CHAPTER 3

Some more homological algebra

1. Induction and Coinduction

PROPOSITION 3.1. *Let R and S be rings, A a right S-module, B a right R-module and a left S-module and C a right R-module. Then there is a natural isomorphism*

$$\mathrm{Hom}_R(A \otimes_S B, C) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_R(B, C)),$$

the so called adjoint isomorphism.

$\mathrm{Hom}_R(B, C)$ is a right S-module via $(\varphi s)(b) = \varphi(sb)$. Contravariance of $\mathrm{Hom}_R(-, C)$ leads to this 'switch from right to left'.

REMARK 3.2. Let $\alpha : S \to R$ be a ring homomorphism. Then every R-module M can be viewed as an S-module via $sm = \alpha(s)m$ for all $s \in S, m \in M$. This is called **Restriction of scalars**.

REMARK 3.3. **Extension of Scalars** Let $\alpha : S \to R$ be a ring homomorphism. As above, R can be viewed as a left S-module vial $sr = \alpha(s)r$ for all $s \in S, r \in R$. Now let M be a right S-module and form a $\mathbb{Z}$-module

$$M \otimes_S R.$$

The right action of R on itself commuted with the left action of S. Hence $M \otimes_S R$ can be viewed as a right R-module via

$$(m \otimes r)r' = m \otimes rr'.$$

We now apply the adjoint isomorphism 3.1 to obtain a natural isomorphism

$$\mathrm{Hom}_R(M \otimes_S R, N) \cong \mathrm{Hom}_S(M, N).$$

We say extension of scalars is left adjoint to restriction of scalars.

REMARK 3.4. **Coextension of scalars** This construction is dual to that in 3.3. Let M be a right S-module. Then

$$\mathrm{Hom}_S(R, M)$$

is a right R-module via $f^{r'}(r) = f(rr')$. Now it follows from 3.1 that for all R-modules N and S-modules M there is a natural isomorphism

$$\mathrm{Hom}_R(N, \mathrm{Hom}_S(R, M)) \cong \mathrm{Hom}_S(M, N).$$

We say Coextension of scalars is right adjoint to restriction of scalars.

EXAMPLE 3.5. Let $S = \mathbb{Z}G$ for a group G and $R = \mathbb{Z}$. Consider the augmentation map $\epsilon : \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$, which is a ring homomorphism. extension of scalars sends a G-module M to

$$M \otimes_{\mathbb{Z}G} \mathbb{Z} \cong M_G,$$

where $M_G = M/L$, where L is the submodule generated by all $mg - m$. Also note, that

$$M_G \cong M/\mathfrak{g}.$$

On the other hand, coextension of scalars gives $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) = M^G = \mathrm{H}^o(G, M)$.

We will be insterested under which circumstances these constructions preserve exactness, send projectives to projectives or injectives to injectives. Note, that so far, it is only clear that restriction preserves exactness.

Lemma 3.6. (1) *Extension of scalars sends projective S-modules to projective R-modules.*
(2) *Coextension of scalars sends injective S-modules to injective R-modules.*
(3) *Let R be flat as an S-module. Then under restriction, injective R-modules become injective S-modules.*
(4) *Let R be projective as an S-module. Then under restriction projective mR-modules become projective S-modules.*

Lemma 3.7. *Let G be a group. Then every right G-module can be viewed as a left G-module and vice versa. The operation is given by $gm = mg^{-1}$ for all $g \in G, m \in M$.*

From now on let's consider group rings again. Let $H \leq G$ be a subgroup. Then the inclusion induces a ring-homomorphism

$$\mathbb{Z}H \hookrightarrow \mathbb{Z}G.$$

Extension of scalars becomes **Induction from H to G.** Let M be an H-module. Then.

$$Ind_H^G M = M \otimes_{\mathbb{Z}H} \mathbb{Z}G = M{\uparrow}_H^G$$

Coextension of scalars becomes **Coinduction from H to G.** Let M be an H-module. Then:

$$Coind_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

Let N be a G-module. Then restriction of scalars is usually denoted by

$$Res_H^G N = N{\downarrow}_H^G .$$

Proposition 3.8. *The G-module $M{\uparrow}_H^G$ contains M as a H-submodule. Furthermore,*

$$M{\uparrow}_H^G \cong \bigoplus_{g \in E} Mg$$

where E is a system of representatives for the right cosets Hg.

Note that $\mathbb{Z}{\uparrow}_H^G \cong \mathbb{Z}[H \backslash G]$ is a permutation module.

Proposition 3.9. **Frobenius reciprocity** *Let $H \leq G$ be a subgroup of the group G. Let M be an H-module and N be a G-module. Then there is an isomorphism of G-modules*

$$N \otimes M{\uparrow}_H^G \cong (N{\downarrow}_H^G \otimes M){\uparrow}_H^G .$$

This implies that for every H-module N

$$N \otimes \mathbb{Z}[H \backslash G] \cong N \otimes_{\mathbb{Z}H} \mathbb{Z}G,$$

where on the left we have a diagonal G-action, wheras on the right hand side the G-action only comes from the action on $\mathbb{Z}G$. In particular, if M is a G-module with underlying abelian group M_0 then

$$M \otimes \mathbb{Z}G \cong M_0 \otimes \mathbb{Z}G.$$

In particular, if M_0 is a free abelian group, $M \otimes \mathbb{Z}G$ is a free G-module.

PROPOSITION 3.10. **Mackey's formula** *Let $H \leq G$ and $K \leq G$ and let E denote a system of representatives for the double cosets KgH. For each K-module M there is a K-module isomorphism:*

$$(M\uparrow_H^G)\downarrow_K^G \cong \bigoplus_{g \in E}(Mg\downarrow_{K\cap H^g}^{H^g})\uparrow_{K\cap H^g}^{K}.$$

In particular, if N is a normal subgroup of G then

$$(M\uparrow_H^G)\downarrow_H^G \cong \bigoplus_{g \in H\backslash G} Mg.$$

We can identify $Mg\downarrow_{K\cap H^g}^{H^g}$ with $M\downarrow_{K\cap H^g}^{H}$ whereby the second restriction is with respect to the map: $K \cap g^{-1}Hg \to H$ mapping $k \mapsto gkg^{-1}$.

PROPOSITION 3.11. *Let $|G:H| < \infty$. Then*

$$Ind_H^G M \cong Coind_H^G M$$

for every H-module M.

EXERCISE 10. (1) Show that induction is invariant under conjugation,i.e. show that for every H-module M and $g \in G$

$$M\uparrow_H^G \cong Mg\uparrow_{H^g}^G.$$

(2) Let $|G:H| = \infty$. Show that for any H-module M:

$$(M\uparrow_H^G)^G = 0.$$

THEOREM 3.12. **Eckmann-Shapiro Lemma** *Let $H \leq G$ and let M be an H-module. Then*

$$\mathrm{H}^*(H, M) \cong \mathrm{H}^*(G, Coind_H^G M).$$

REMARK 3.13. Let $|G:H| < \infty$. Then
(1) $\mathrm{H}^*(H,\mathbb{Z}) \cong \mathrm{H}^*(G, \mathbb{Z}[H\backslash G])$ and
(2) $\mathrm{H}^*(H,\mathbb{Z}H) \cong \mathrm{H}^*(G,\mathbb{Z}G)$.

Finally we will make a remark on the exactness of induction:

PROPOSITION 3.14. *Let $A \hookrightarrow B \twoheadrightarrow C$ be a short exact sequence of $\mathbb{Z}H$-modules. Then*

$$A\uparrow_H^G \hookrightarrow B\uparrow_H^G \twoheadrightarrow C\uparrow_H^G$$

is an exact sequence of $\mathbb{Z}G$-modules.

EXERCISE 11. Let k be a field and let G be a finite group. Prove that a kG-module is projective if and only if it is injective. (Hint: every k-module is free).

2. Exact Colimits

In this section we will give a homological criterion for a R-module to be of type FP_n. This is sometimes called the Bieri-Eckmann criterion. The results of this section are taken from [**2**].

Let Γ be a directed graph without loops. A Γ-diagram in the category of R-modules is given by

(a) For all vertices v of Γ, a R-module M_v;
(b) For every edge from v to w, a R-module homomorphism $\varphi_{v,w} : M_v \to M_w$.

For every Γ-diagram M_*, we define the colimit, $colimM_*$ to be the R-module satisfying the following universal property:

- For every vertices v and w, here are R-module maps $f_v : M_v \to colimM_*$ such that $f_w \circ \varphi_{v,w} = f_v$.
- For every R-module X such that there are R-module maps $\varphi_v : M_v \to colimM_*$ such that $\varphi_w \circ \varphi_{v,w} = \varphi_v$, there is a unique R-module map $\psi : colimM_* \to X$ making the diagram commute.

Exercise 12. Show that $colimM_*$ exists and is unique.

Now let

$$F : R - mod \to Ab$$

be a covariant functor from the category of R-modules to the category of abelian groups. For example, let M be an R-module, then $Hom_R(M, -)$ and $Ext^k(M, -)$ are such functors. Then the universal property for colimits gives a well-defined homomorphism

$$colim(F(M_*)) \to F(colimM_*).$$

We say F $c$ommutes with colimits, if this map is an isomorphism.

We will be interested in graphs such that the colimit is an exact functor.

Example 3.15. (1) Let Γ be the graph consisting of a set of vertices I and no edges. Then

$$colimM_i = \bigoplus_{i \in I} M_i.$$

This is exact.

(2) Let Γ be a graph with the following property: For all vertices u and v, there is a vertex w such that there are directed edges from u to w and from v to w. Then

$$colimM_* = \varinjlim M_*$$

is the directed limit. This is also exact.

Proposition 3.16. *Let A be an R-module of type FP_n. Then for every exact colimit, the natural homomorphism*

$$colimExt^k(A, M_*) \to Ext^k(A, colimM_*)$$

is an isomorphism for $k < n$, and a monomorphism for $k = n$.

And now follows the Bieri-Eckmann criterion for Ext:

Theorem 3.17. *The following are equivalent for a left R-module A:*

(1) *A is of type FP_n.*
(2) *For every exact colimit, the natural homomorphism*

$$colimExt^k(A, M_*) \to Ext^k(A, colimM_*)$$

is an isomorphism for $k < n$, and a monomorphism for $k = n$.
(3) *For any direct system of R-modules M_* such that $\varinjlim M_* = 0$, one has*

$$\varinjlim Ext^k(A, M_*) = 0 \text{ for all } k \leq n.$$

The following proposition is now an easy consequence of the Bieri-Eckmann criterion, using the long exact sequence of Ext-functors.

PROPOSITION 3.18. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then the following holds:*
(1) *If L is of type FP_{n-1} and M is of type FP_n, then N is of type FP_n.*
(2) *If M is of type FP_{n-1} and N is of type FP_n, then L is of type FP_n.*
(3) *If L and N are of type FP_n, then M is of type FP_n.*

3. Proof of Brown's First Theorem

We now have enough to prove Theorem 2.35. Here is a sketch: We take the augmented cellular chain complex $C_*(X) \twoheadrightarrow \mathbb{Z}$ of X. Since X is n-acyclic, we have that

$$C_n(X) \to C_{n-1} \to ... \to C_0(X) \to \mathbb{Z} \to 0$$

is exact. Also recall that

$$C_i(X) = \bigoplus_{\sigma \in X^{(i)}} \mathbb{Z}[G/G_\sigma \cong \bigoplus_{\sigma \in X^{(i)}} \mathbb{Z}\uparrow_{G_\sigma}^G,$$

where σ are orbit representatives. Since X has a finite n-skeleton mod G, these sums are finite sums. Furthermore, n-good implies that G_σ is of type FP_{n-i}, and since induction takes projectives to projectives, it implies that $\mathbb{Z}\uparrow_{G_\sigma}^G$ is of type FP_{n-i}. Hence, as the sums are finite $C_i(X)$ is of type FP_{n-i}. Now apply Proposition 3.18 successively to the short exact sequences $0 \to K_i \to C_i(X) \to K_{i-1} \to 0$ for all $i \leq n-1$, noting that K_{n-1} is finitely generated. $\square$