Z})$. We've seen in example 2.6 (c) that G acts by Möbius transformations on $\mathbb{H}$ the upper half plane. This is a 2-dimensional model for $\underline{E}G$.

    Considering the two generators, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ we can see that $G \cong C_2 * C_3$ the free product of a cyclic group of order 2 and a cyclic group of order 3. Hence, the dual tree T is a 1-dimensional model for $\underline{E}G$.

For the interested reader I will include a very brief overview of some of the homotopy theory behind the above construction:

DEFINITION 2.21. A G-space X is called proper of for each pair of points $x, y \in X$ there are open neighbourhoods V_x of x and V_y of y such that the closure of $\{g \in G \,|\, gV_x \cap V_y \neq \emptyset\}$ is a compact subset of G.

If G is discrete this means that the above set is finite. Hence a G-CW complex X is proper if and only if all stabilizers are finite.

THEOREM 2.22. *(**J.H.C. Whitehead**, see* [**?**]*, Chapter I)*
A G-map $f : X \to Y$ between two G-CW-complexes is a G-homotopy equivalence if for all $H < G$ and all $x_0 \in X^H$ the induced map

$$\pi_*(X^H, x_0) \to \pi_*(Y^H, f(x_0))$$

is bijective.

Now, the following theorem explains why we call $\underline{E}G$ the classifying space for proper actions:

THEOREM 2.23. *(See* [**?**, Theorem 2.4]*) Let X be a proper G-CW-complex. Then, up to G-homotopy, there is a unique G-map $X \to \underline{E}G$.*

EXERCISE 8. Show that any two models for $\underline{E}G$ are G-homotopy equivalent.

4. Projective resolutions

In this section we will construct projective resolutions by considering classifying spaces. We will look at EG, the classifying space for free actions. But let us begin with the augmented cellular chain complex.

Let X be a G-CW-complex. It's augmented cellular chain complex is a chain complex of G-modules

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_o(X) \twoheadrightarrow \mathbb{Z},$$

in which each $C_i(X)$ is the free abelian group on the orbits of i-cells. Hence

$$C_i(X) \cong \bigoplus_{orbit-reps\,\sigma} \mathbb{Z}[G/G_\sigma].$$

Recall that a G-CW-complex X is a model for EG if X is contractible and G acts freely on X.

Proposition 2.24. *Let X be a model for* EG. *Then the augmented cellular chain complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Remark 2.25. Let $\mathbf{C}$ be a chain complex. We say $\mathbf{C}$ is acyclic of and only if $H_*(\mathbf{C}) = \mathbf{0}$. A G-CW-complex X is called acyclic if it has the homology of a point. Hence an acyclic G-CW-complex with a free G-action would also give us a free resolution of $\mathbb{Z}$.

Definition 2.26. We say a group G has finite geometric dimension ($\mathrm{gd}G < \infty$) if it admits a finite dimensional model for EG. The smallest such dimension is called the geomtric dimension of G.

We can now state the following corollary to Proposition 2.24:

Corollary 2.27. *For each group G:*

$$\mathrm{cd}G \leq gdG.$$

The converse is almost true and involves rather more than we can cover here. It comes in two parts, which were proved using very different methods.

Theorem 2.28. [**18, 19**][**Stallings-Swan**] *Let G be a group. Then*

$$\mathrm{cd}G = 1 \iff \mathrm{gd}G = 1 \iff G is\ free.$$

Theorem 2.29. [**8**][**Eilenberg-Ganea**] *Let G be a group such that* $\mathrm{cd}G \geq 3$. *Then*

$$\mathrm{cd}G = \mathrm{gd}3.$$

This leaves the case when G is a group with $\mathrm{cd}G = 2$. It is still unknown whether there is a group G, which does not admit a 2-dimensional model for EG, although there are some candidates for examples [**?**] (Bestvina).

Example 2.30. Let G be a free group on S. We now construct a model X for EG . We take a fixed vertex x_0 and we then have a unique orbit of vertices . Hence

$$C_0(X) = \mathbb{Z}V = \mathbb{Z}G.$$

As basis for $C_1(X) = \mathbb{Z}E$ we take for each $s \in S$ an oriented 1-cell e_s. We assume the initial vertex is x_0. Otherwise we translate by a suitable g. Hence the initial vertex of e_s is x_0 and the terminal vertex is sx_0. and we get a map

$$\begin{array}{rccl} \delta: & \mathbb{Z}E & \to & \mathbb{Z}V \\ & e_s & \mapsto & sx_0 - x_0 = (s-1)x_0 \end{array}$$

$\mathbb{Z}E = C_i(X) = \mathbb{Z}G^{(S)}$ and we have a free resolution of length 1:

$$0 \longrightarrow \mathbb{Z}G^{(S)} \xrightarrow{\delta} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

This is the resolution we have seen in chapter **??**.

(a) Let $G = \langle t \rangle$ be infinite cyclic. Then $X = \mathbb{R}$ and we recover the resolution in section 1:

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

(b) We can also view X as a G-CW-comples with one orbit of 0-cells and s orbits of one cells $\{g, gs\}$ with the action induced by left translation of G on itself. E.g for $G = \langle s, t \rangle$ we get the tree seen in chapter **??**

Example 2.31. Let G be a finite cyclic group, i.e. $G = \langle t \,|\, t^n - 1 \rangle$. The circle S^1 is a G-CW-complex with n vertices and n 1-cells. There is one orbit of vertices $\{v, tv, ..., t^{n-1}v\}$ and one orbit of 1-cells $\{e, te, ..., t^{n-1}e\}$ and $(t-1)(e + te + ... + t^{n-1}e) = 0$. Consider

$$\mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0\ .$$

Hence $H_1(S^1)$ is generated by 1 element $e + te + .. + t^{n-1}e = Ne$ and we get an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\eta(1) = N$. We now splice these sequences together to obtain a free resolution

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of infinite length, as seen in Section **??** in Chapter 1.

And we retrieve the following results from Section **??** in Chapter 1:

Proposition 2.32. *Let G be a finite cyclic group. Then*

$$\mathrm{H}^{2k}(G, \mathbb{Z}) \neq 0)$$

for all $k > 0$. In particular

$$\mathrm{cd}G = \infty.$$

Corollary 2.33. *Let G be a group of finite cohomological dimension. Then G is torsion-free.*

5. K.S. Brown's first condition

Here we state a sufficient condition for a group to be of type FP_n . We will need some more homological algebra before we can prove it, which will be done in the next chapter.

We begin by defining the reduced homology of a G-CW-complex X. We have seen that the homology of X is given by

$$H_*(X) = H_*(C_*(X))$$

the homology of the cellular chain complex

$$...C_n(X) \to C_{n-1}(X) \to ... \to C_0(X) \to 0.$$

The reduced homology $\tilde{H}_*(X)$ is computed by computing the homology of the augmented cellular chain complex

$$...C_n(X) \to C_{n-1}(X) \to ... \to C_0(X) \to \mathbb{Z} \to 0.$$

Recall that these are homotopy invariants.

Exercise 9. Show that, for a contractible G-CW-complex X we have $H_i(X) = \tilde{H}_i(X) = 0$ for all $i > 0$, and that $H_0(X) = \mathbb{Z}$ and $\tilde{H}_0(X) = 0$.

Definition 2.34. A G-CW-comlex is said to be acyclic if $\tilde{H}_i(X) = 0$ for all $i \in \mathbb{Z}$.

Definition 2.35. [**4**] Let X be a G-CW-complex. We say X is *n-good* if

(a) X is acyclic in dimension $< n$, i.e. $\tilde{H}_i(X) = 0$ for all $i < n$.
(b) For $0 \leq p \leq n$, the stabiliser G_σ of any p-cell σ is of type FP_{n-p}.

Example 2.36. Any model for EG is n-good for all $n \in \mathbb{N}$. Furthermore, for a family of subgroups $\mathfrak{F}$ of type FP_n, any model for $E_{\mathfrak{F}}G$ is an n-good complex.

Theorem 2.37. [**4**, Proposition 1.1] *Suppose G admits a n-good G-CW-complex X such that X has finitely many G-orbits of cells in all dimensions $i \leq n$. Then G is of type* FP_n .