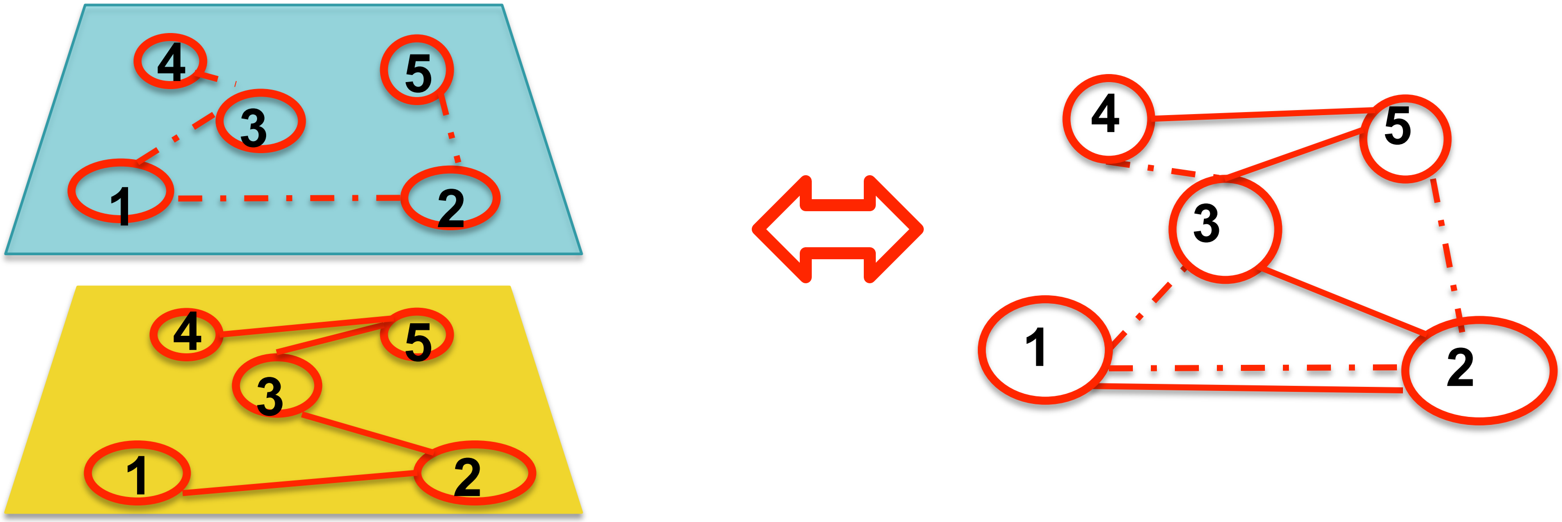

Multiplex Networks: structure of multiplex networks

LTCC Course Multilayer Networks
23-24 November 2016

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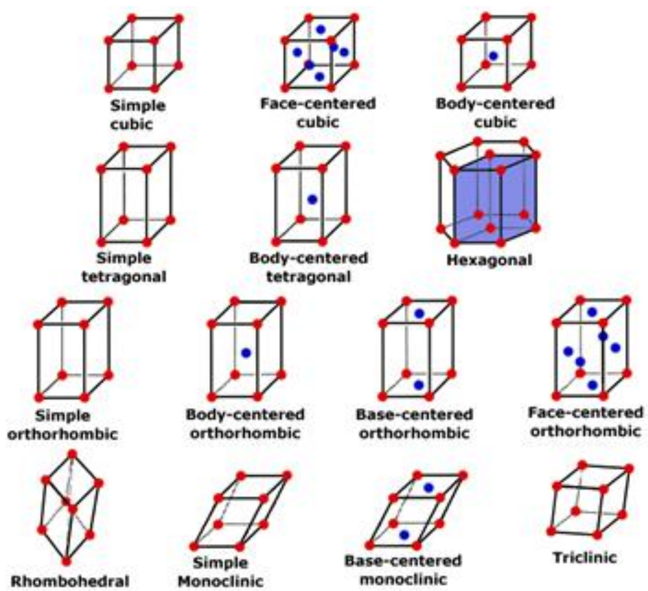
Queen Mary
University of London

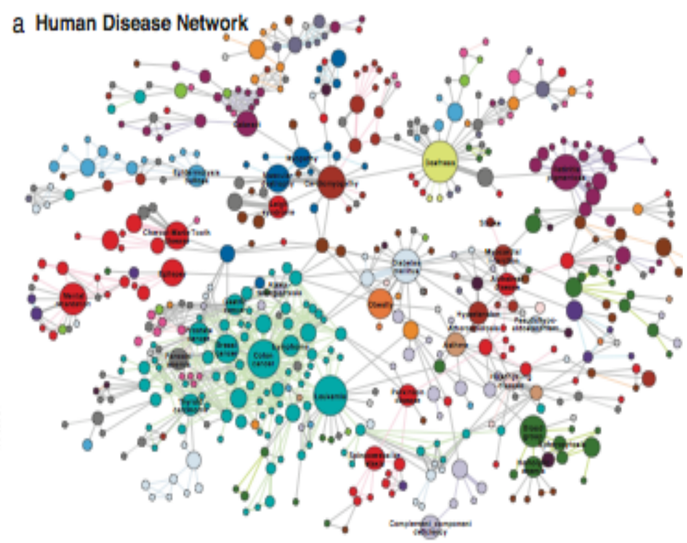
Networks
Encode Information

LATTICES



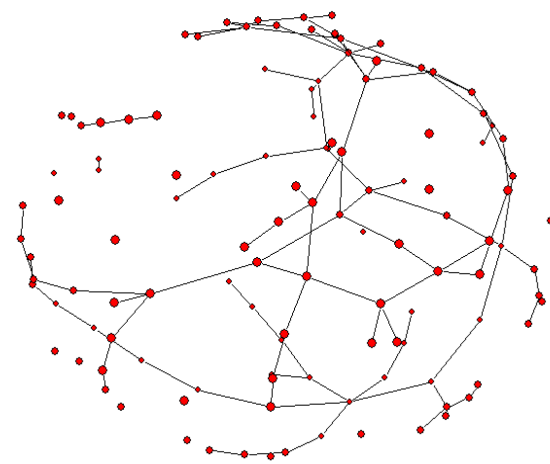
Regular networks
Symmetric

COMPLEX NETWORKS



Scale free networks
Small world
With communities
ENCODING
INFORMATION IN THEIR
STRUCTURE

RANDOM GRAPHS



Totally random
Poisson degree
distribution

Multilayer networks encode more information than single layers

Multilayer networks are not equivalent to a larger single network

Different types of links
describe different types of interactions,
therefore multilayer networks
encode more information than
their single layers
taken in isolation

Multilayers networks

In order to

progress in our understanding of

complex systems

we need to

develop new tools to

extract information

from

multilayer networks

Representation of a multiplex

A multiplex network of N nodes formed by M layers

is fully specified by

M adjacency matrices

$$\boxed{a^{[\alpha]}}$$

with α=1, 2, … M

of matrix elements

$$\boxed{a_{ij}^{[\alpha]} = \begin{cases} 1 \ \ if \ \ node \ i \ and \ node \ \ j \ are \ linked \ in \ layer \ \alpha \\ 0 \qquad\qquad\qquad otherwise \end{cases}}$$

Aggregated network

The aggregated network

is the network in which we consider every interaction on the same footing,

i.e. we neglect information about the layers.

The adjacency matrix of the

aggregated network is

$$\tilde{a}_{ij} = \begin{cases} 1 & if \displaystyle\sum_{\alpha=1,..M} a_{ij}^{[\alpha]} > 0 \\ 0 & otherwise \end{cases}$$

Multiplex degree

The degree of a node

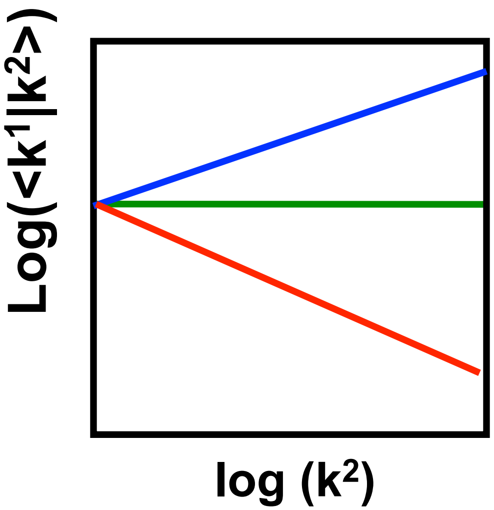
in a multiplex network is a vector

$$\mathbf{k_i} = (k_i^{[1]}, k_i^{[2]}, \ldots k_i^{[M]})$$

with

$$k_i^{[\alpha]} = \sum_{j=1,N} a_{ij}^{[\alpha]}$$

Detecting degree correlations between two layers



Positive degree correlations
(Hubs are hubs in both layers,
low degree nodes have low degree in both layers)

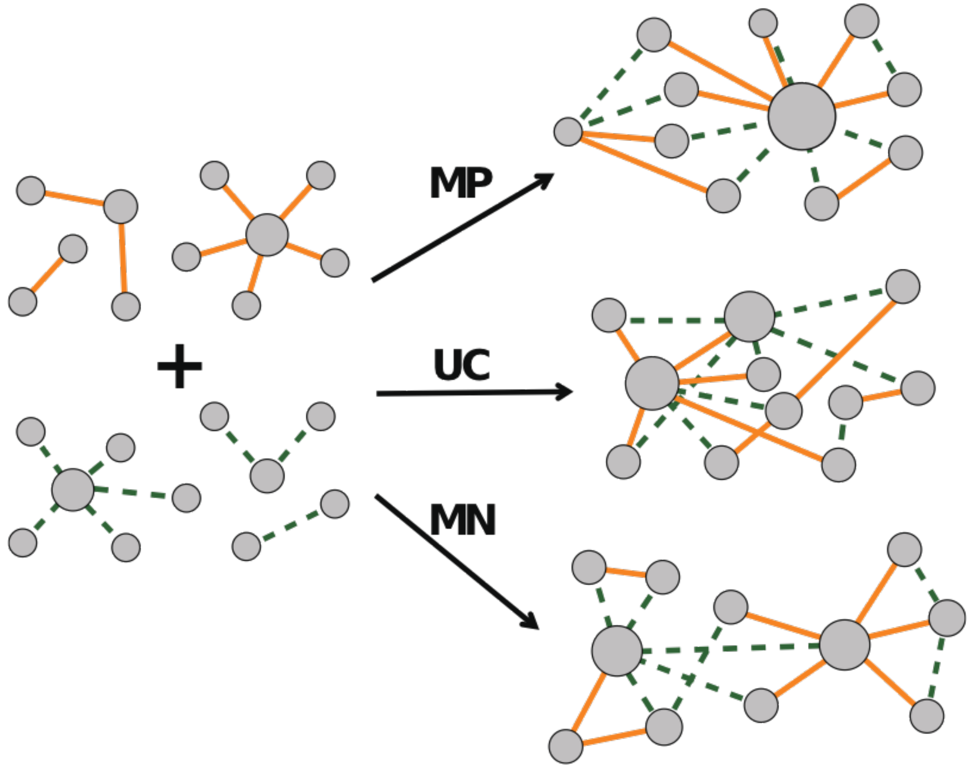
No degree correlations

Negative degree correlations
(Hubs in one layer are low degree nodes in the other)

$$\left\langle k^{[1]} \middle| k^{[2]} \right\rangle = \sum_{k^1} k^{[1]} P(k^{[1]} \mid k^{[2]})$$

$P(k^{[1]}|k^{[2]})$
probability that a node has degree $k^{[1]}$ in layer 1
given that
it has degree $k^{[2]}$ in other layer 2

Tuning the degree correlations across two layers

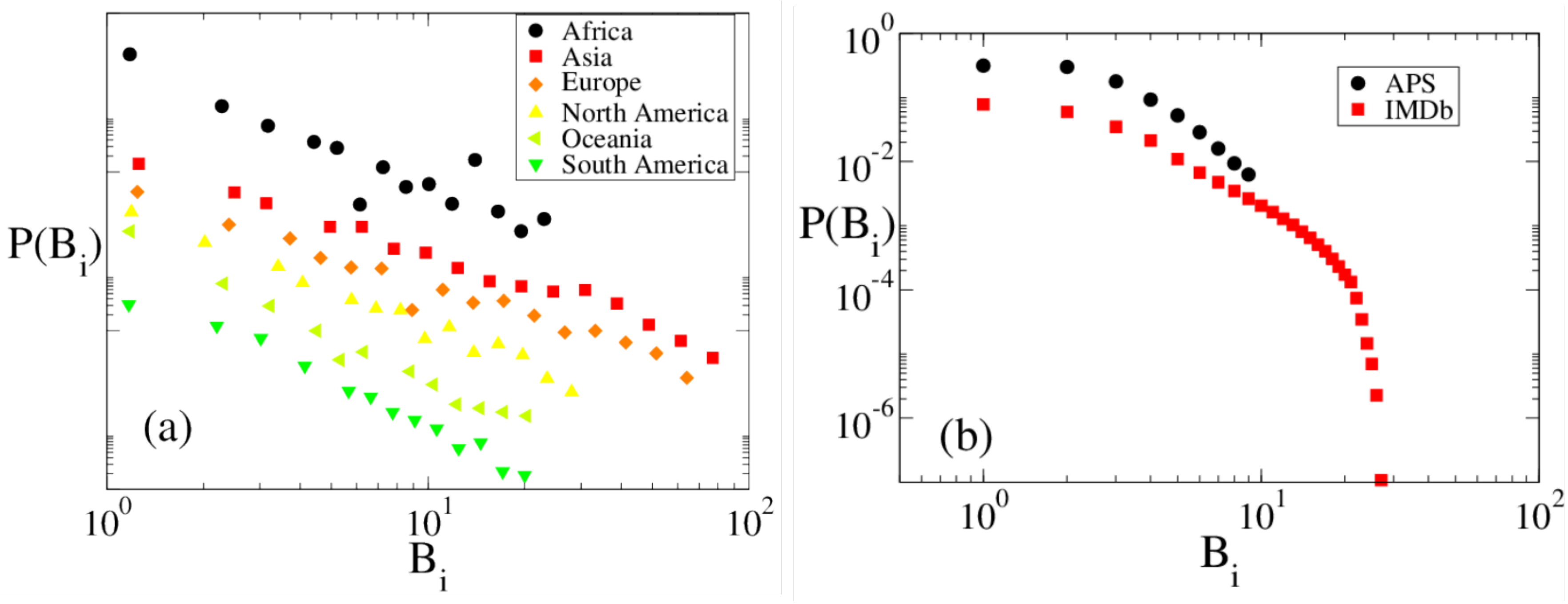


By relabelling the nodes of two layers it is possible to build
Maximum positive (MP)
Maximum negative (MN)
and Uncorrelated (UC)
Multiplex Networks.

B. Min et al. PRE (2014)

Activity of a node

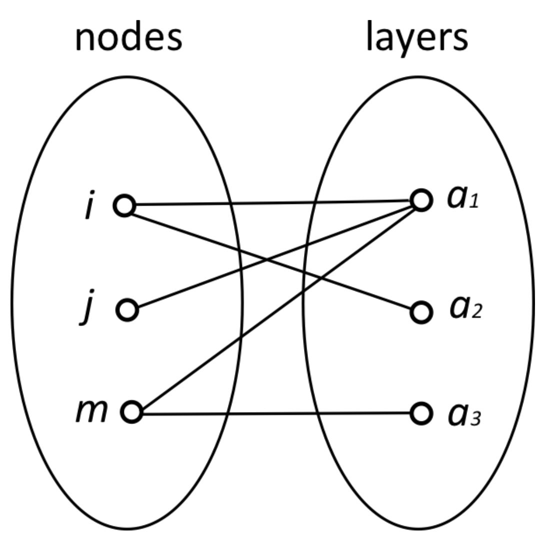
The activity B_i of a node i is equal to the number of layers in which the node is connected



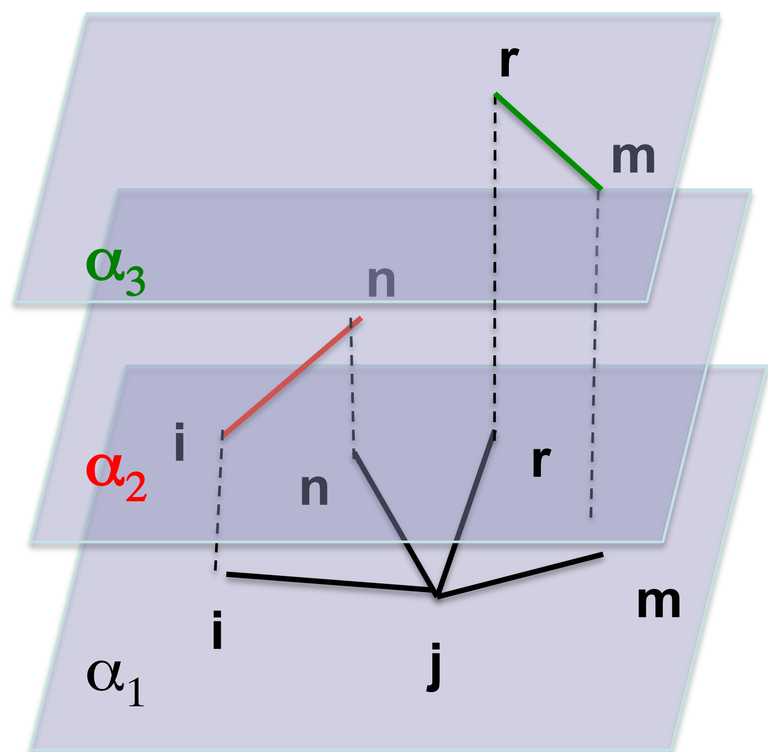
V. Nicosia, V.Latora PRE (2015)

Multiplex networks with heterogeneous activity of the nodes

Bipartite network:
Nodes and Layers



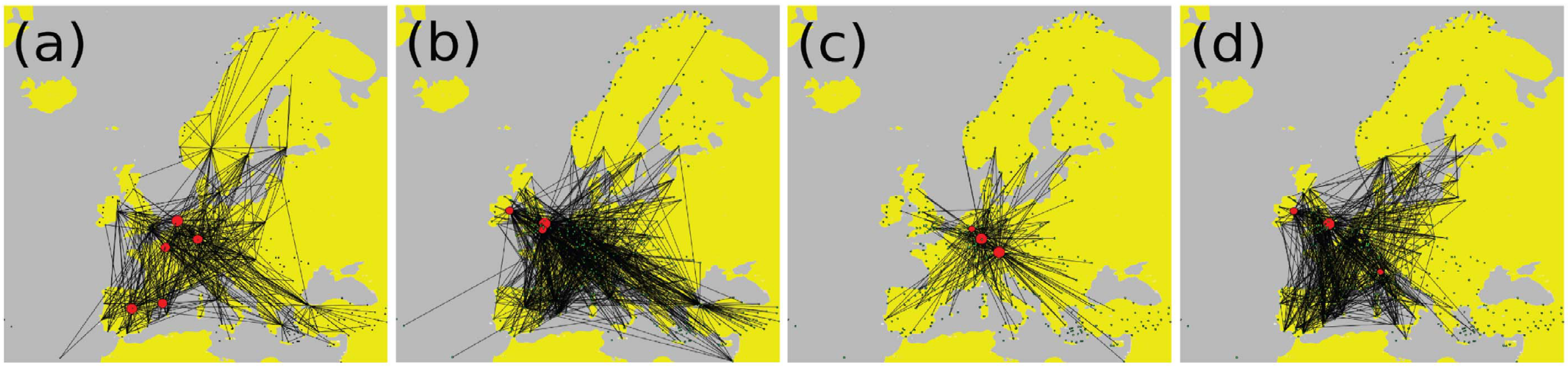
Multiplex network



D. Cellai et al., PRE 93, 032302 (2016)

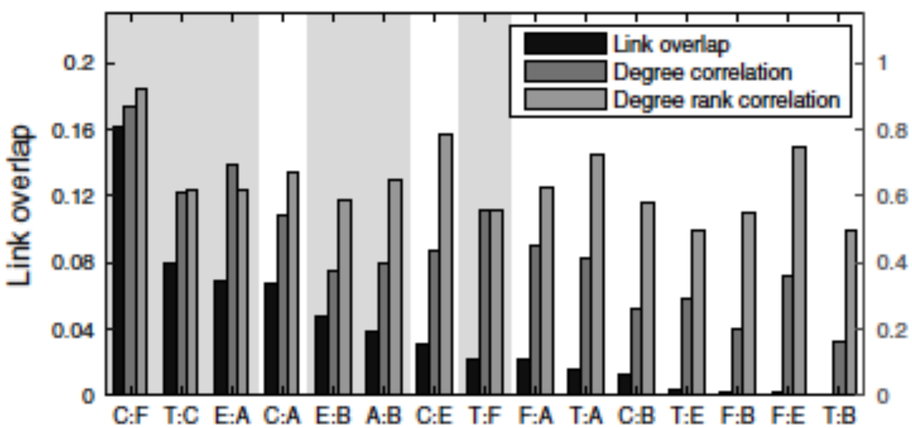
Multiplex networks with link overlap

Overlap in multiplex networks



- (a) Only links belonging to more than one airline company are plotted

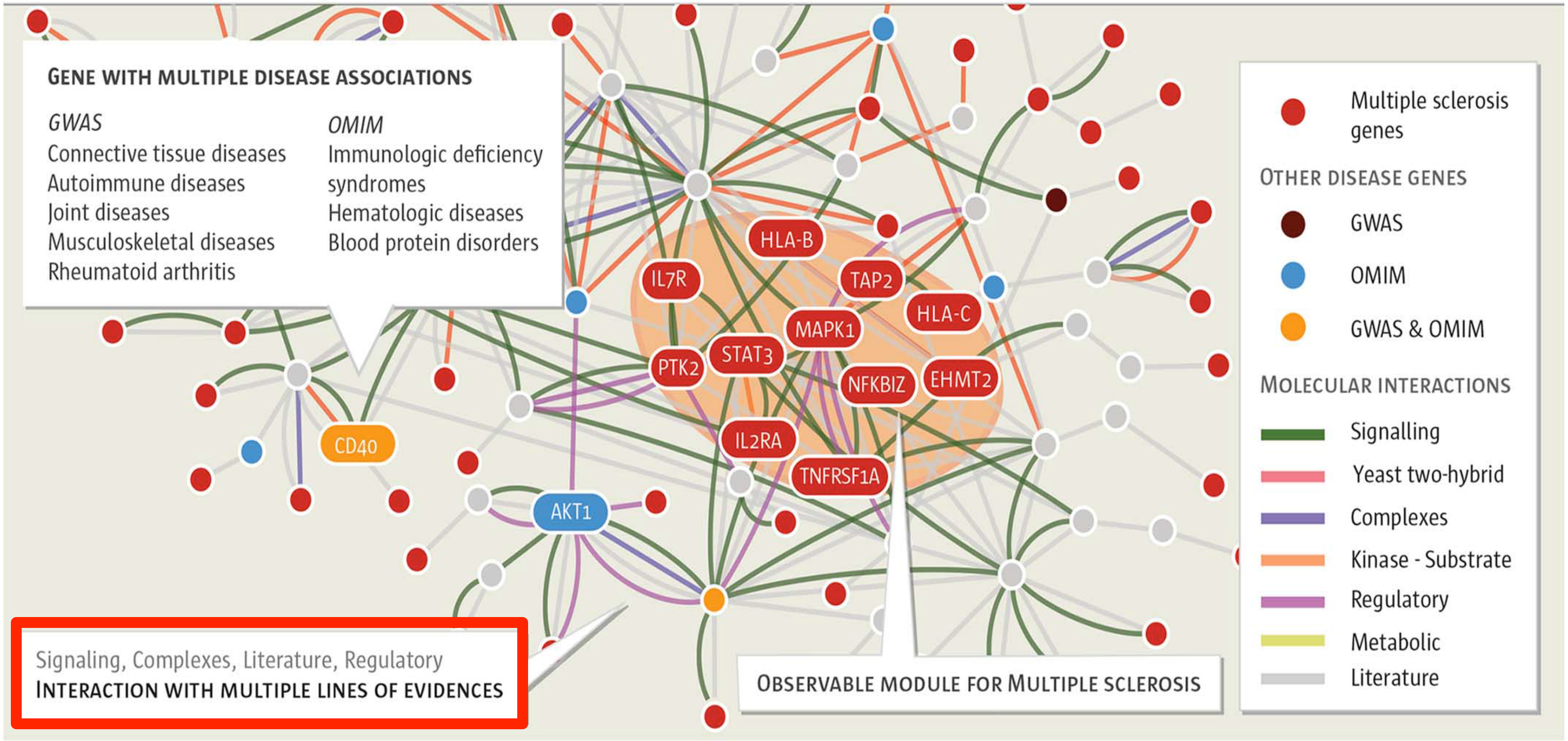
Cardillo et al. Scientific Reports (2013).



Social network of online social game

Szell et al . PNAS 2010

Interactions with multiple lines of evidences



Overlap

The total overlap $O^{\alpha\alpha}$

between layer α and layer α'

is given by'

$$\boxed{O^{\alpha,\alpha'} = \sum_{i<j} a_{ij}^{\alpha} a_{ij}^{\alpha'}}$$

The local overlap $o_i^{\alpha,\alpha}$

of node i between layer α and layer α'

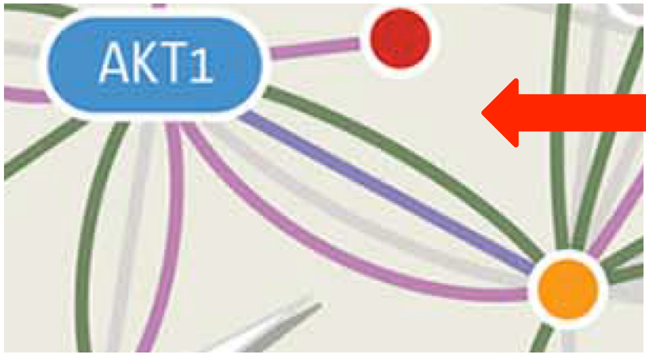
is given by

$$\boxed{o_i^{\alpha,\alpha'} = \sum_{j} a_{ij}^{\alpha} a_{ij}^{\alpha'}}$$

Multiplicity of link overlap

The multiplicity of link overlap is the number of layers in which a given link is present

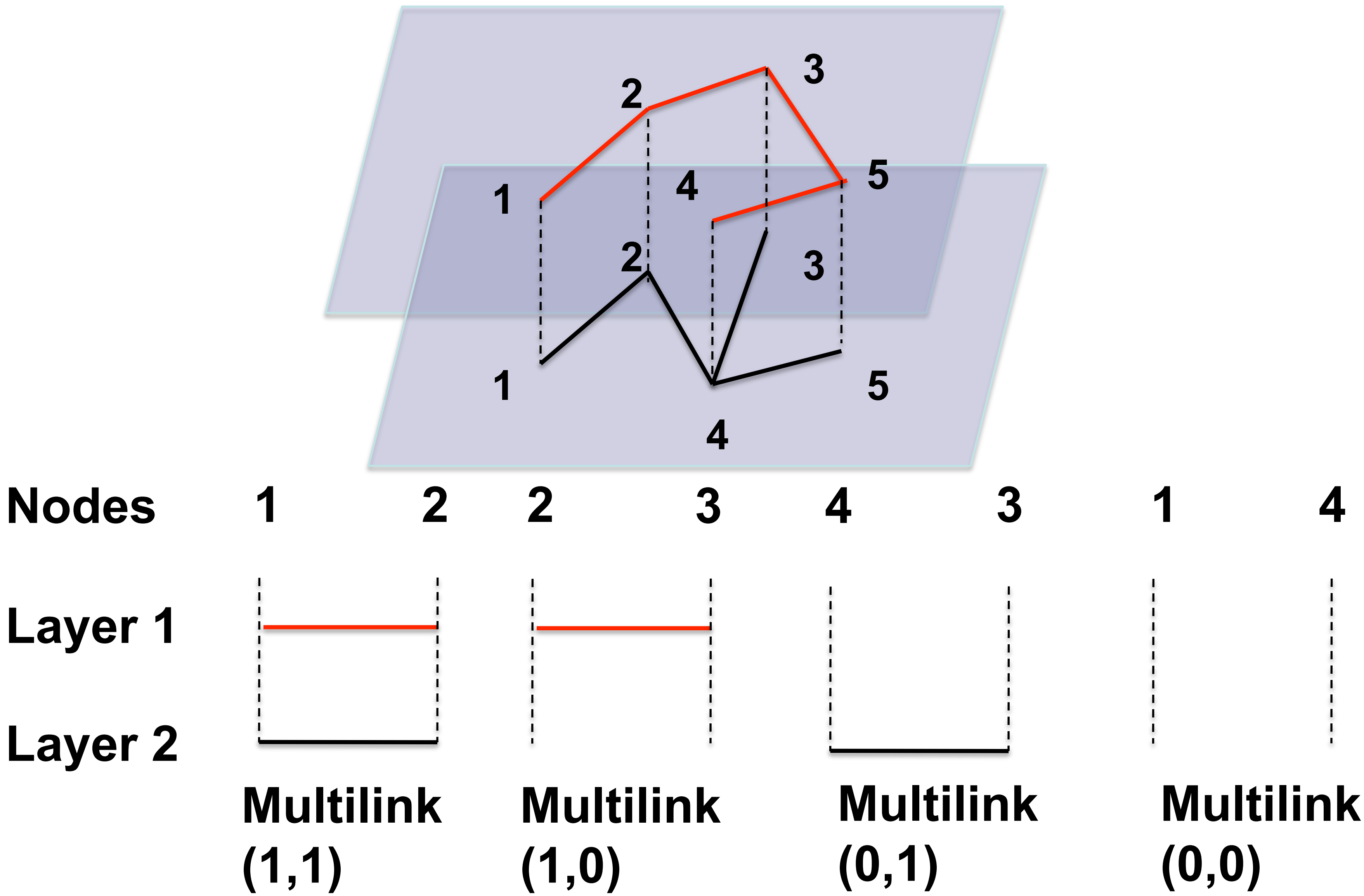
$$\boxed{\mu_{ij} = \sum_{\alpha} a_{ij}^{\alpha}}$$



$$\mu_{ij} = 4$$

Multilinks

G. Bianconi
PRE (2013)



Case of two layers

Multiadjacency matrices

$$A_{ij}^{10} = \begin{cases} 1 & if \;\; node\ i\ and\ node\ j\ are\ linked\ in\ layer\ 1\ and\ not\ linked\ in\ layer\ 2 \\ 0 & otherwise \end{cases}$$

$$A_{ij}^{01} = \begin{cases} 1 & if \;\; node\ i\ and\ node\ j\ are\ linked\ in\ layer\ 2\ and\ not\ linked\ in\ layer\ 1 \\ 0 & otherwise \end{cases}$$

$$A_{ij}^{11} = \begin{cases} 1 & if \;\; node\ i\ and\ node\ j\ are\ linked\ in\ layer\ 1\ and\ in\ layer\ 2 \\ 0 & otherwise \end{cases}$$

$$A_{ij}^{00} = \begin{cases} 1 & if \;\; node\ i\ and\ node\ j\ are\ not\ linked\ in\ layer\ 1\ and\ not\ linked\ in\ layer\ 2 \\ 0 & otherwise \end{cases}$$

Constraints on the multiadjacency matrices

$$\boxed{A_{ij}^{10} + A_{ij}^{01} + A_{ij}^{11} + A_{ij}^{00} = 1}$$

Multidegree

The multidegree $k_i^{\vec{m}}$ of a node i is defined as the number of multilinks

$$\vec{m} = (m_1, \ldots m_M)$$

incident to it

It is given by

$$\boxed{k_i^{\vec{m}} = \sum_{j=1,..N} A_{ij}^{\vec{m}}}$$

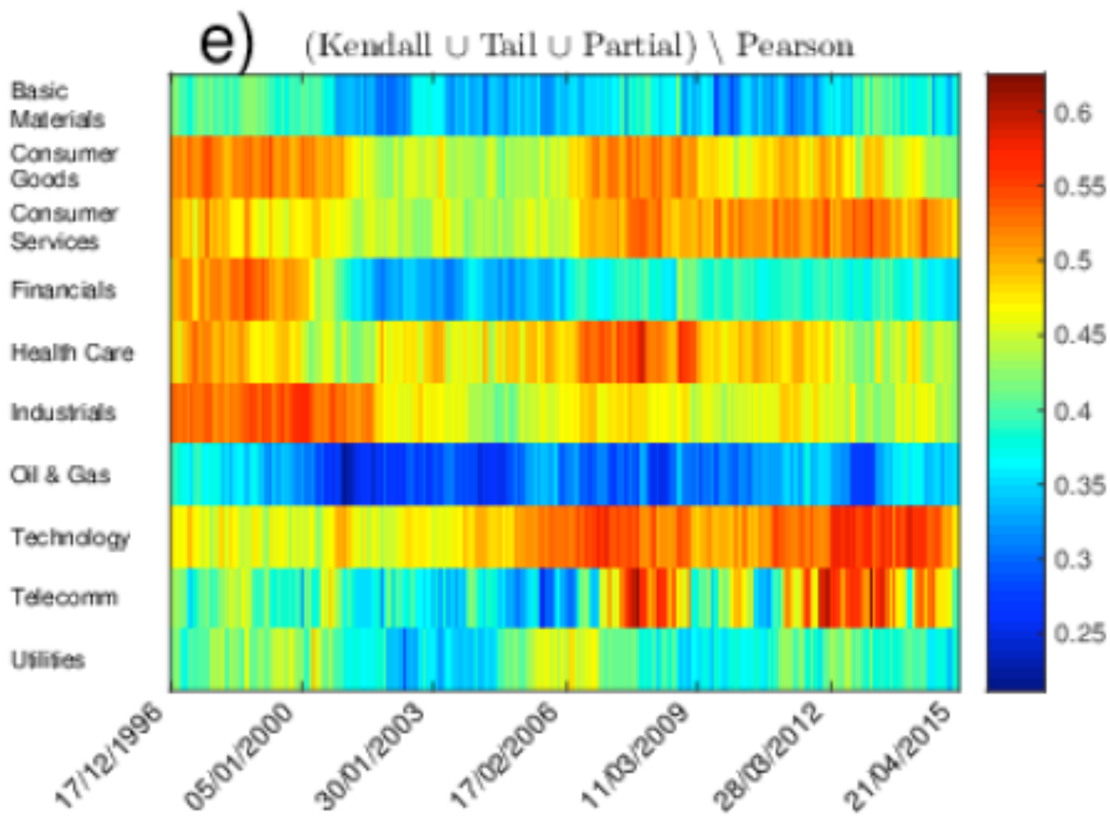
In the case of two layers we have

$$k_i^{10} = \sum_j a_{ij}^1 (1 - a_{ij}^2)$$

$$k_i^{01} = \sum_j (1 - a_{ij}^1) a_{ij}^2$$

$$k_i^{11} = \sum_j a_{ij}^1 a_{ij}^2$$

Multidegrees in financial networks



Musmeci et al. (2016)

Weighted multiplex networks with link overlap

Strength vs degree

The strength s_i of a node i is equal
to the sum of the weights

$$s_i = \sum_j w_{ij}$$

The average strength s_k of nodes of degree k
can either grow
linearly
(homogeneous distribution of the weights)
or
non-linearly
(hubs have in average links with stronger weights)

Barrat et al. PNAS (2004)

Multi-strength

The multi-strength $s_i^{\vec{m},[\alpha]}$

evaluates the sum of the weights

of multi-links $\vec{m}$

of node i in layer α

The multi-strength allows

to condition on

the presence of the absence of the

link overlap

G. Menichetti et. al. Plos One (2014)

Multistrength in a Duplex network

Strength on the first layer restricted to links

with no overlap - with overlap

$$s_i^{(1,0),[1]} = \sum_j w_{ij}^{[1]}(1 - a_{ij}^{[2]}) \qquad s_i^{(1,1),[1]} = \sum_j w_{ij}^{[1]} a_{ij}^{[2]}$$

Strength on the second layer restricted to links

with no overlap - with overlap

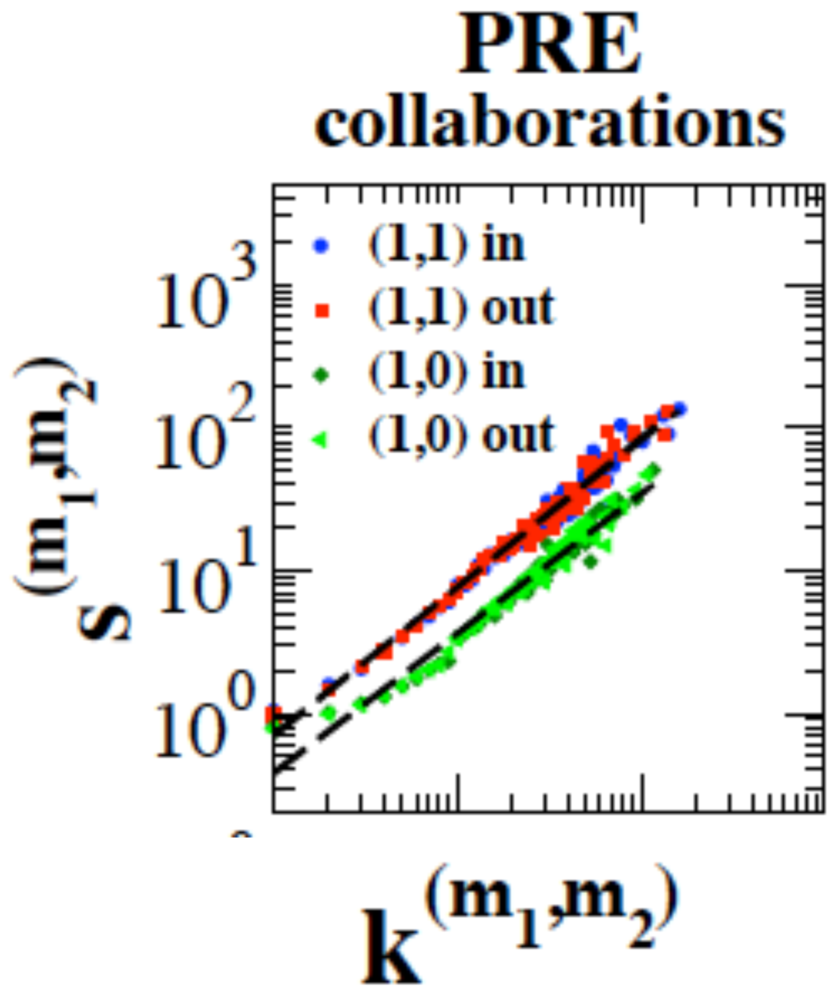
$$s_i^{(0,1),[2]} = \sum_j (1 - a_{ij}^{[1]}) w_{ij}^{[2]} \qquad s_i^{(1,1),[2]} = \sum_j a_{ij}^{[1]} w_{ij}^{[2]}$$

G. Menichetti et. al. Plos One (2014)

Multi-strength in the collaboration layer of the citation/collaboration duplex

The average weight of a link in the collaboration network depends on the existence of a link in the citation network.

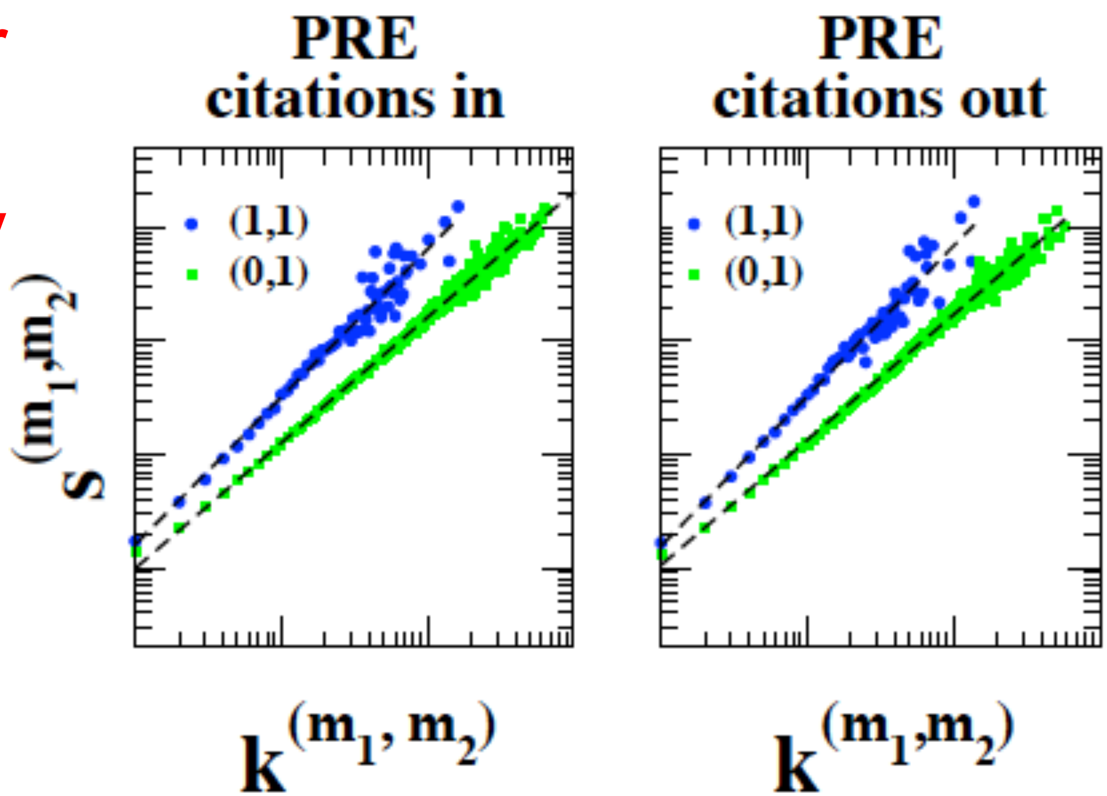
The dependence of the multistrenth vs. multidegree remains linear in both cases.



Multistrength vs multidegree in the citation layer of the citation/collaboration duplex

The way you cite your collaborators is different from the way you cite the other scientists.

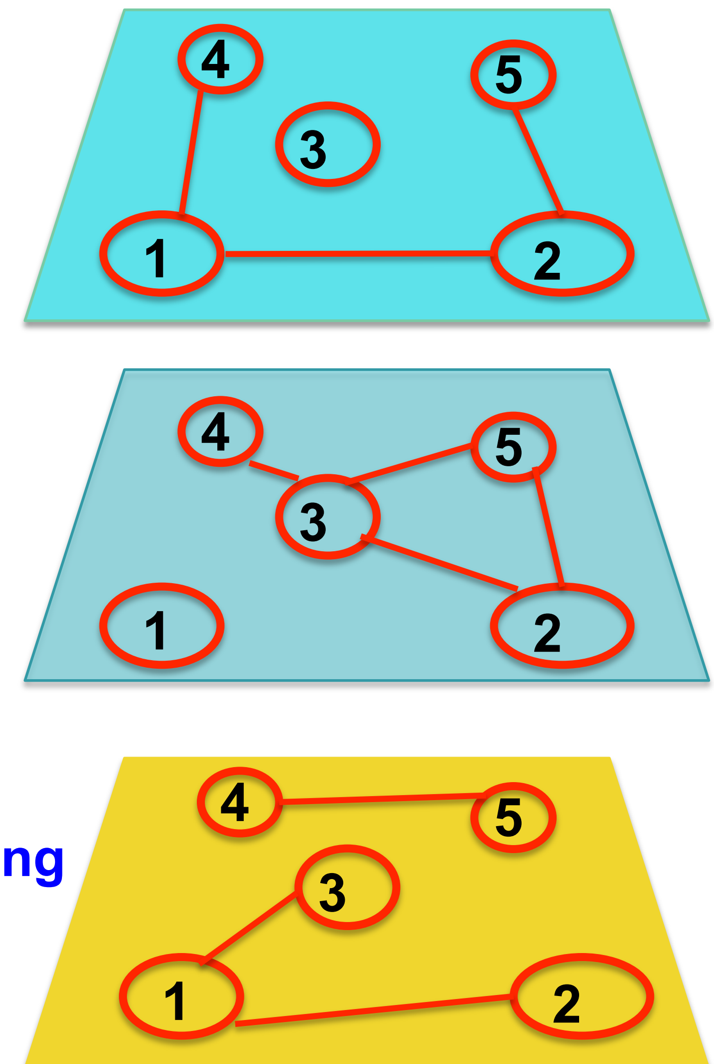
People tend to cite more the hubs with whom they have collaborated.



G. Menichetti et. al. Plos One (2014)

Clustering coefficient among three layers

$$C_i^{[\alpha,\alpha',\alpha'']} = \frac{\sum_{j\neq i, m\neq i} a_{ij}^{[\alpha]} a_{jm}^{[\alpha']} a_{mi}^{[\alpha'']}}{\sum_{j\neq i, m\neq i} a_{ij}^{[\alpha]} a_{mi}^{[\alpha'']}}$$



Fraction of pair of friends that are friend with each other across different layers

Keeps track of all the layers
Can become computationally demanding

Clustering coefficient

The clustering coefficients of a multiplex networks consider all the layers on the same footing

$$C_{i,1} = 2\frac{\sum_{\alpha=1}^{M}\sum_{\mu|\mu\neq\alpha}\sum_{j,k} a_{ij}^{[\alpha]}a_{jk}^{[\mu]}a_{ki}^{[\alpha]}}{(M-1)\sum_{\alpha=1}^{M} k^{[\alpha]}(k^{[\alpha]}-1)/2},$$

$$C_{i,2} = 2\frac{\sum_{\alpha=1}^{M}\sum_{\kappa|\kappa\neq\alpha}\sum_{\mu|\mu\neq\alpha,\kappa}\sum_{j,k} a_{ij}^{[\alpha]}a_{jk}^{[\mu]}a_{ki}^{[\kappa]}}{(M-2)\sum_{\alpha=1}^{M}\sum_{\kappa\neq\alpha} k^{[\alpha]}k_i^{[\kappa]}},$$

$C_{i,1}$ ($C_{i,2}$) evaluates the normalized number of triangles of node i belonging to two (three) layers

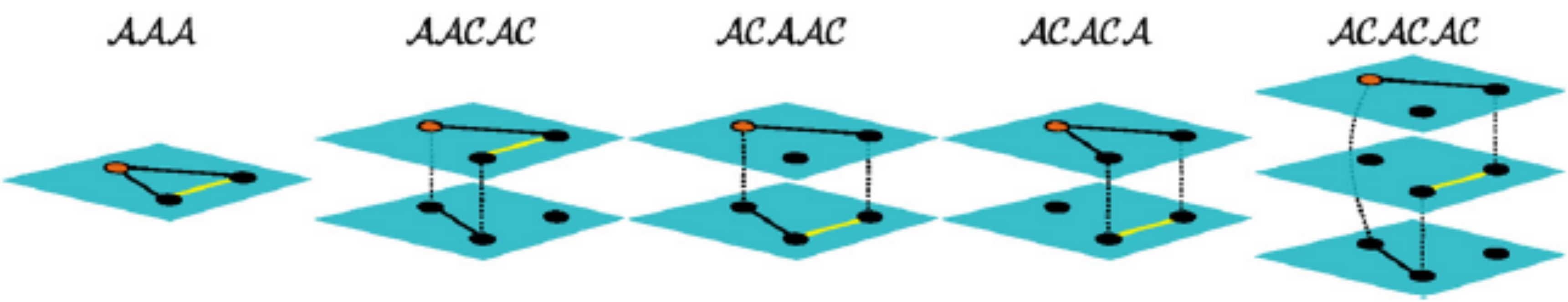
Battiston et al (2013).

Multilayer clustering coefficient

By associating a "cost" *t* to changing layers,

it is possible to define a functional clustering coefficient depending on *t* and encoding

different ways in which triadic closure is achieved



Cozzo et al. New Journal of Physics (2015)

Multilayer communities

Modularity of a single layer

The Modularity is a measure to evaluate the significance of a certain community structure

$$M = \frac{1}{2\mu} \sum_{ij} \left[\left(a_{ij} - \frac{k_i k_j}{\langle k \rangle N} \right) \right] \delta(g_i, g_j)$$

it measure how dense is a community with respect to the uncorrelated network structure with the same degree sequence

Multilayer modularity

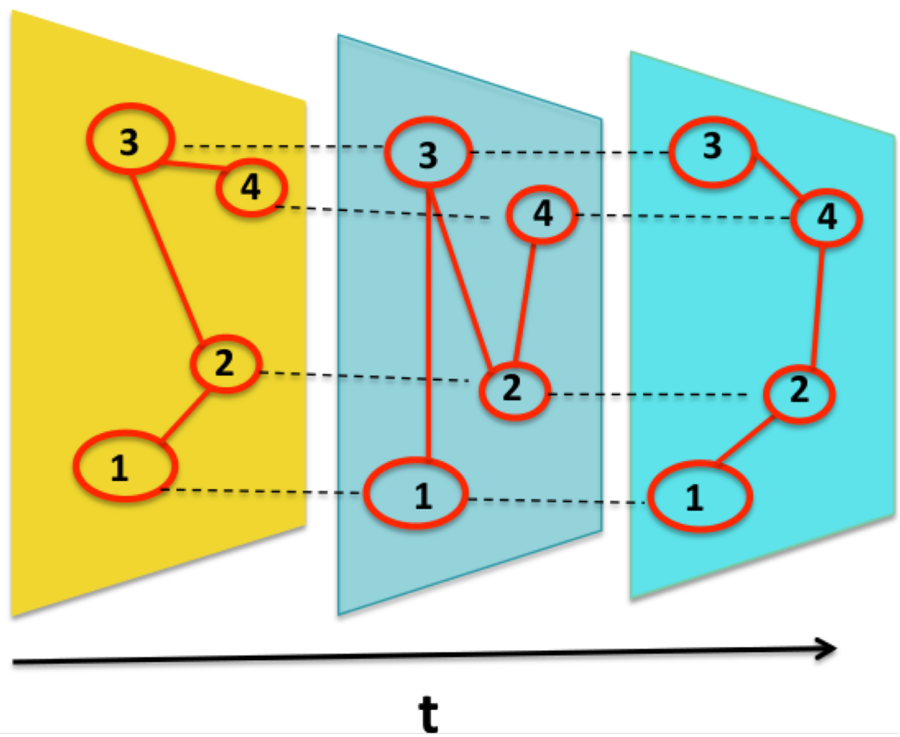
Communities can spam across different layers,

they can be found by optimizing the multilayer modularity $Q_{\text{multislice}}$

$$Q_{multislice} = \frac{1}{2\mu}\sum_{ij\alpha\beta}\left[\left(a_{ij}^{[\alpha]} - \gamma^{[\alpha]}\frac{k_i^{[\alpha]}k_j^{[\alpha]}}{\left\langle k^{[\alpha]}\right\rangle N}\right)\delta_{\alpha,\beta} + \delta_{ij}C_{jj}^{[\alpha,\beta]}\right]\delta(g_i^{[\alpha]}, g_j^{[\beta]})$$

P. J. Mucha, et al. Science (2010)

Temporal or multi-slice networks



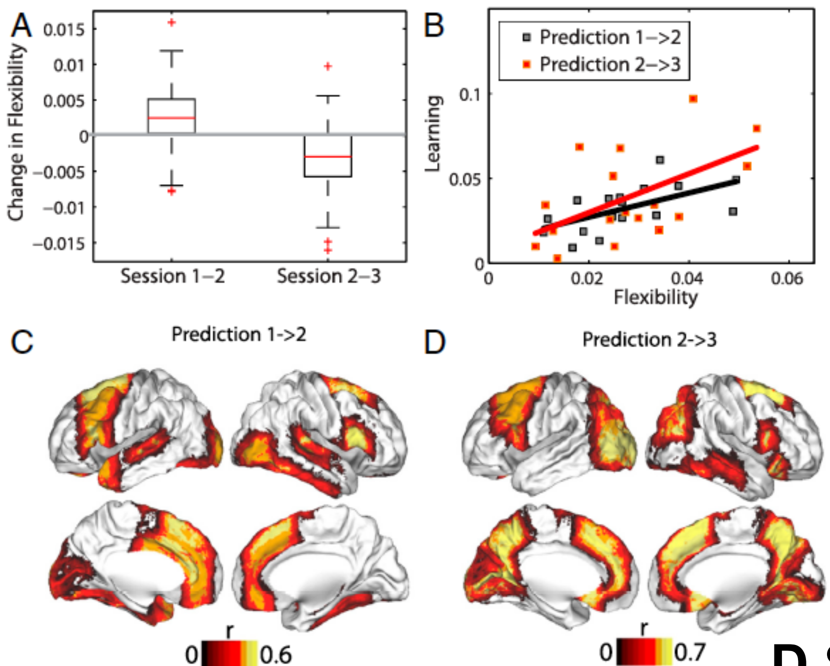
Temporal networks can be seen as a multi-slice network where each slice is a temporal snapshot

Flexibility

The flexibility f_i of a node i

is the number of times

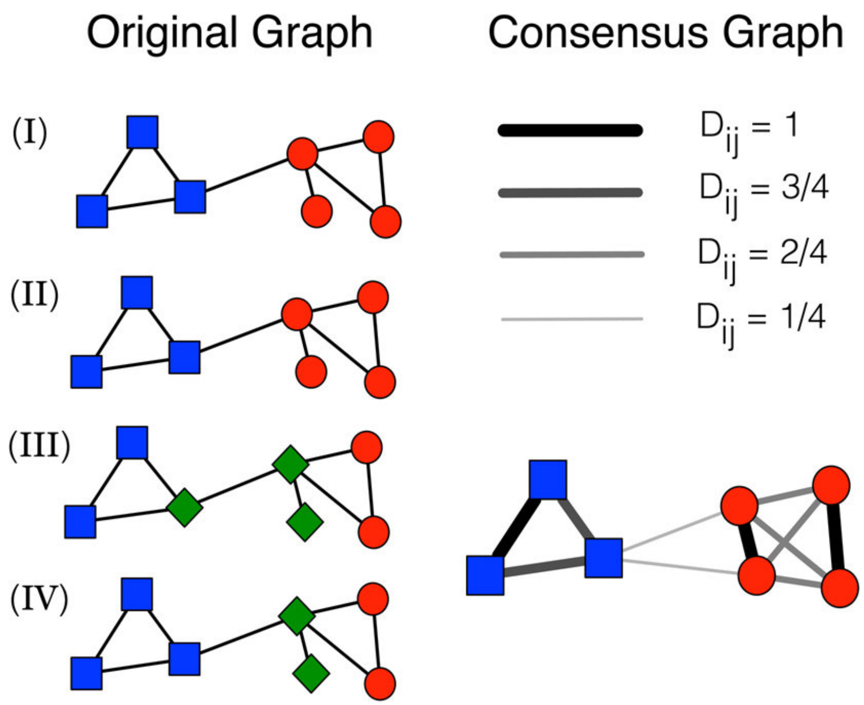
the node changes community assignment



Correlation between flexibility and learning in brain functional networks

D.S. Bassett PNAS (2011)

Consensus clustering for detects multilayer communities

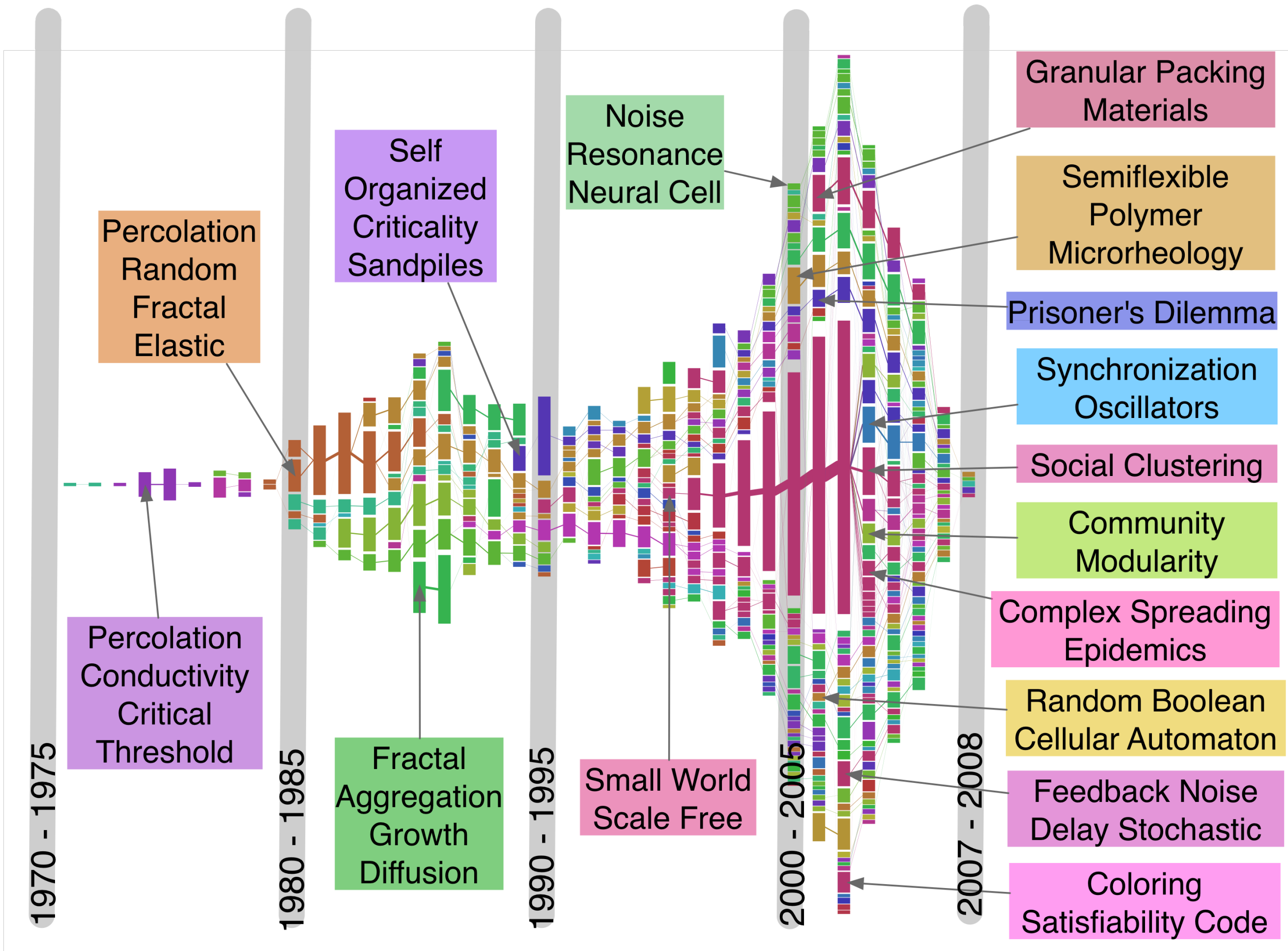


The consensus graph is constructed by comparing the communities in different layers of a multiplex network.

The consensus graph reveals the multilayer communities

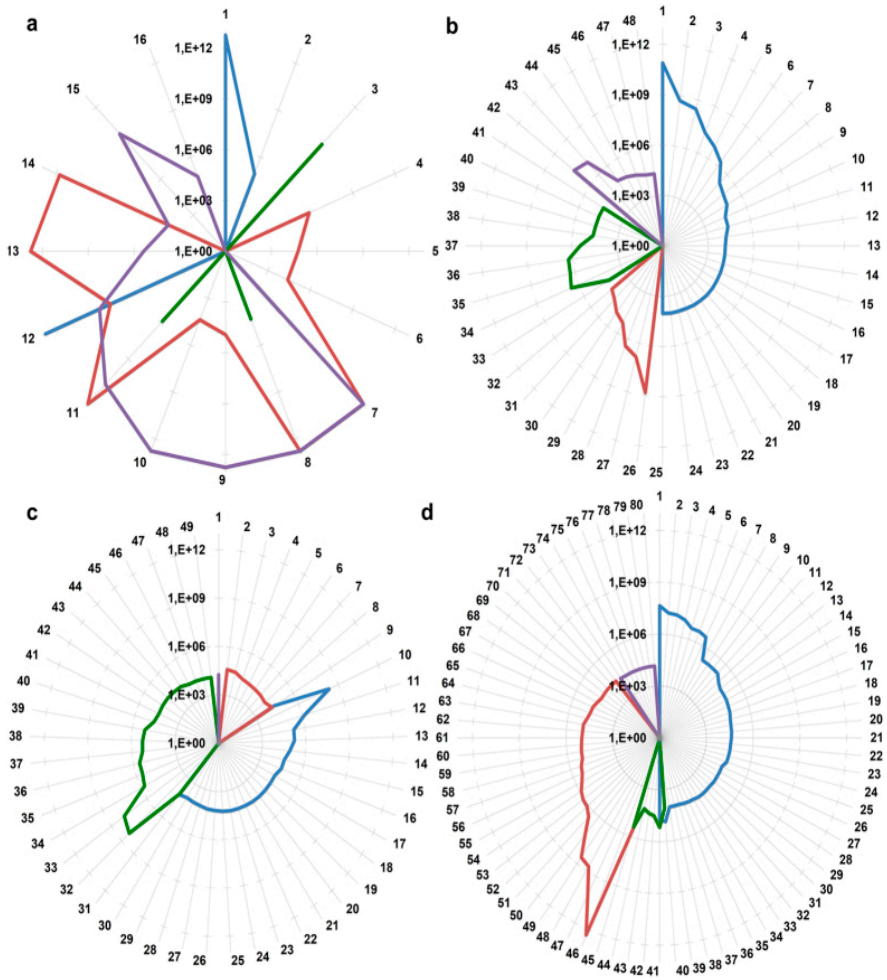
Lancichinetti Fortunato (2012)

Evolution of communities in temporal networks



Lancichinetti Fortunato (2012)

Enrichment in oncogenic biological components



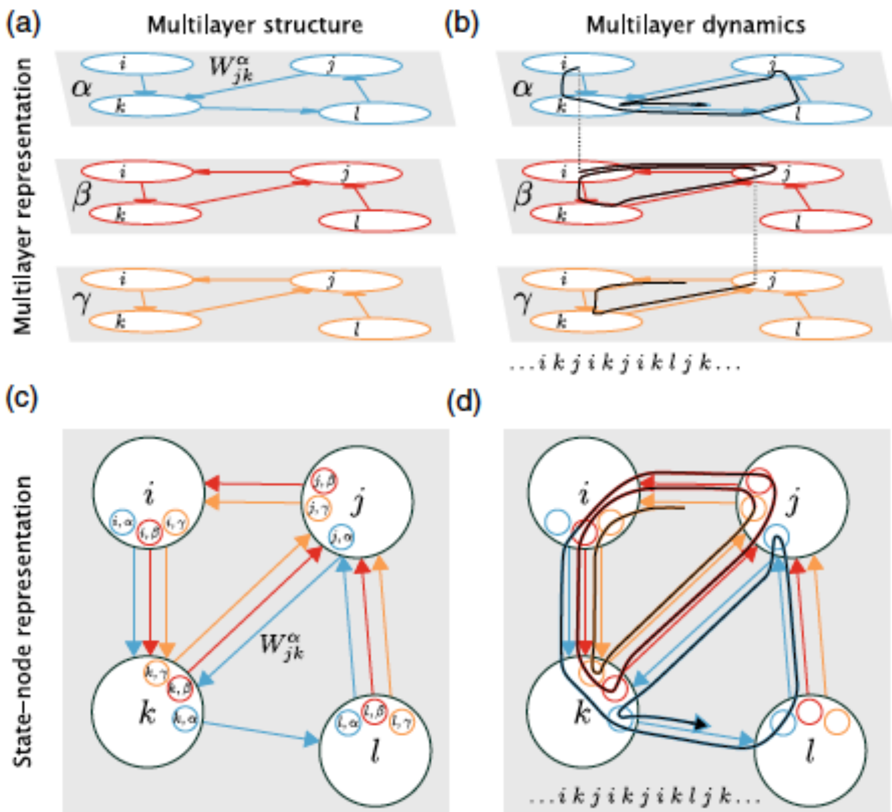
Multiplex network of four layers:

- co-expression network,
- transcription factor (TF) co-targeting network,
- microRNA co-targeting network
- protein-protein interaction network (PPI)

The enrichment p-values for (**a**) chromosomes, (**b**) pathways, (**c**) TF/microRNAs motifs and (**d**) GO. The four tissues are indicated by different colors: gastric (blue), lung (red), pancreas (green) and colon (violet).

Cantini et al. Scietific Reports 2015

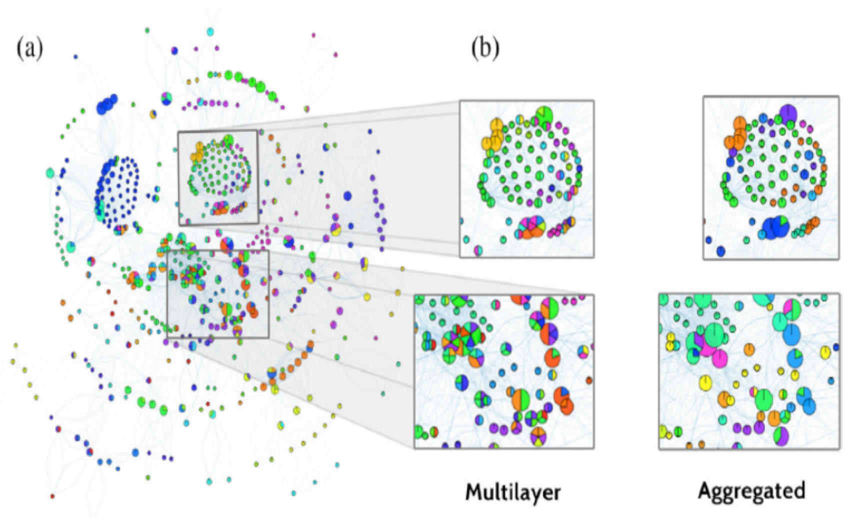
Community detection using diffusion properties



Diffusion along the interlinks can be used to characterize communities as the random walk tends to be localized on communities for short-meso timescales

De Domenico et al. PRX (2015)

Multilayer communities do not reduce to single layer communities

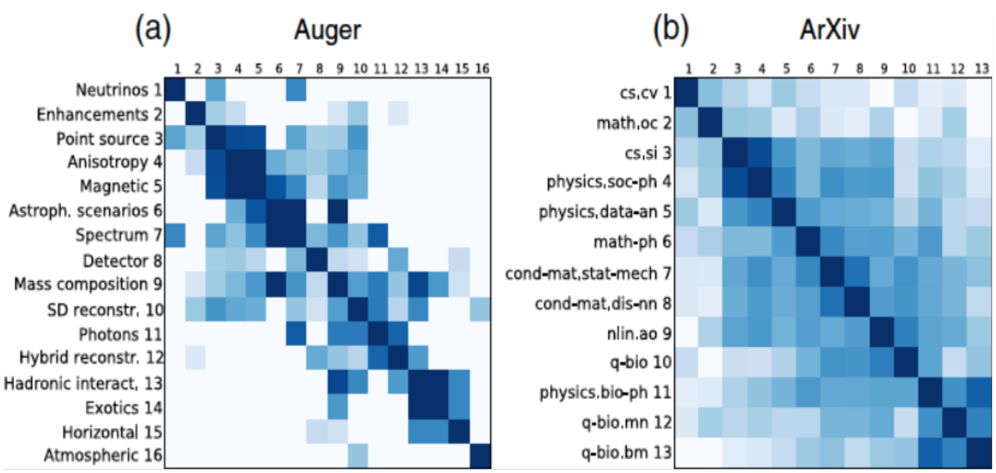


Each node might belong to more than one community

Auger collaboration network

De Domenico et al. PRX (2015)

Similarities between the layers

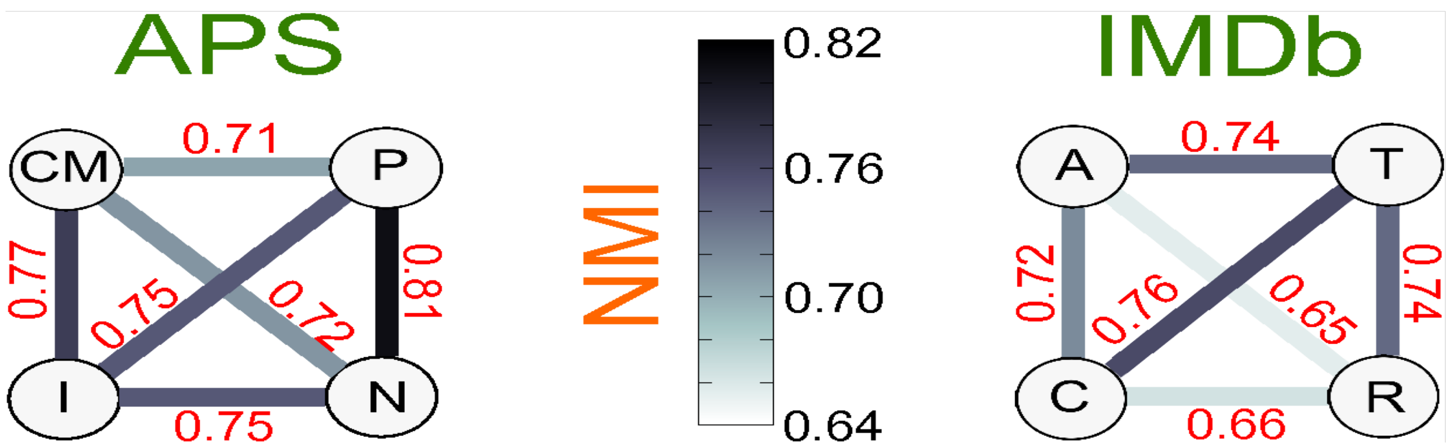


De Domenico et al. PRX (2015)

The layer similarity can be taken to be the number of replica nodes of the two layers belonging to the same community

Correlated mesoscale structure of the multiplex layers

The community structure of different layers is correlated



The Normalized Mutual Information

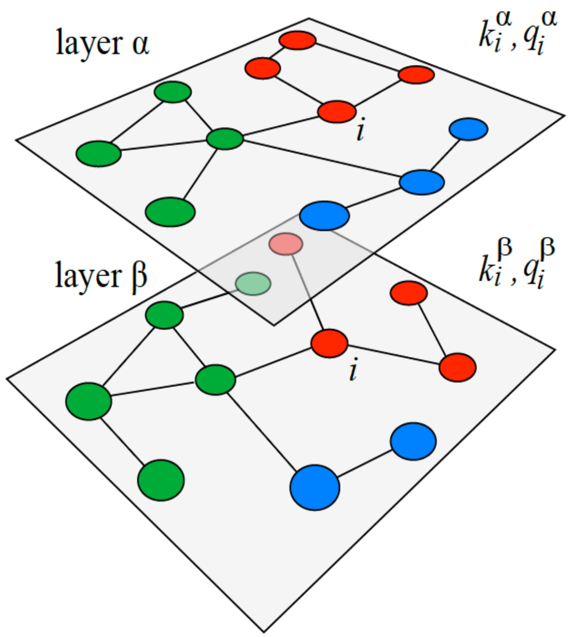
$$NMI = \frac{-2\sum_{m}\sum_{m'} N_{mm'} \log\left(\frac{N_{mm'}N}{N_m N_{m'}}\right)}{\sum N_m \log\left(\frac{N_m}{N}\right) + \sum N_{m'} \log\left(\frac{N_{m'}}{N}\right)}$$

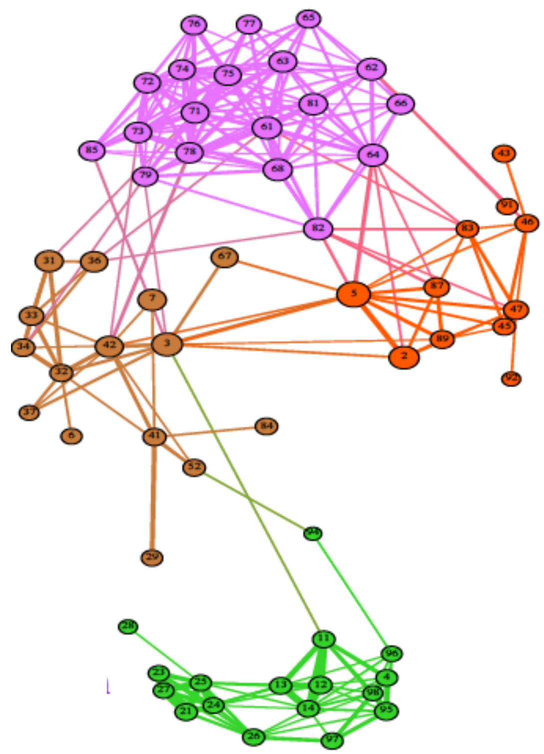
APS	N	$\langle k \rangle$	C
Nuclear (N)	1238	4.75	0.27
Particle (P)	1238	4.66	0.30
Cond. Matt. I (CM)	1238	10.29	0.24
Interdisciplinary (I)	1238	7.37	0.26

IMDb	N	$\langle k \rangle$	C
Action (A)	55797	83.56	0.61
Crime (C)	55797	82.30	0.58
Romance (R)	55797	86.00	0.59
Thriller (T)	55797	77.75	0.56

Battiston et al. PlosOne 2016

Network entropy reveals the network between the layers of multiplex datasets





The community structure of the layers of a multiplex network reveals the "networks of layers". Case of collaboration networks revealing the network between the PACS.

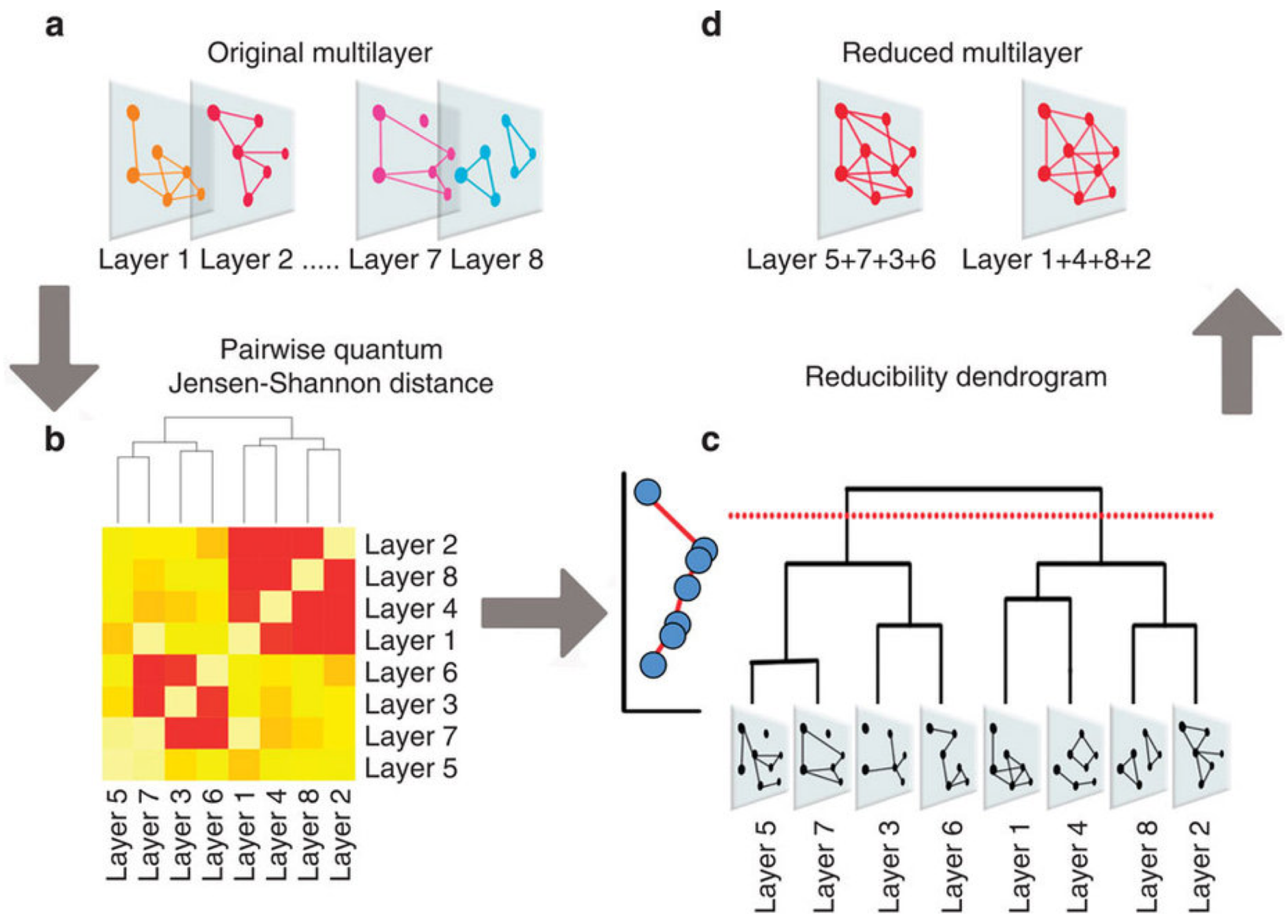
J. Iacovacci, et al. PRE (2015)

To aggregate

or

to disaggregate?

Reducibility of networks



De Domenico et al Nature Com. 2015

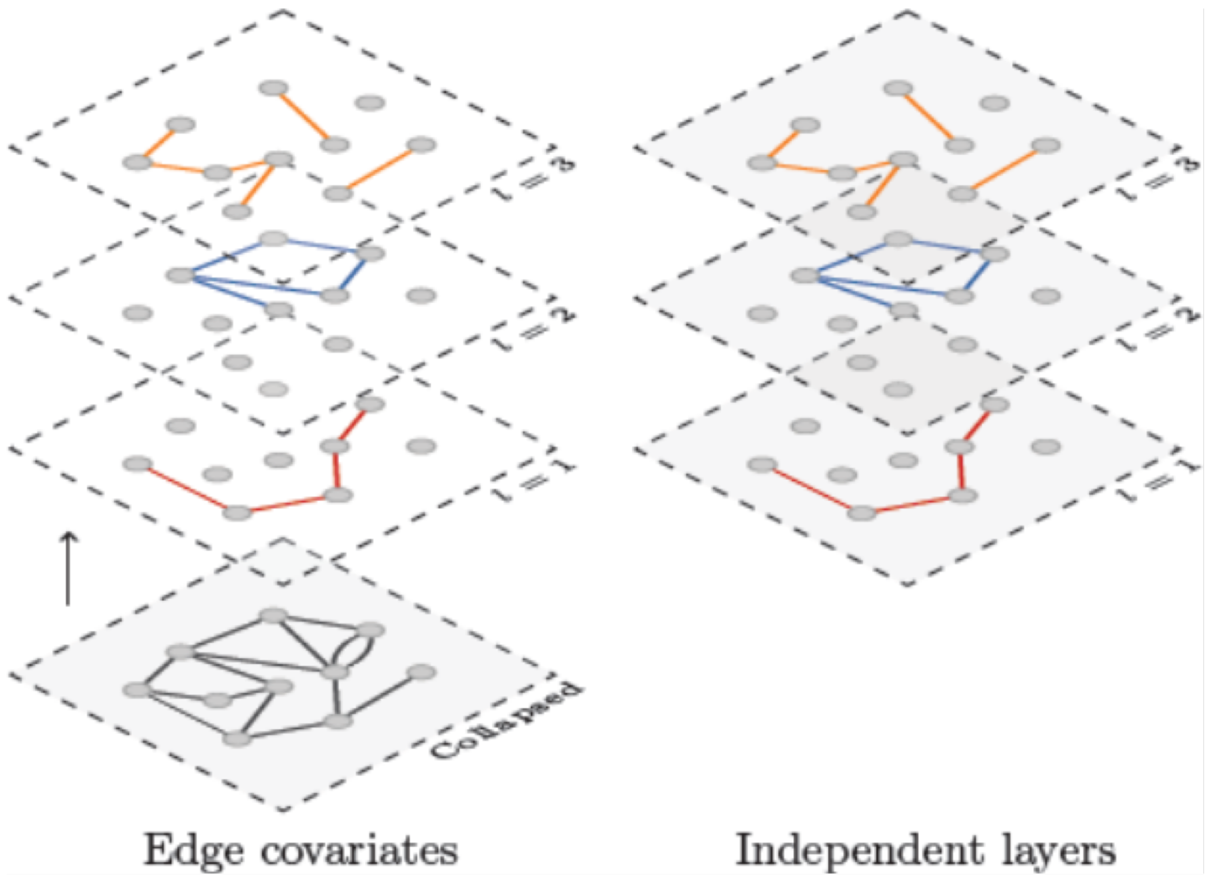
Reducibility of different datasets

Table 1 | Reducibility of empirical multilayer networks.

Network	N	M	M_{opt}	max[$q(\bullet)$]	χ
Arabidopsis	6981	7	5	0.436	0.33
Bos	326	4	3	0.494	0.33
Candida	368	7	4	0.527	0.50
C. elegans	3880	6	4	0.390	0.40
Drosophila	8216	7	5	0.426	0.33
Gallus	314	6	4	0.505	0.40
Human HIV-1	1006	5	2	0.499	0.75
Mus	7748	7	6	0.376	0.17
Plasmodium	1204	3	2	0.500	0.50
Rattus	2641	6	4	0.504	0.40
S. cerevisiae	6571	7	4	0.115	0.50
S. pombe	4093	7	4	0.197	0.50
Xenopus	462	5	3	0.424	0.50
Arxiv coauthorship	14065	13	11	0.231	0.17
Terrorist network	78	4	2	0.239	0.67
FAO Trade network	184	340	182	0.354	0.47
London Tube	369	13	12	0.441	0.08
Airports Europe	1064	175	165	0.667	0.06
Airports Asia	1130	213	202	0.653	0.05
Airports North America	2040	143	136	0.686	0.05

De Domenico et al Nature communication 2015

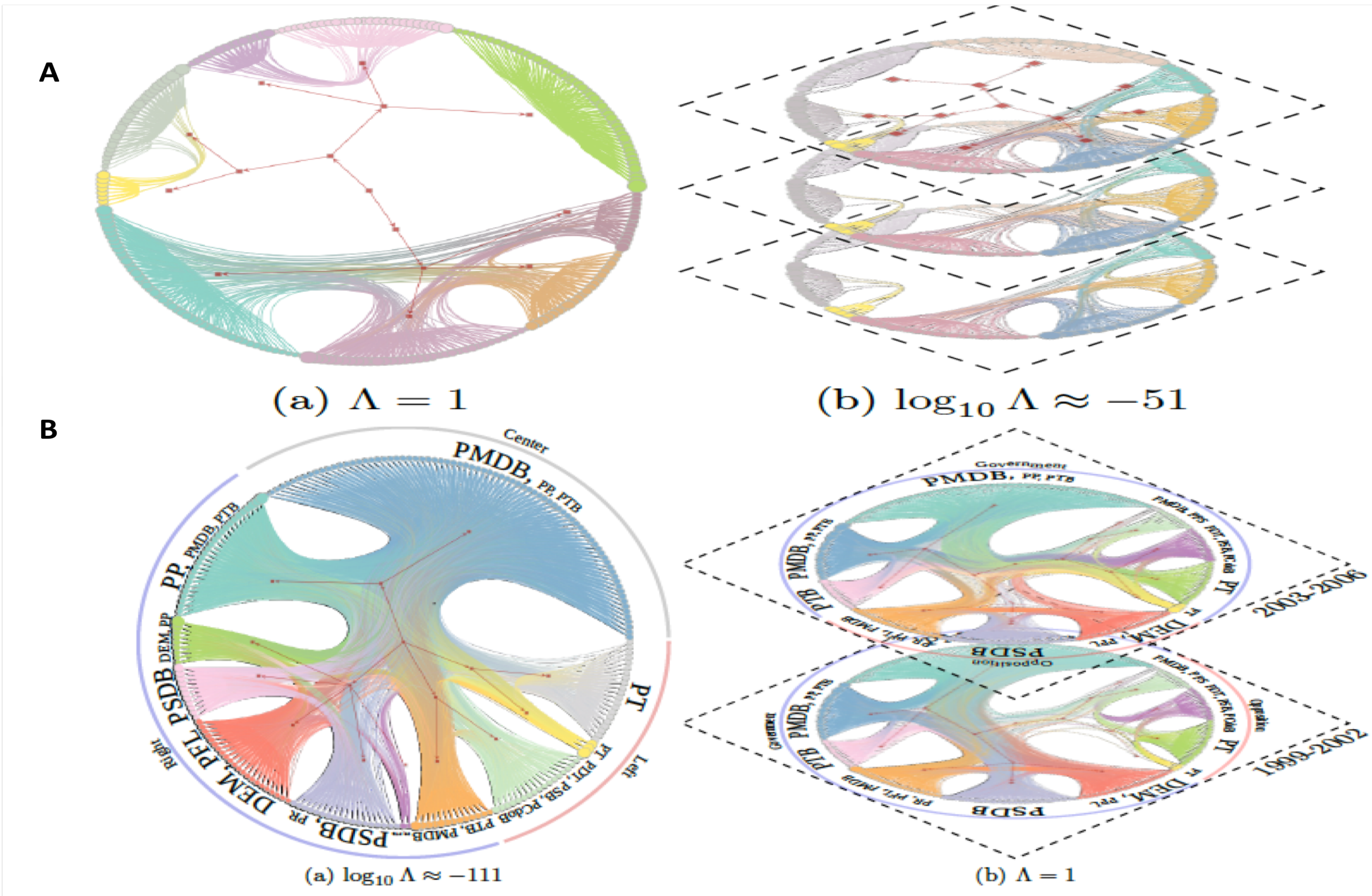
Inference models



Peixoto PRE (2015)

To aggregate or to disaggreate:

the answer might depend on the dataset!



Peixoto PRE (2015)

Conclusions

Extracting information from multilayer networks is essential to make progress in our understanding of multilayer networks

Network theory is providing new tools to meet the challenge

- Multiplex networks can have a highly correlated structure that encodes relevant information.
- Degree correlations and the overlap are fundamental to investigate multiplex networks
- Weights in multiplex networks can be correlated with the overlap of the links providing a straightforward way to extract information not present in their single layers
- The community structure of multilayer network can include communities spanning and overlapping across multiple layers

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J. Iacovacci et al. PRE 92, 042806 (2015)

M. De Domenico et al. Nature Comm. 6, 6864 (2015)

M. De Domenico et al. PRX 5, 011027 (2015).

Mucha et al. Science 328 876 (2010)

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Lancichinetti and Fortunato Scientific Reports 2, 336 (2012)

Cantini et al. Scientific Reports 5, 17386 (2015)

T. Peixoto PRE 92, 042807 (2016)