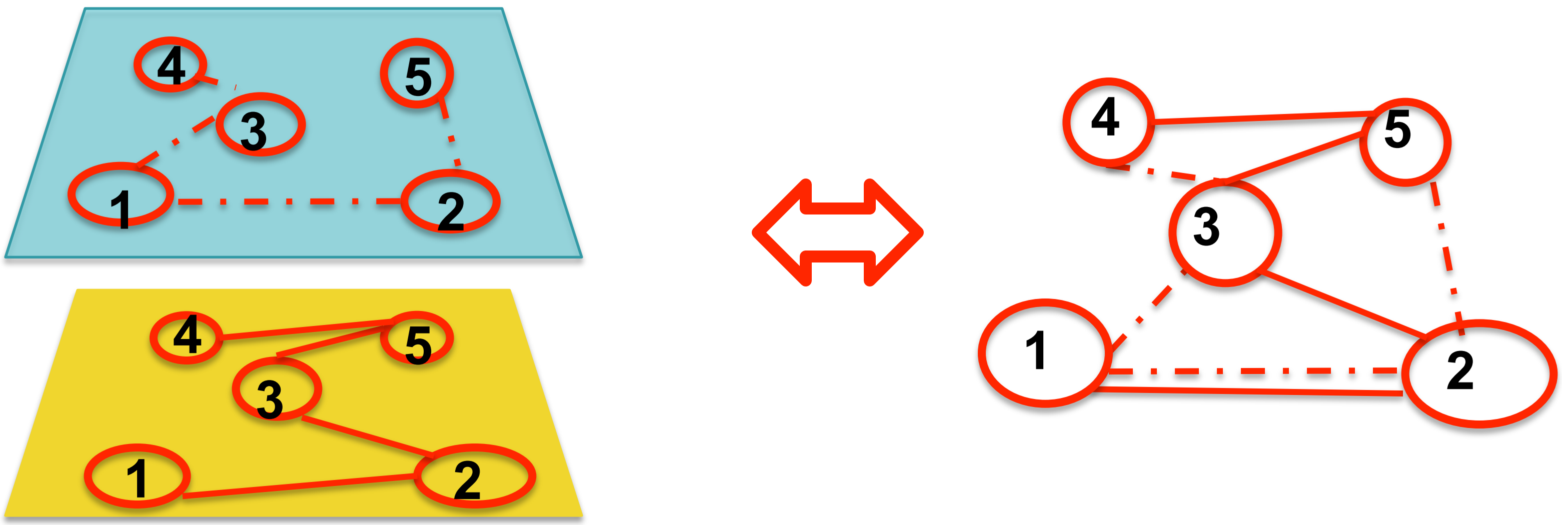

Multiplex Networks: generative models

LTCC Course Multilayer Networks
23-24 November 2016

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Queen Mary
University of London

Ensembles
of
single networks
(null models)

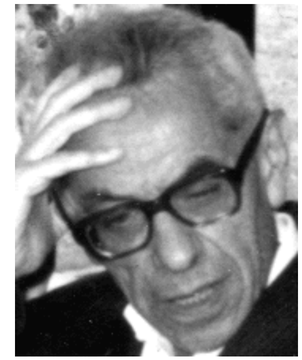
Chance: Random graphs

G(N,L) ensemble

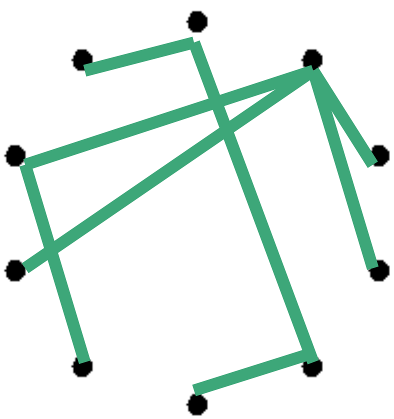
Graphs with exactly N nodes and L links

G(N,p) ensemble

Graphs with N nodes
Each pair of nodes linked with probability p

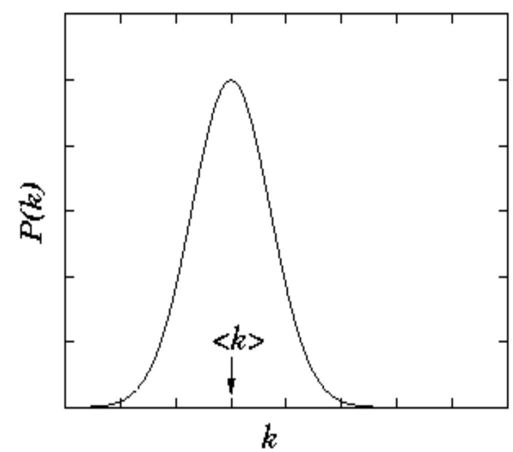


Poisson distribution



$$L/N = c$$

$$p = c/(N-1)$$



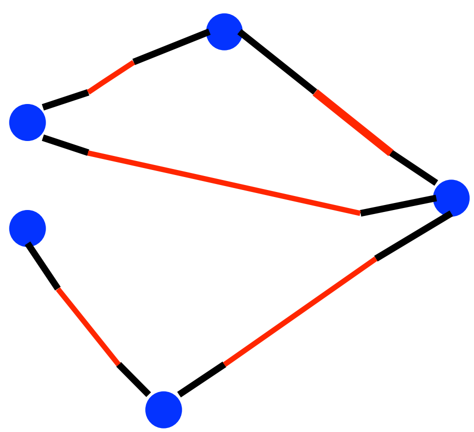
Statistical mechanics
and
random graphs

Statistical mechanics		Random graphs	
Microcanonical Ensemble	Configurations with fixed energy E	G(N,L) Ensemble	Graphs with fixed # of links L
Canonical Ensemble	Configurations with fixed average energy <E>	G(N,p) Ensemble	Graphs with fixed average # of links <L>

Networks with given degree sequence

Microcanonical ensemble

$$P(G)=\frac{1}{\aleph}\prod_i \delta\left(k_i-\sum_j a_{ij}\right)$$

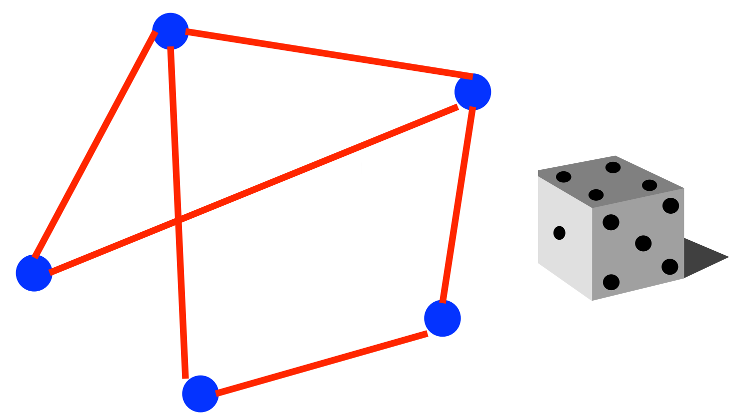


Ensemble of network with exact degree sequence

Configuration model

Canonical ensemble

$$P(G)=\frac{1}{Z}e^{-\sum_i \lambda_i k_i}=\prod_{i<j} p_{ij}^{\,a_{ij}}(1-p_{ij})^{1-a_{ij}}$$



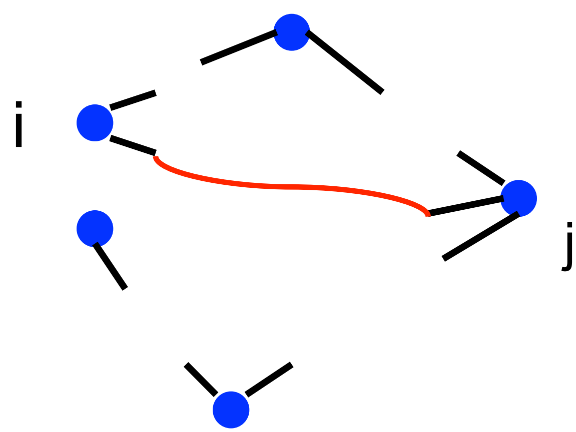
Ensemble of networks given expected degree sequence

Hidden variables model

Link probability in uncorrelated networks

In uncorrelated networks *the probability that a node i is linked to a node j is given by*

$$p_{ij} = \frac{k_i k_j}{<k> N}$$



$$p_{ij} = 2\frac{3}{\langle k \rangle N}$$

Network model with given degree sequence are only uncorrelated if they have a structural cutoff

$$k_i < K_S = \sqrt{\langle k \rangle N}$$

for

$$i = 1,2\ldots,N$$

Entropy of network ensembles

Entropy of a **canonical network ensemble** with expected degree sequence

$$S=-\left[\sum_{ij} p_{ij}\ln p_{ij}+(1-p_{ij})\ln(1-p_{ij})\right]$$

Entropy of a **microcanonical network ensemble** with given degree sequence is given by

$$\Sigma=\log[\aleph]=S-\Omega \qquad \Omega=-\sum_i \ln\frac{1}{k_i!}k_i^{k_i}e^{-k_i}$$

Where $\aleph$ is the total number of networks in the ensemble

Bianconi et al. PRE 2008

There is no equivalence of the ensembles as long as the number of constraints are extensive

-Example

Microcanonical esemble	**Canonical ensemble**
Regular networks	Poisson networks
$p_{ij} = \frac{c}{N}$	$p_{ij} = \frac{c}{N}$

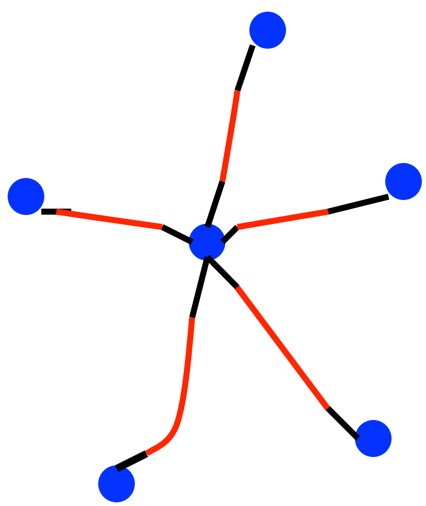
but

$$\Sigma < S$$

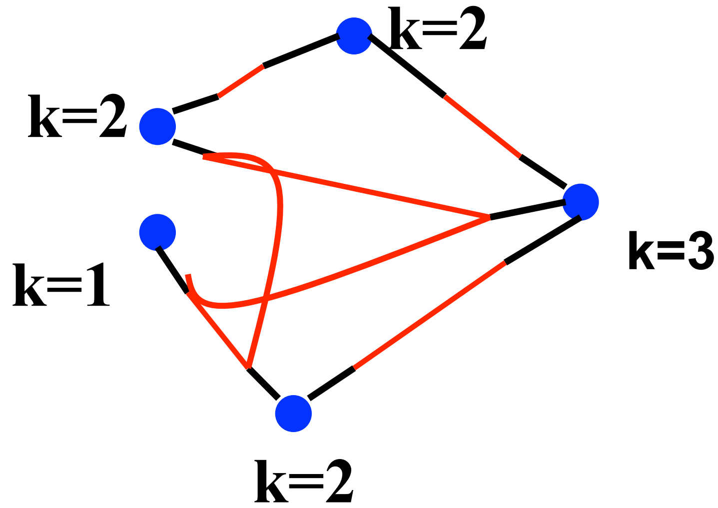
K. Anand, G. Bianconi PRE 2009

Two examples of given degree sequence

Zero entropy

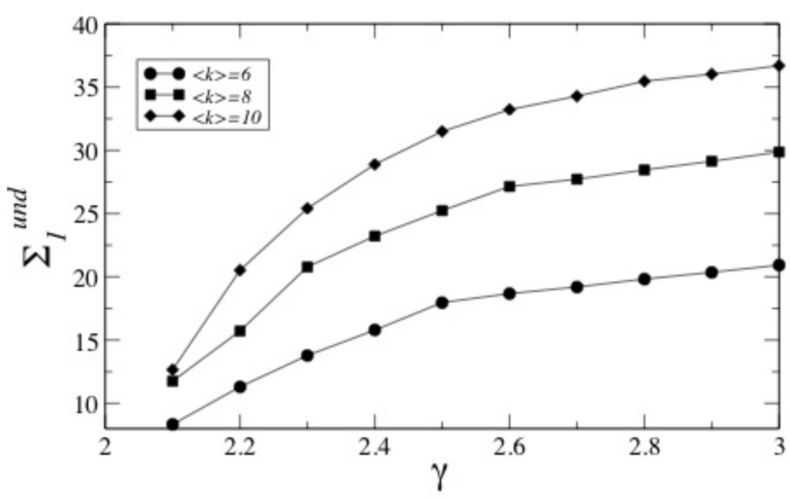


Non-zero entropy



The entropy of random scale-free networks

$$P(k) \propto k^{-\gamma}$$



The entropy decreases as γ **decrease toward** 2
quantifying a higher order in networks with fatter tails

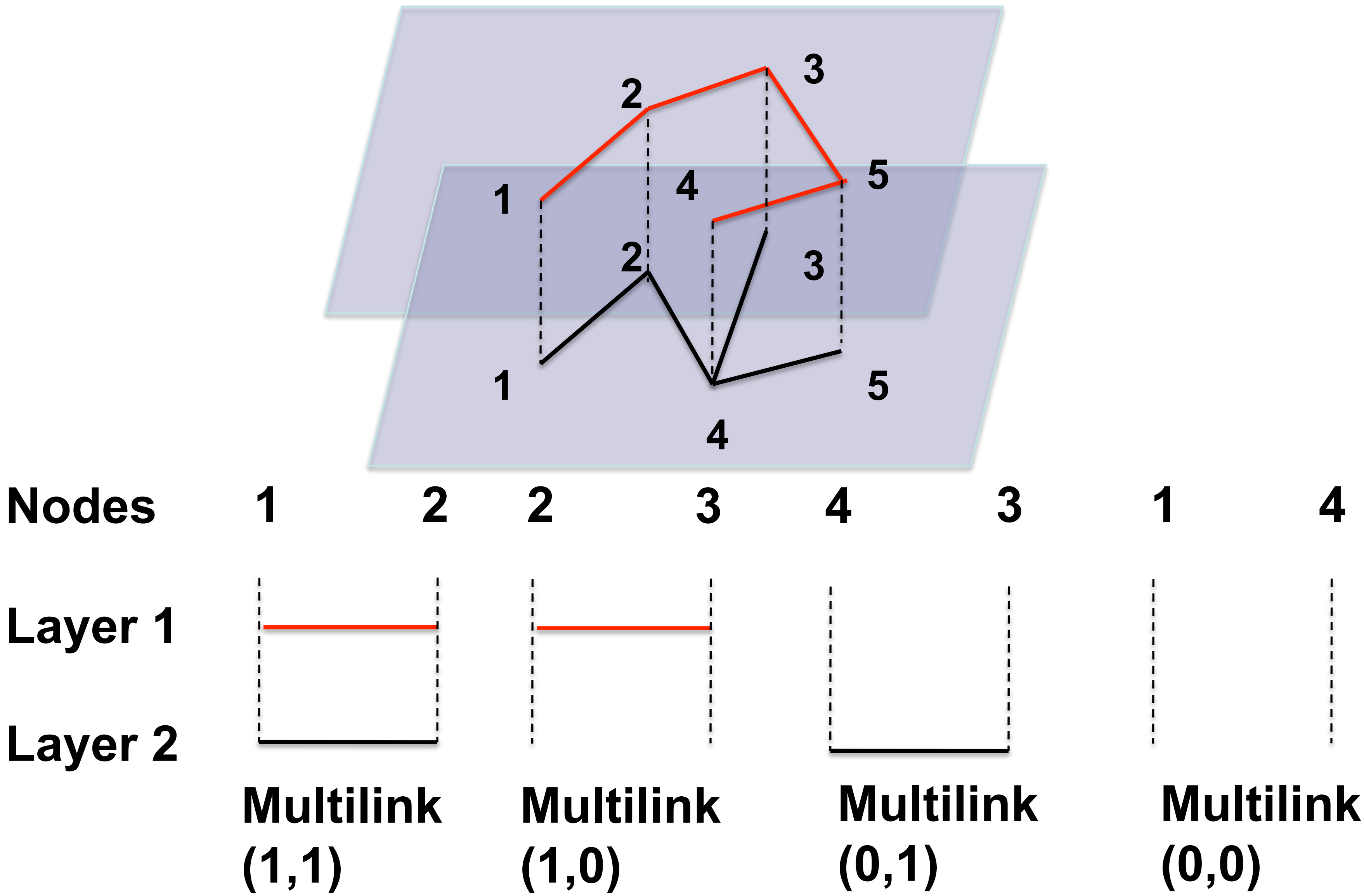
Generative models of multiplex networks with multilinks

If we generate
in each layer independently
with the configuration model
we will get
a negligible total and local overlap
For a null model that preserves
the overlap we need
multilinks.

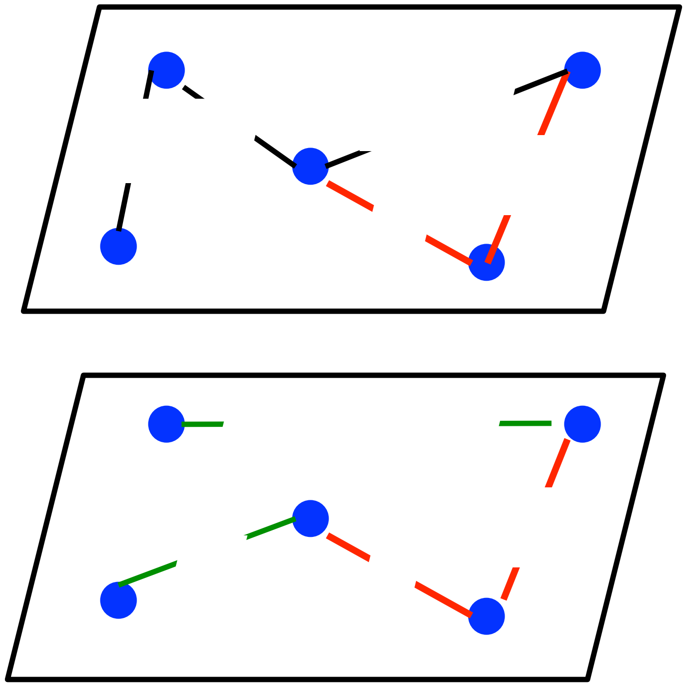
G. Bianconi PRE (2013)

Multilinks

G. Bianconi
PRE (2013)

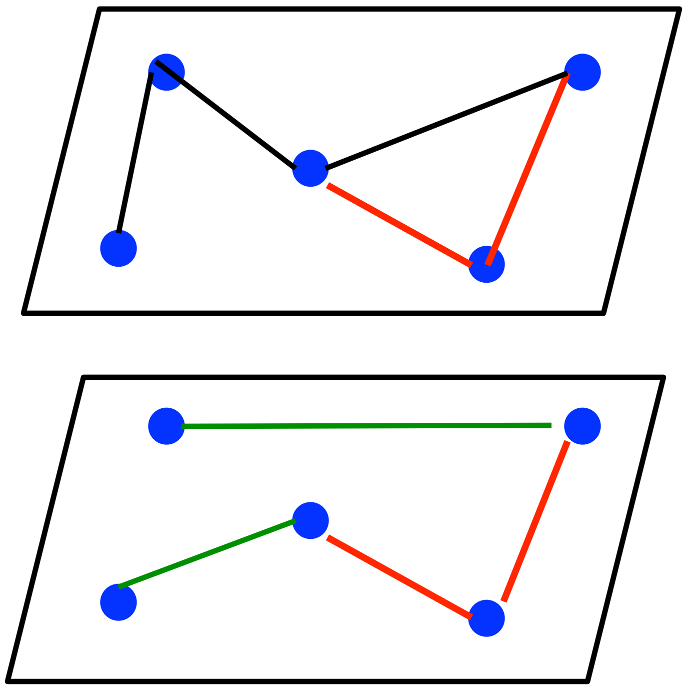


Configuration model with multilinks



$$P(\vec{G}) = \frac{1}{Z}\prod_i \delta(k^{10}{}_i - \sum_j A_{ij}^{10})\delta(k^{01}{}_i - \sum_j A_{ij}^{01})\delta(k^{11}{}_i - \sum_j A_{ij}^{11})$$

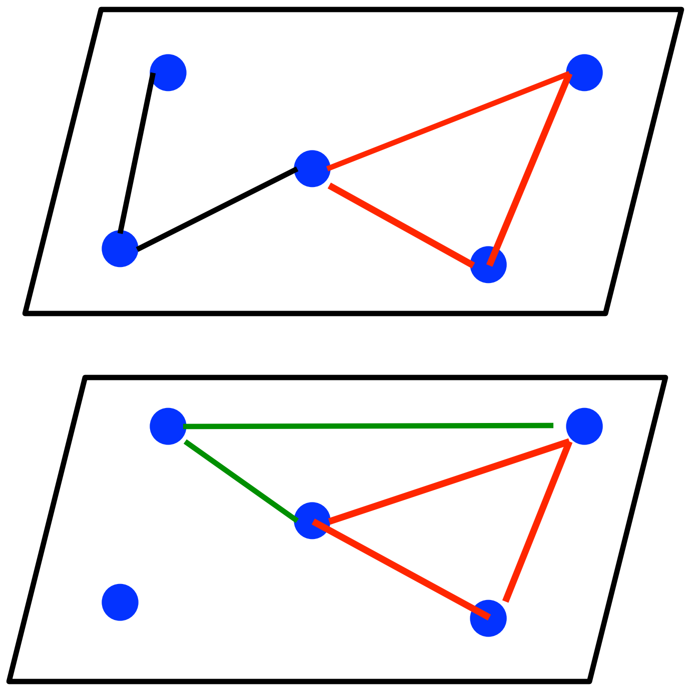
Configuration model with multilinks



$$P(\vec{G}) = \frac{1}{Z} \prod_i \delta(k^{10}{}_i - \sum_j A_{ij}^{10}) \delta(k^{01}{}_i - \sum_j A_{ij}^{01}) \delta(k^{11}{}_i - \sum_j A_{ij}^{11})$$

Exponential random multiplex model with multilinks

$$P(\vec{G}) = \prod_{i<j} (p_{ij}^{10} A_{ij}^{10} + p_{ij}^{01} A_{ij}^{01} + p_{ij}^{11} A_{ij}^{11} + p_{ij}^{00} A_{ij}^{00})$$



Constructive algorithm

For every pair of nodes (i,j)

Draw a multilink $\vec{m}$

with probability $p_{ij}^{\vec{m}}$,

i.e. put a link in every layer

where m_α**=1.**

G. Bianconi PRE (2013)

Multilinks probabilities in a duplex with structural multidegree cutoffs

Probabilities of the multilinks

$$p_{ij}^{10} = \frac{k_i^{10} k_j^{10}}{\langle k^{10} \rangle N}$$

$$p_{ij}^{01} = \frac{k_i^{01} k_j^{01}}{\langle k^{01} \rangle N}$$

$$p_{ij}^{11} = \frac{k_i^{11} k_j^{11}}{\langle k^{11} \rangle N}$$

Structural cutoff

$$k^{10} < \sqrt{\langle k^{10} \rangle N}$$

$$k^{01} < \sqrt{\langle k^{01} \rangle N}$$

$$k^{11} < \sqrt{\langle k^{11} \rangle N}$$

G. Bianconi PRE 2013

Entropy of correlated multiplex ensembles

Entropy of a **canonical multiplex ensemble** with linear constraints

$$S=-\left[\sum_{\vec{m}}\sum_{ij}p_{ij}^{\vec{m}}\ln p_{ij}^{\vec{m}}\right]$$

Entropy of a **microcanonical mutliplex ensemble with linear constraints con be found by the cavity method**, if we fix only the multi degree sequence in the sparse network limit, we get

$$\Sigma=S-N\Omega$$

$$\Omega=-\frac{1}{N}\sum_{\vec{m}}\sum_{i}\log\left[\frac{\left(k_i^{\vec{m}}\right)^{k_i^{\vec{m}}}e^{-k_i^{\vec{m}}}}{k_i^{\vec{m}}!}\right]$$

G. Bianconi PRE 2013

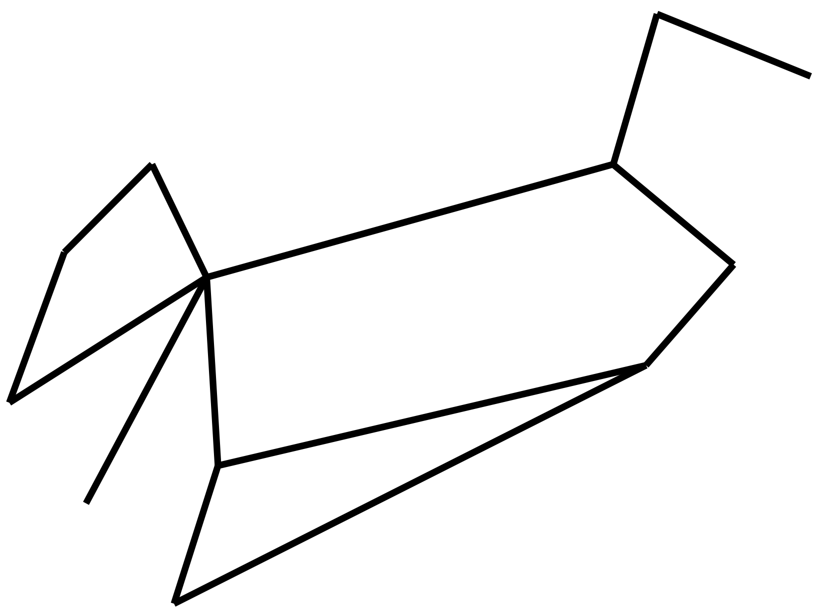
Randomization algorithms

Swap randomization for single networks

If we randomize the networks
in each layer independently
we will get
a negligible total and local overlap
For a null model that preserves
the overlap we need
multilinks.

G. Bianconi PRE (2013)

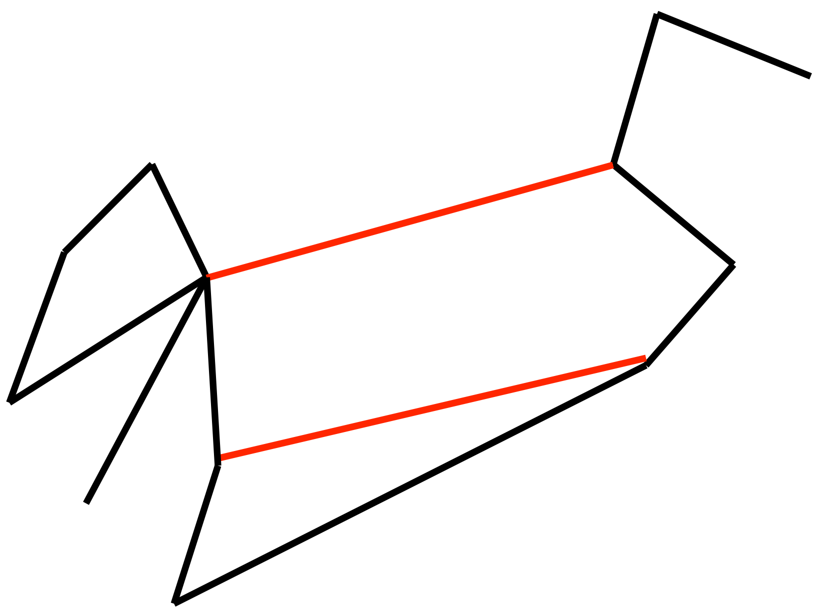
How to build a null model from a given network: swap of connections



- *Choose two random links linking four distinct nodes*

Maslov & Sneppen 2002

How to build a null model from a given network: swap of connections



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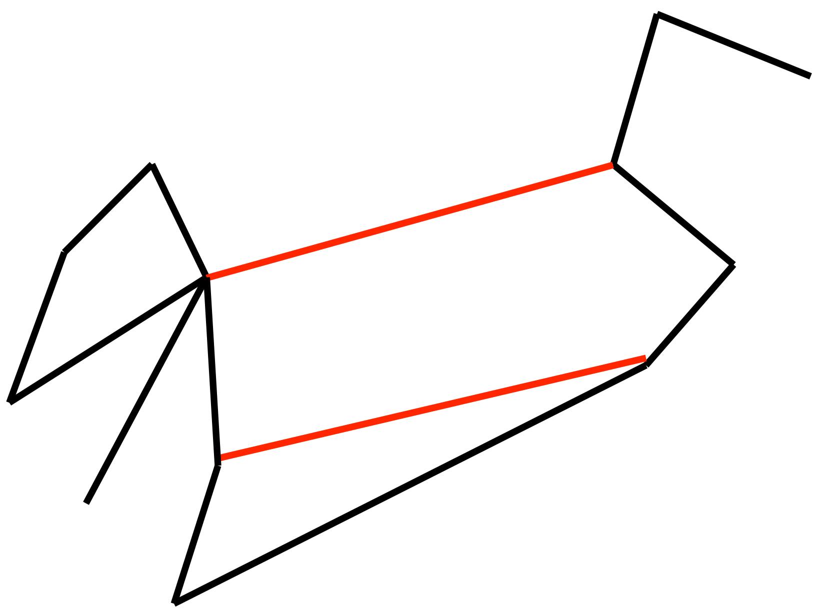
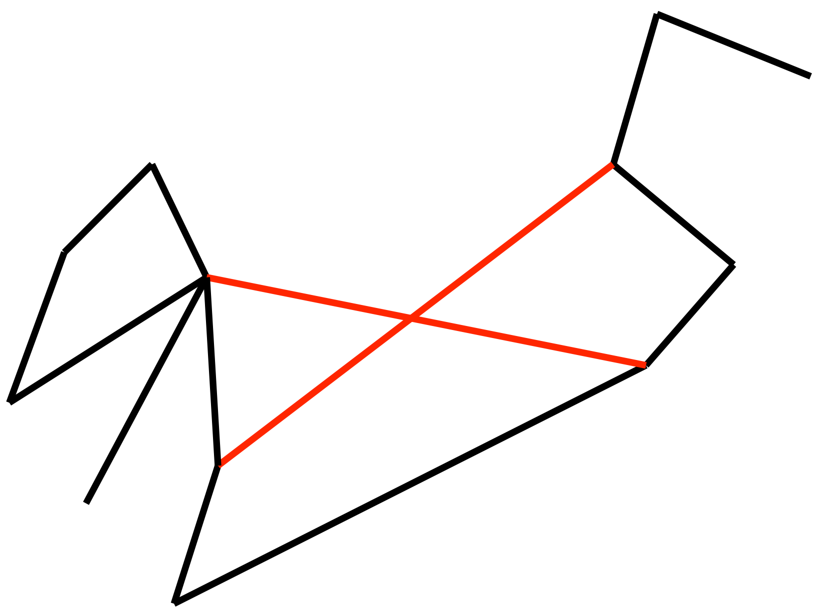


- *Choose two random links linking four distinct nodes*
- *If possible (not already existing links) swap the ends of the links*

Maslov & Sneppen 2002

How to build a null model from a given network: swap of connections



- *Choose two random links linking four distinct nodes*
- *If possible (not already existing links) swap the ends of the links*

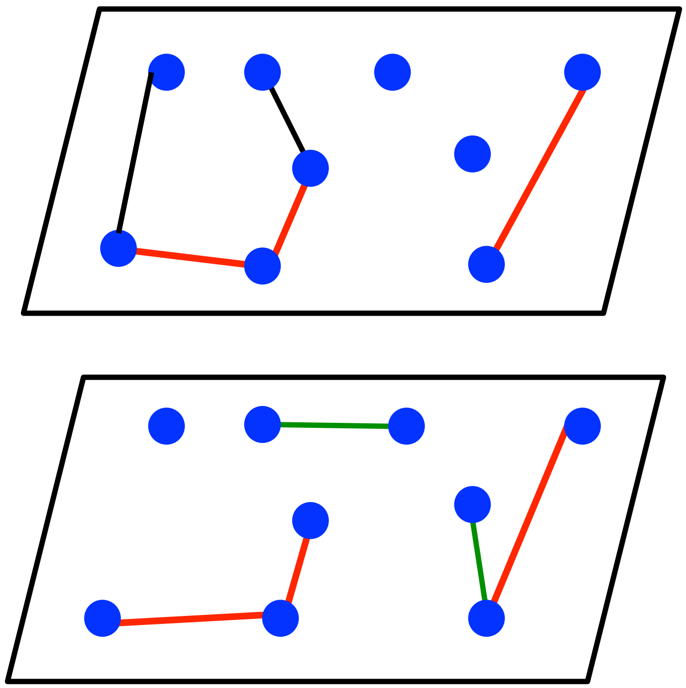
Maslov & Sneppen 2002

If we randomize the networks
in each layer independently
we will get
a negligible total and local overlap
For a null model that preserves
the overlap we need
multilinks.

G. Bianconi PRE (2013)

Swap algorithm

Only multilinks of the same type can swap!



Swap algorithm

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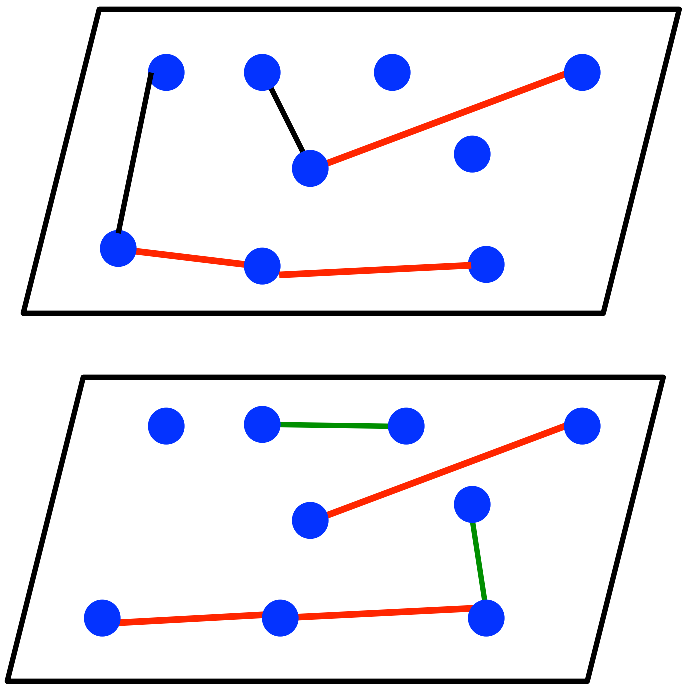

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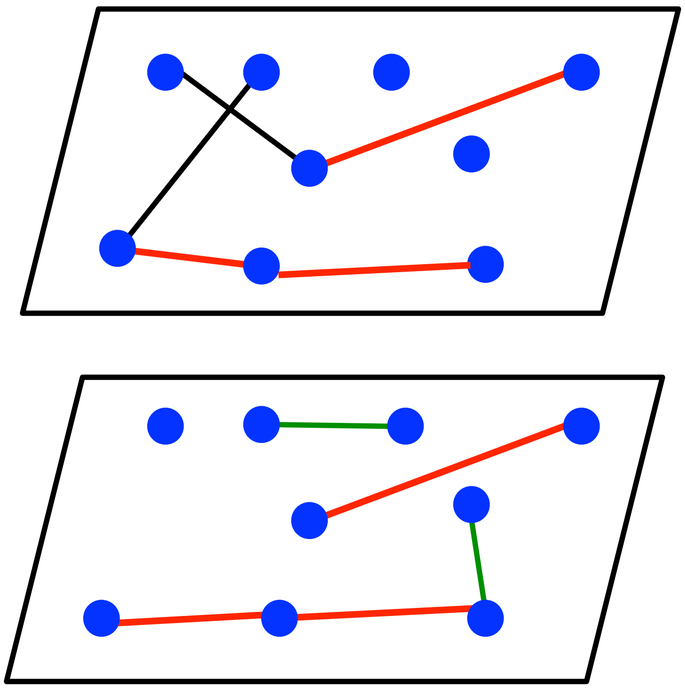
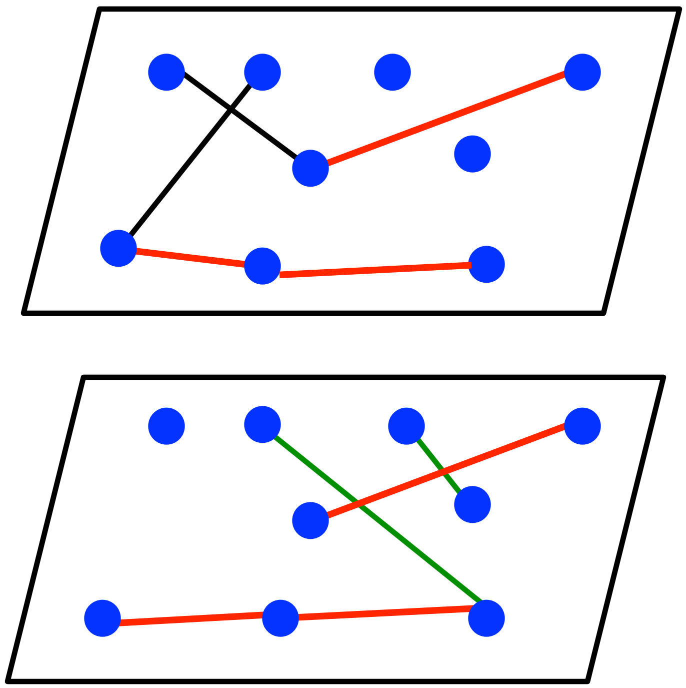
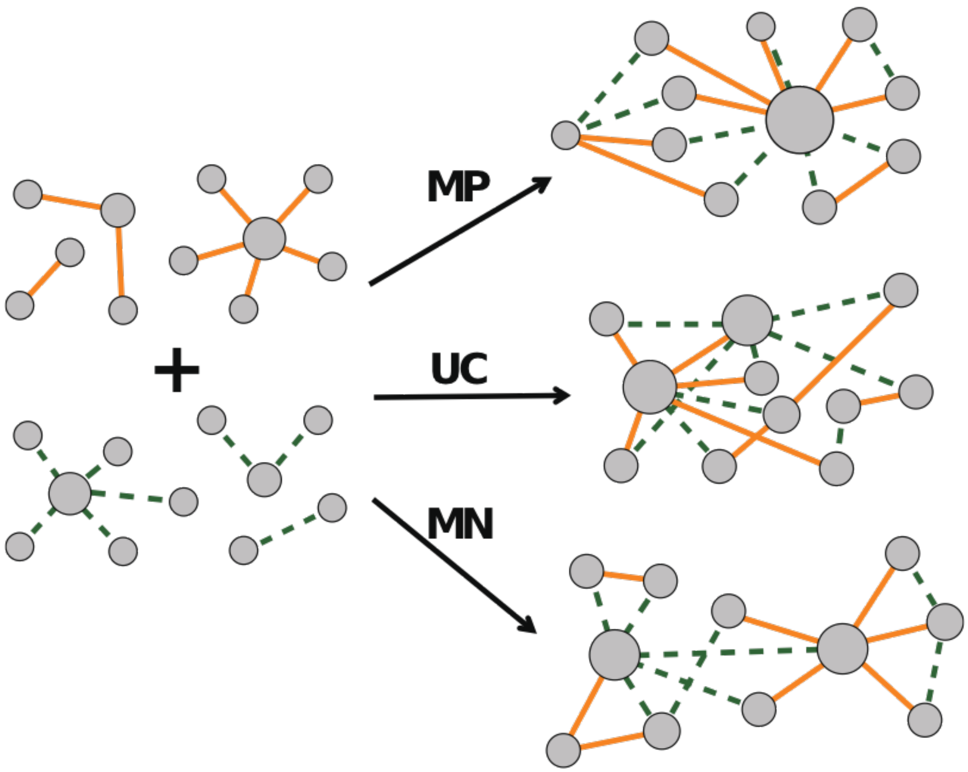


Swap algorithm

Only multilinks of the same type can swap!



Correlated degree across two layers



By relabelling the nodes of two layers it is possible to build

Maximum positive (MP)

Maximum negative (MN)

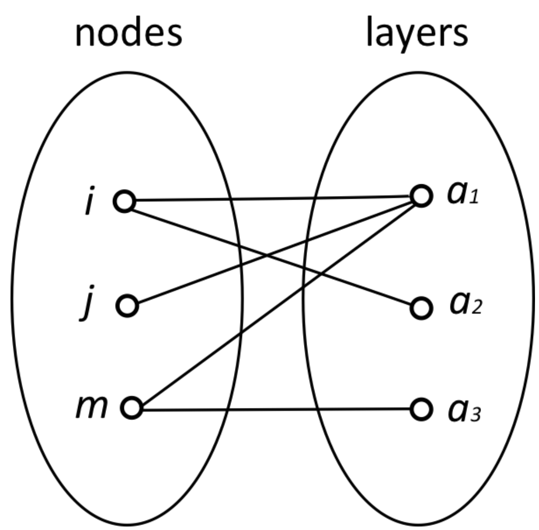
and Uncorrelated (UC)

Multiplex Networks.

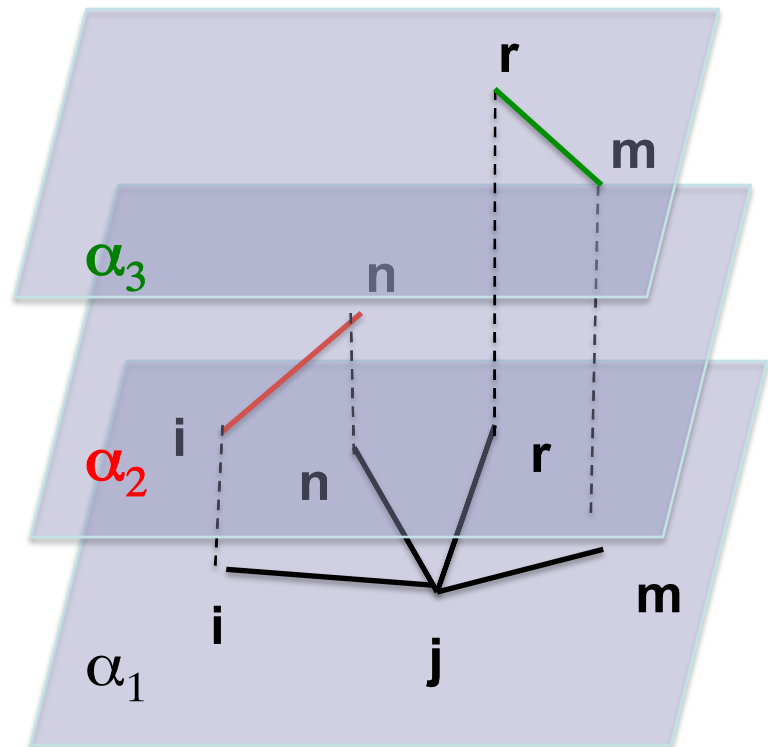
B. Min et al. PRE (2014)

Multiplex networks with heterogeneous activity of the nodes

Bipartite network: Nodes and Layers



Multiplex network



D. Cellai and G. Bianconi, PRE 93, 032302 (2016)

Spatial Multiplexes

The nodes in a spatial multiplex have a position $\vec{r}$ in their real or hidden embedding space

$$\boxed{P\left(\vec{G} \mid \{\vec{r}_i\}\right) = \prod_{\alpha=1,M} P_\alpha\left(G_\alpha \mid \{\vec{r}_i\}\right)}$$

In these ensembles we can observe a significant overlap of the links because nodes that are "close in space" are more likely to be linked in every network

(A. Halu, S. Mukherjee and G. Bianconi PRE 2014)

Class of network models

- Static networks:
  - **Hidden variables mechanism**

    Bollobas 1979, Chung & Lu 2002,

    Caldarelli et al. 2002, Park & Newman 2003
- **Growing networks:**
  - **Preferential attachment**

    Barabasi & Albert 1999,

    Dorogovtsev & Mendes 2000,

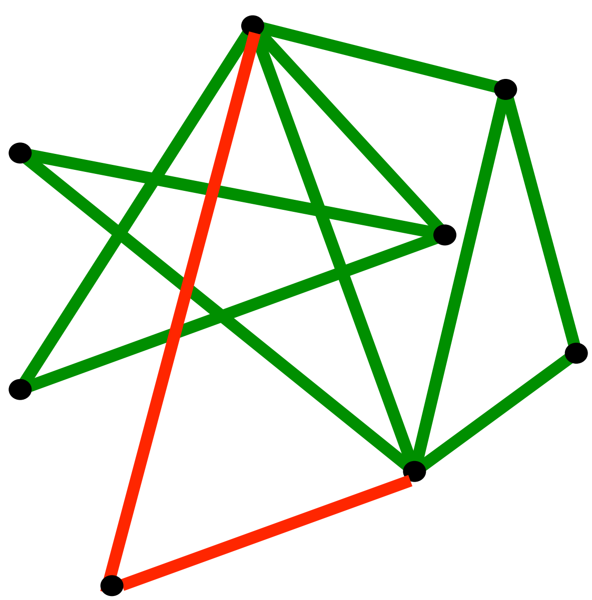
    Bianconi & Barabasi 2001

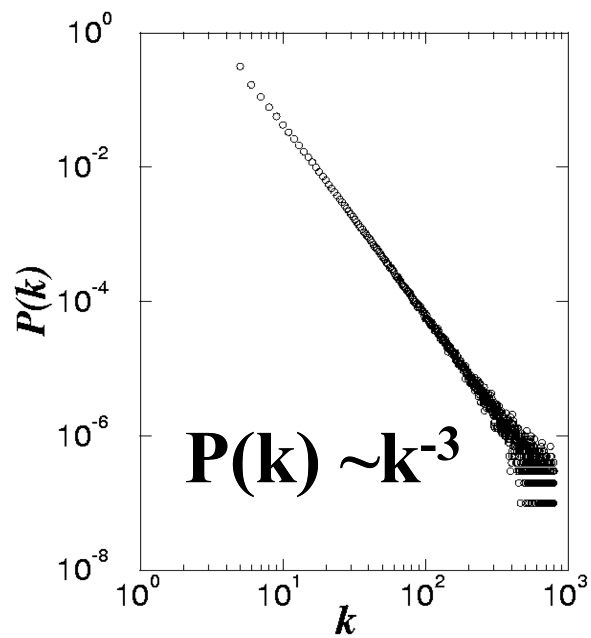
BA model

(1) GROWTH : At every timestep we add a new node with m edges (connected to the nodes already present in the system).

(2) PREFERENTIAL ATTACHMENT : The probability Π that a new node will be connected to node i depends on the connectivity k_i of that node

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$





Barabási et al. Science (1999)

Growing multiplex (duplex)

GROWTH

At each time a new node is added to the multiplex.

Every new node has a copy in each layer and has m links in each layer.

LINEAR PREFERENTIAL ATTACHMENT

The probability Π^{α}_{i} that the new link

is added to node i in layer α

is given by

$$\boxed{\begin{aligned} \Pi_i^{[1]} &\propto a k_i^{[1]} + (1-a) k_i^{[2]} \\ \Pi_i^{[2]} &\propto b k_i^{[1]} + (1-b) k_i^{[2]} \end{aligned}}$$

with

a,b $\leq$ 1.

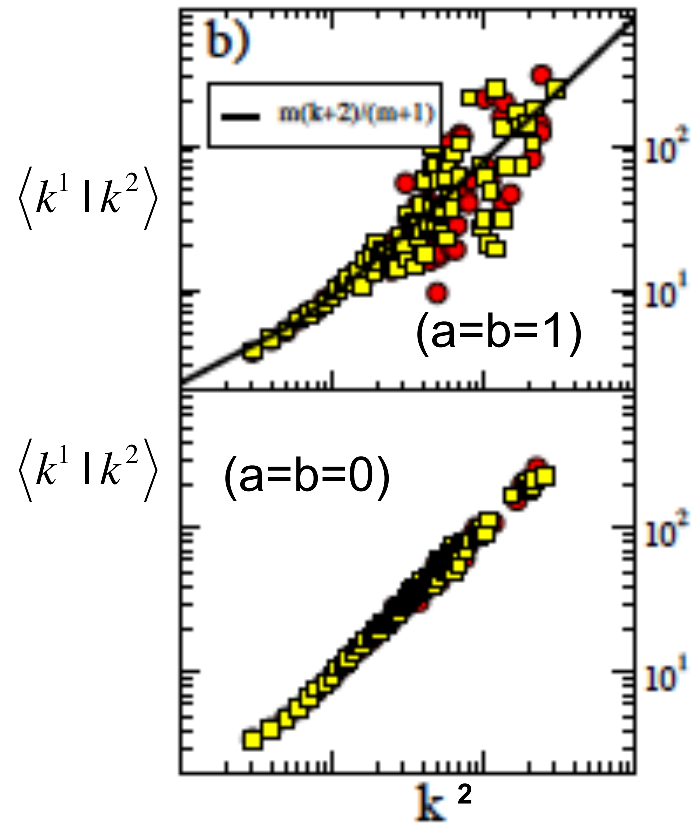
Nicosia et al PRL 2013
Kim et al. PRL 2013

Degree correlations

Spontaneous emergence of positive degree correlations

$$\boxed{\left\langle k^1 \mid k^2 \right\rangle \propto k^2}$$

Old nodes are more connected in both layers yielding positive degree correlations



Nicosia et al PRL 2013

Growing multiplexes with non-linear preferential attachment

- GROWTH

At each time a new node is added to the multiplex.
Every new node has a copy in each layer and has m links in each layer.

- NON-LINEAR PREFERENTIAL ATTACHMENT

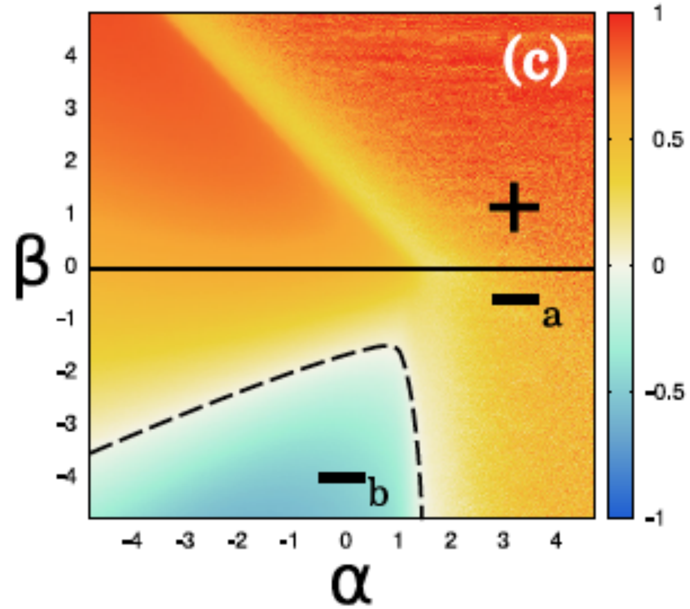
The probability that the new link is added to node i in layer α
is given by Π^{α} with a non-linear preferential attachment

$$\Pi_i^1 \propto (k_i^1)^{\alpha}(k_i^2)^{\beta}$$

$$\Pi_i^2 \propto (k_i^2)^{\alpha}(k_i^1)^{\beta}$$

Nonlinear preferential attachment

Including the nonlinear preferential attachment we can get either positive or negative degree correlations as measured by the Kendall's correlation coefficient of the degrees



Nicosia et al. (2014)

Model enforcing triadic closure in multiplex networks

The models includes

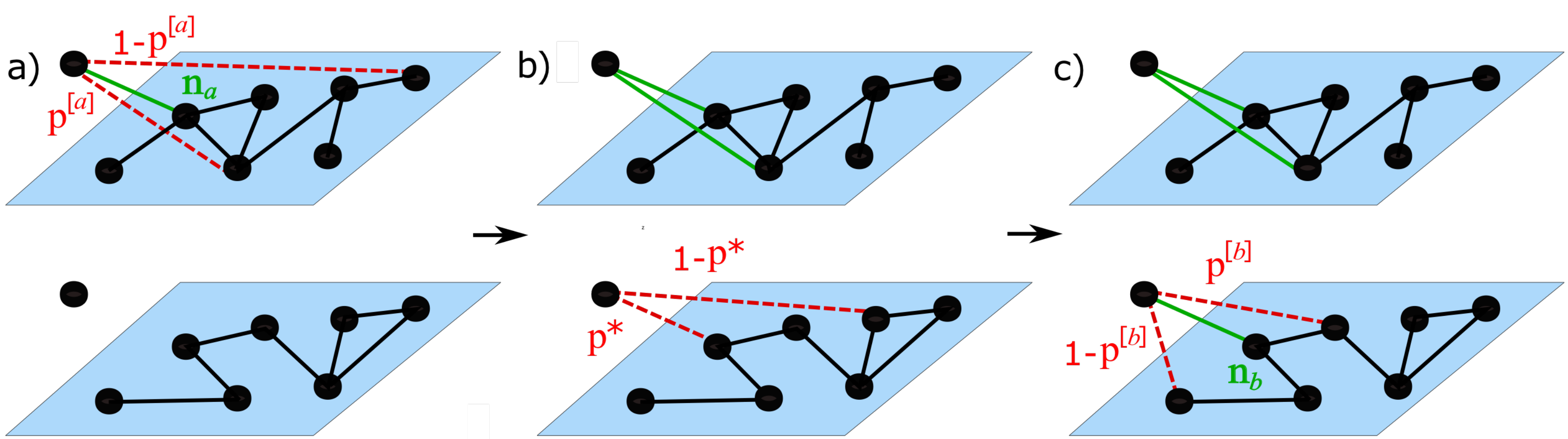
GROWTH

and

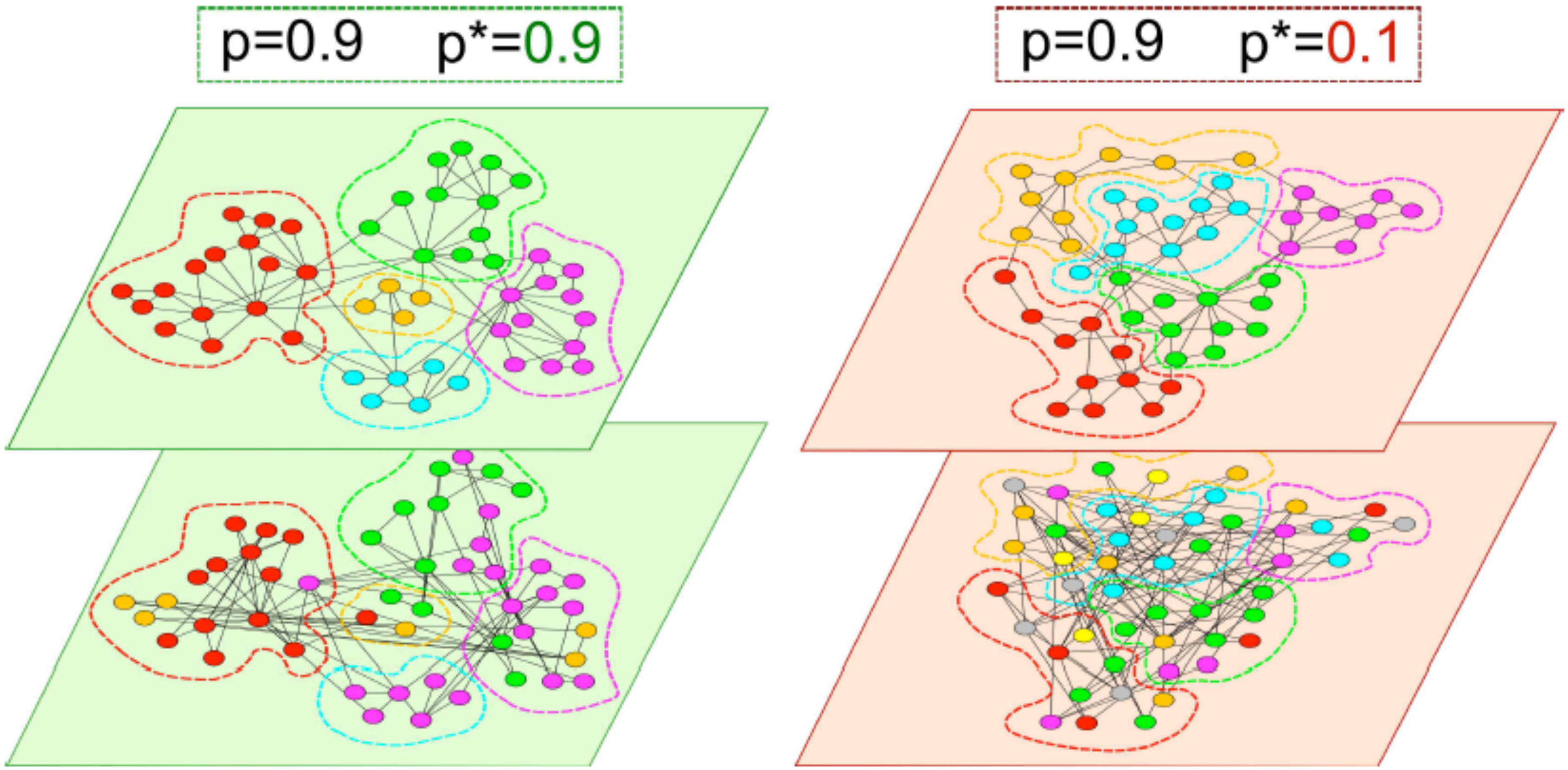
TRIADIC CLOSURE

p probability of triadic closure within a layer

p* probability of linking to the same node in different layers



Emergence of multiplex communities



Battiston et al. Plos One (2016)

Conclusions

Modeling multilayer networks is essential to generate null models to test significance of multiplex network properties and investigate the interlay between structure and dynamical processes

Network theory is providing new tools to model multiplex networks

- Null models of multiplex networks preserving the overlap of the links can be used to generate an artificial multiplex with the given multidegree
- Null models can be used to randomize a given multiplex network dataset
- Non-equilibrium growing multiplex networks can explain the basic mechanisms responsible for generating correlated multiplex networks

References

- **Multiplex ensembles**

  G. Bianconi, PRE 87, 062806 (2013).

  G. Menichetti, et al. PloS one e97857 (2014).

  G. Menichetti, et al. PRE, 90 062817 (2014).

  D. Cellai and G. Bianconi, PRE 93, 032302 (2016)

  A. Halu et al. PRE 89, 012806 (2014)

- **Growing models**

  V. Nicosia et al. PRL 111,058701 (2013)

  V. Nicosia et al. PRE 90, 042807 (2014)

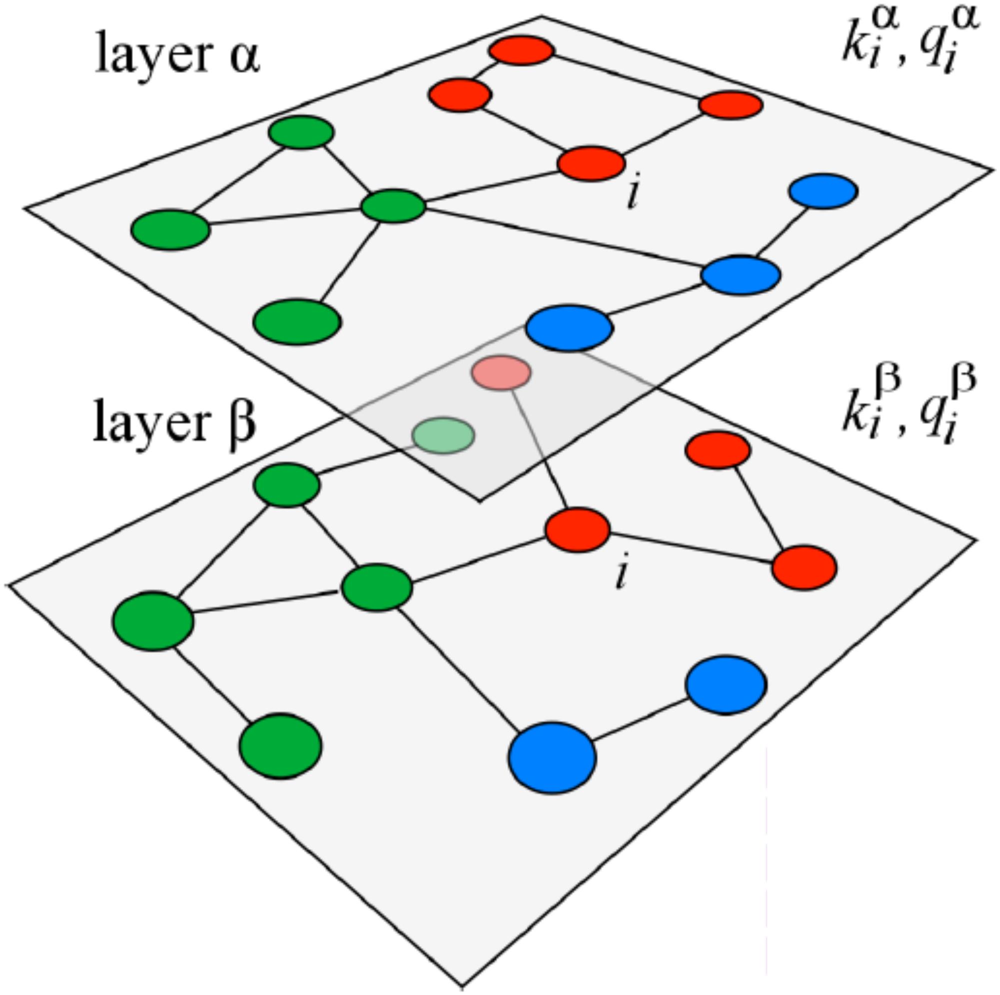
  Battiston et al. PloSOne 11, e0147451 (2016)

  N.Momeni and B.Fotouhi et al. PRE 92, 062812 (2015)

- **DATA and CODES repositories :**
- GitHub page: https://github.com/ginestrab (G. Bianconi)

Mesoscopic Structures Reveal the Network Between the Layers of a Multiplex Network

Mesoscopic structures



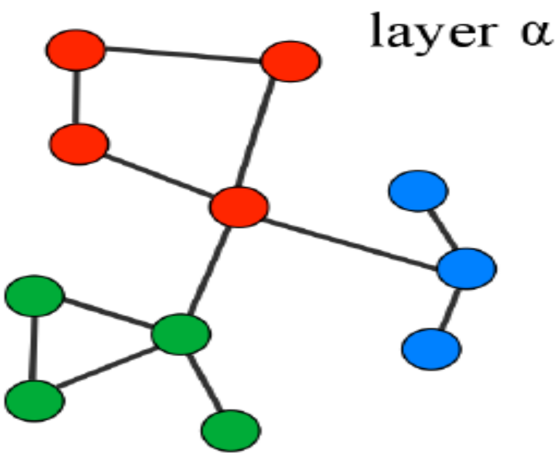
The feature of the nodes

$$q_i^\alpha$$

induce a mesoscopic structure (communities) in the layer α.

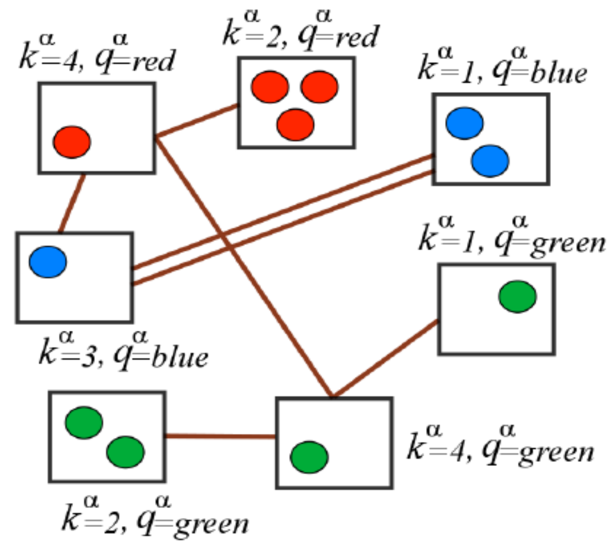
Our aim is to characterize the similarity between the mesoscopic structure of any two layers.

Entropy



$$(k^\alpha, q^\alpha)$$

defines a block structure



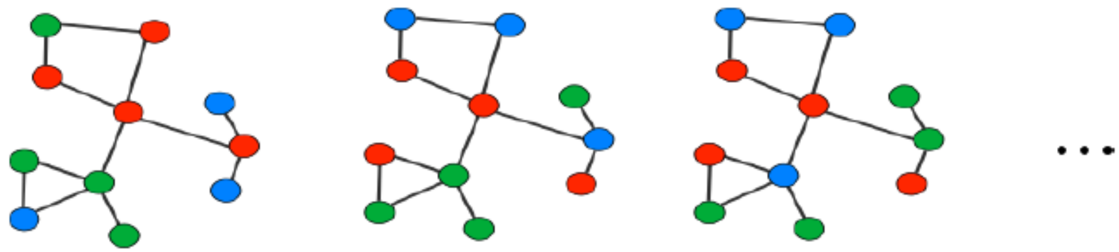
The entropy

$$\Sigma(k^\alpha, q^\alpha)$$

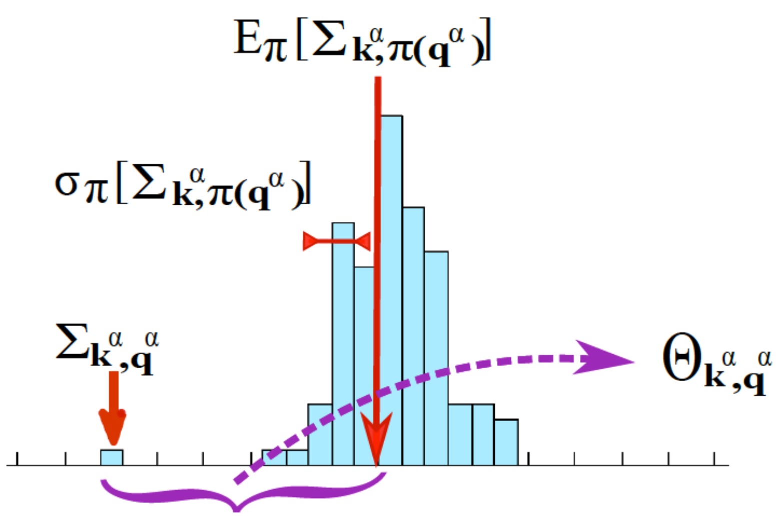
counts all the possible
network configurations
compatible with the block structure

Significance of the features with respect to the network structure

Random permutation of the features π(q)



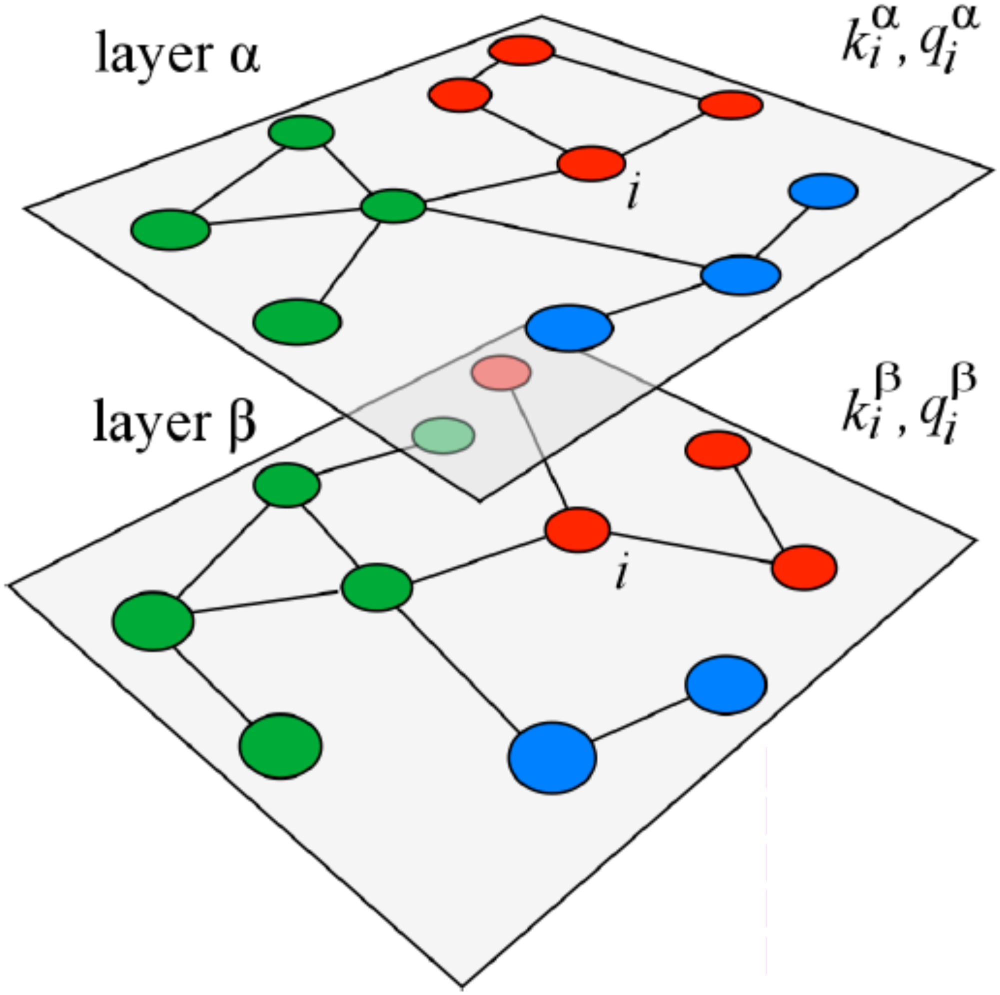
Entropy distribution of the block structure induced by random permutation



The significance of the feature with respect to the network structure is

$$\Theta_{k,q} = \frac{\langle \Sigma(k,\pi(q)) \rangle - \Sigma(k,q)}{\sigma_\pi(\Sigma(k,\pi(q)))}$$

Quantify mesoscopic structural similarities



The indicator

$$\tilde{\Theta}^{\alpha,\beta}$$

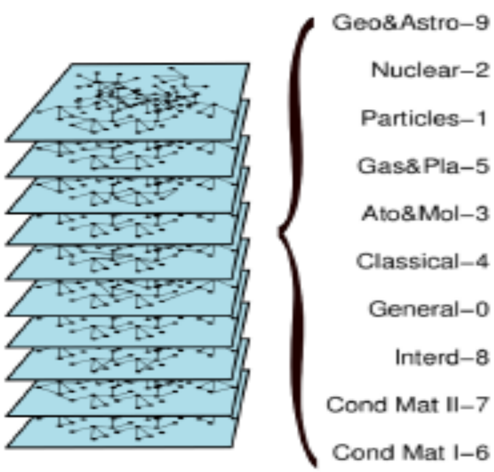
measures the similarity between the mesoscopic structure of two layers α and β

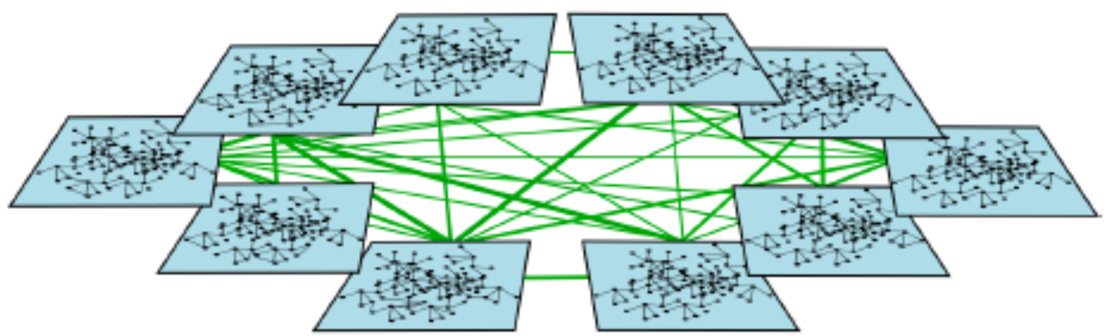
Multiplex community structure of the APS collaboration network

APS collaboration network → 180,538 authors of the American Physical Society papers till 2014 with less than 10 authors

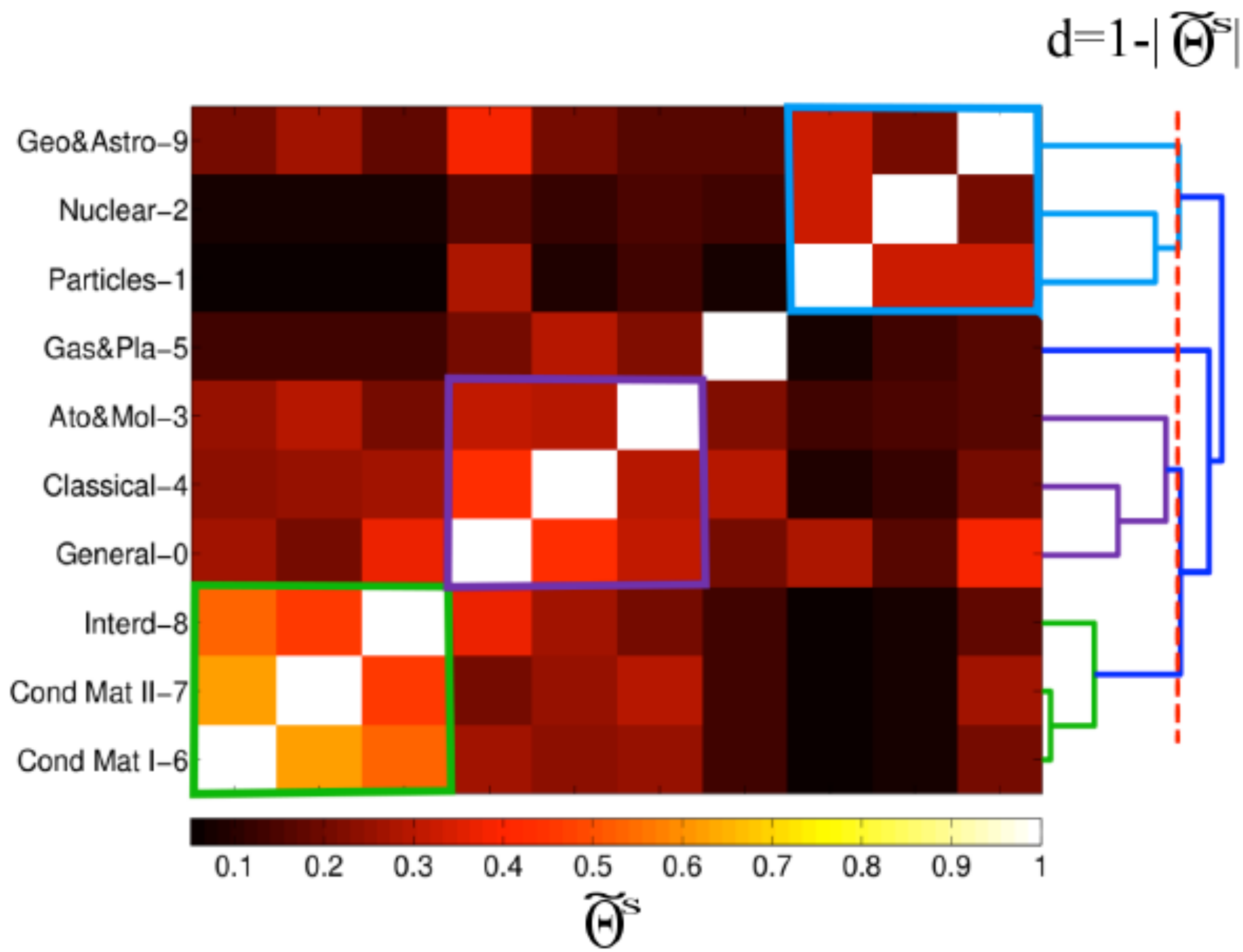
First layer PACS hierarchy (10 layers) → each layer describes the collaboration network in a general field of physics

q_i^{α} → Community label of node i in layer α

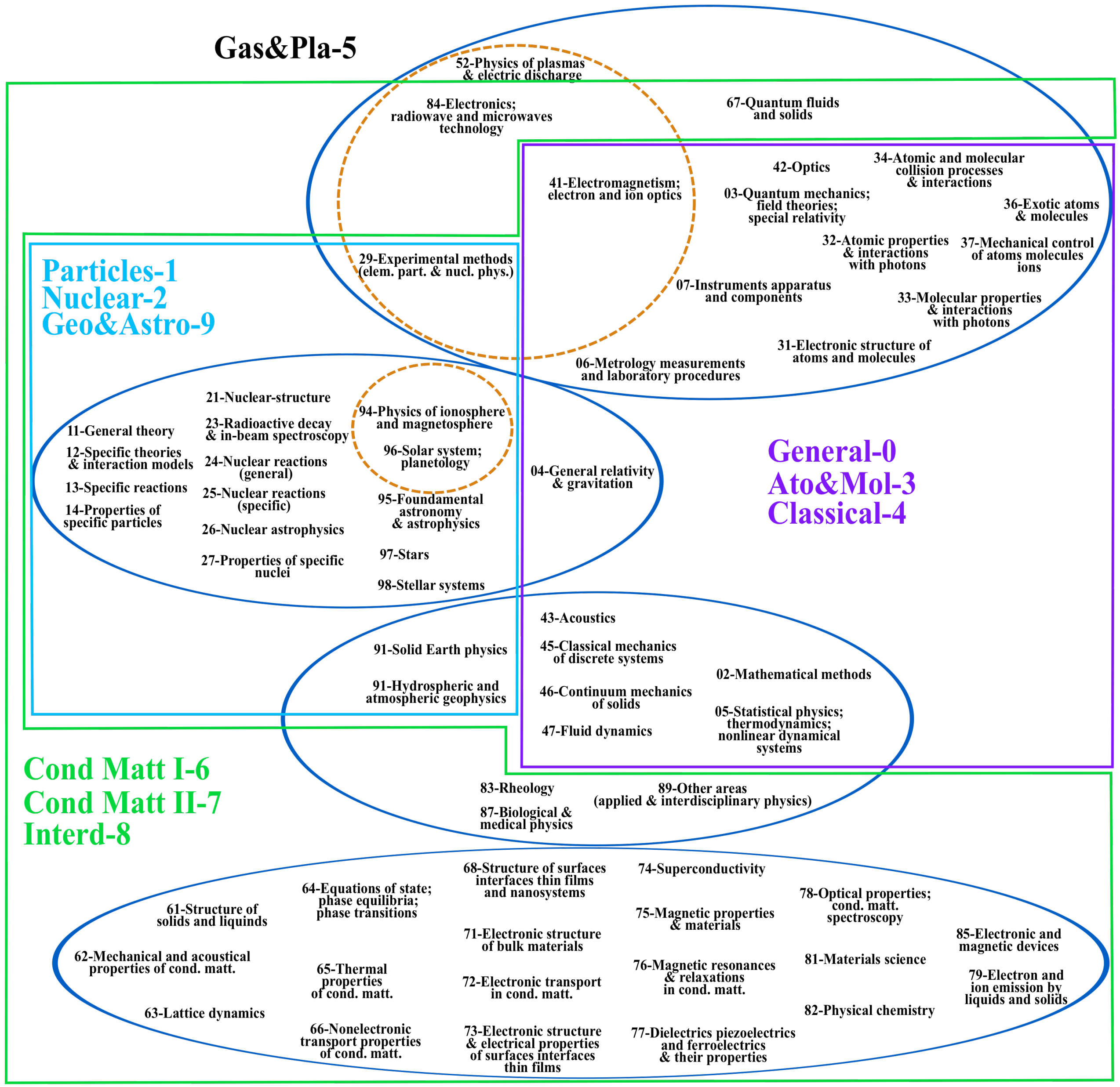




Communities in the network between layers



Second PACS hierarchy (66 layers)



Network between the layers