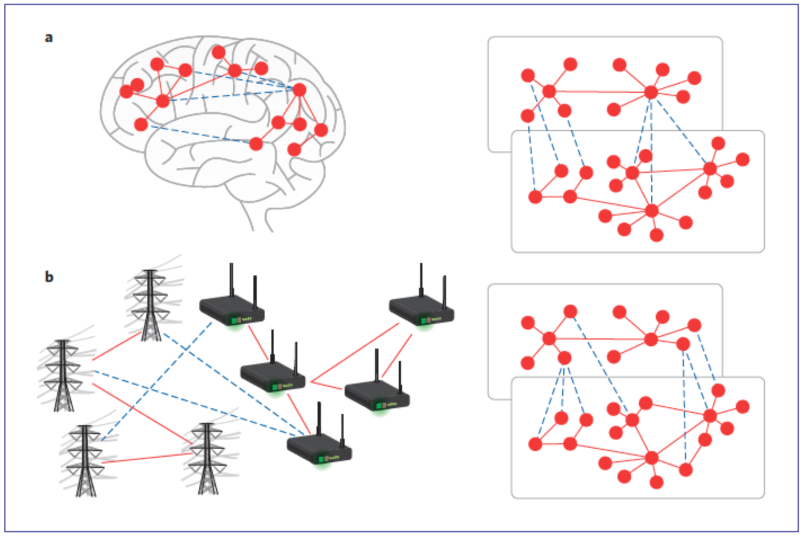

Robustness of Interdependent Multilayer Networks

LTCC Course Multilayer Networks
23-24 November 2016

Ginestra Bianconi

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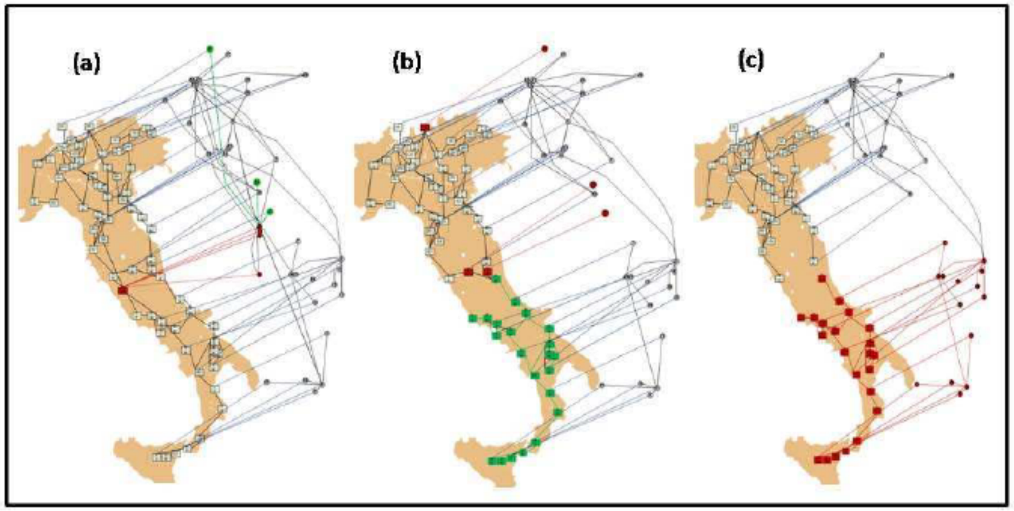
Queen Mary
University of London

Interacting infrastructure networks

Complex infrastructures are interdependent and a failure in one network can generate a cascade of failures in the Interdependent networks



Buldyrev et al. Nature 2010

Interacting Transportation networks

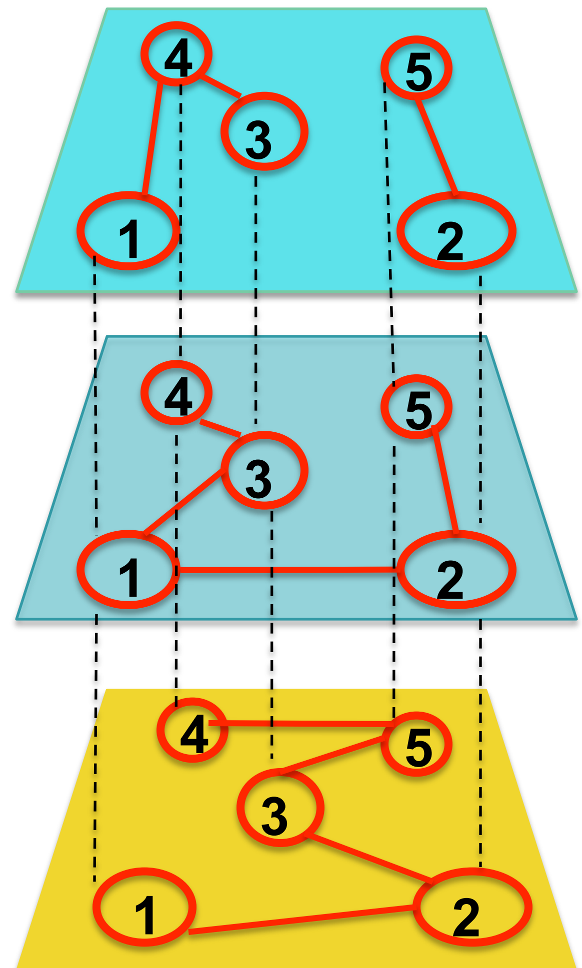
Transportation networks are another major example of interacting networks.

Here
blue lines represent short-range commuting flow by car or train
the red lines indicate airline flow for few selected cities

Vespignani Nature 2010

Multiplex Networks

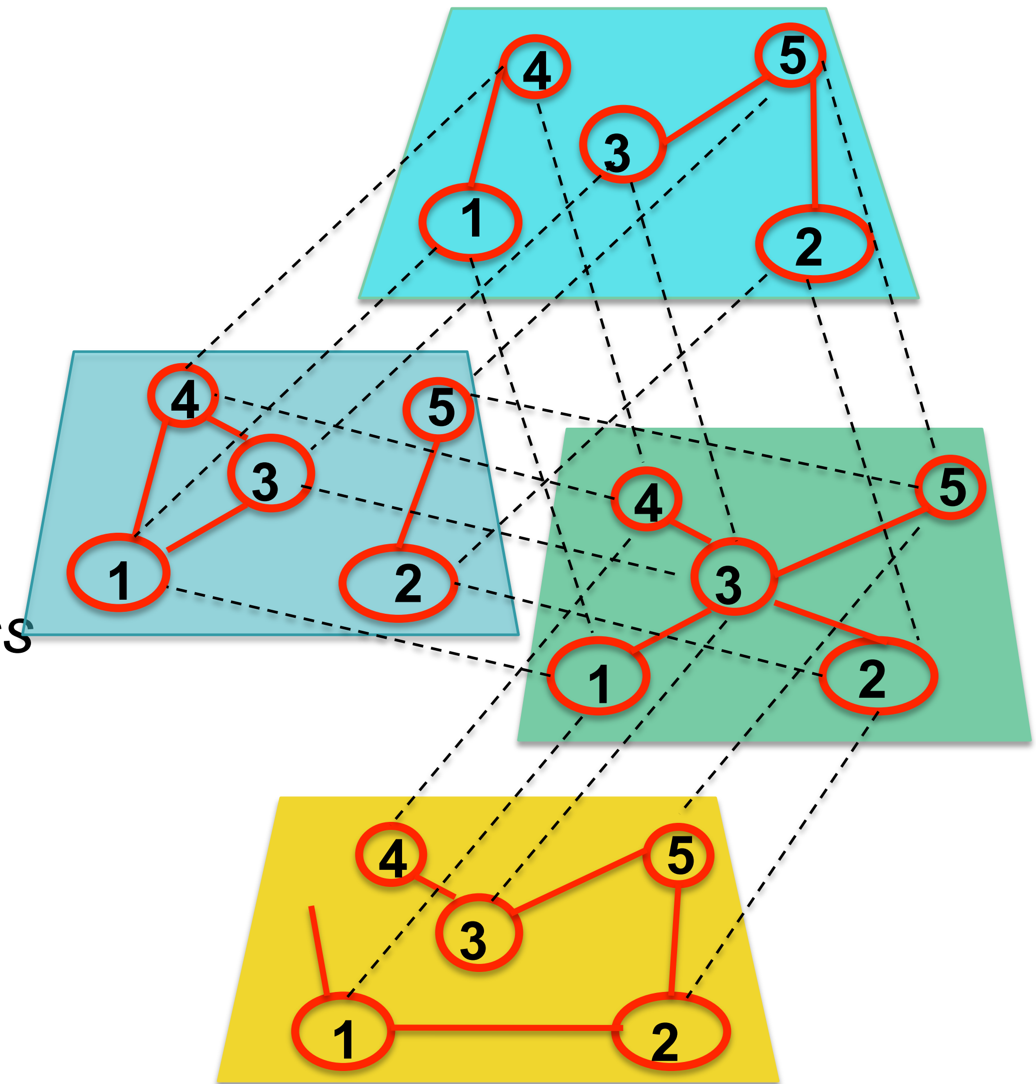
- A multiplex is formed by a set of nodes that are connected in different layers (M layers).
- Each node can be represented by a set of replica nodes present on each layer.
- Replica nodes are connected by interlinks (dashed lines).



Networks of Networks

Network of networks are formed by different sets of nodes.

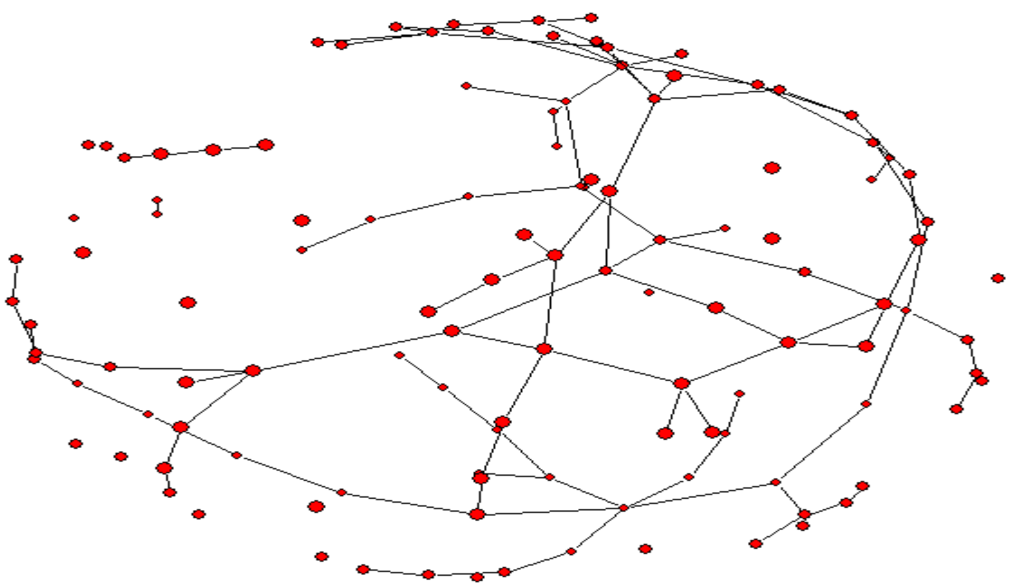
The nodes are connect within each layer and across different layers (interlinks)



Percolation in single networks

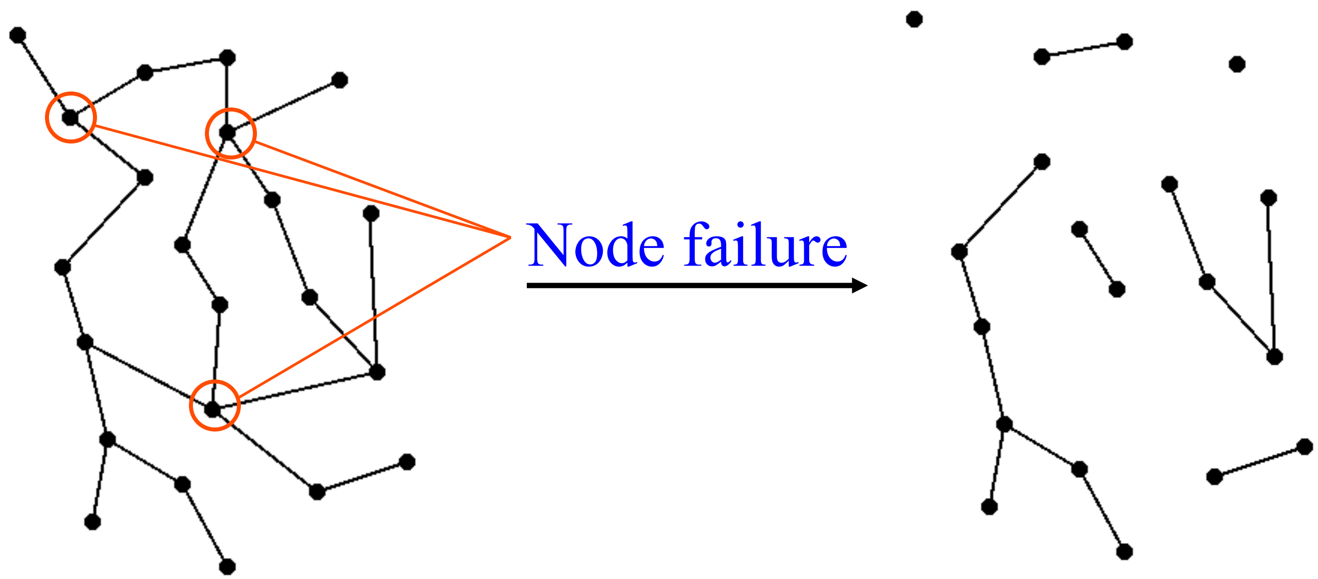
Giant component

- A **connected component** of a network is a subgraph induced by any set of nodes such that for each pair of nodes in the subgraph there is at least one path connecting them and such that no other node is connected to them by any path.
- The **giant component** is the connected component of the network which contains a number of nodes of the same order of magnitude of the total number of nodes in the network.



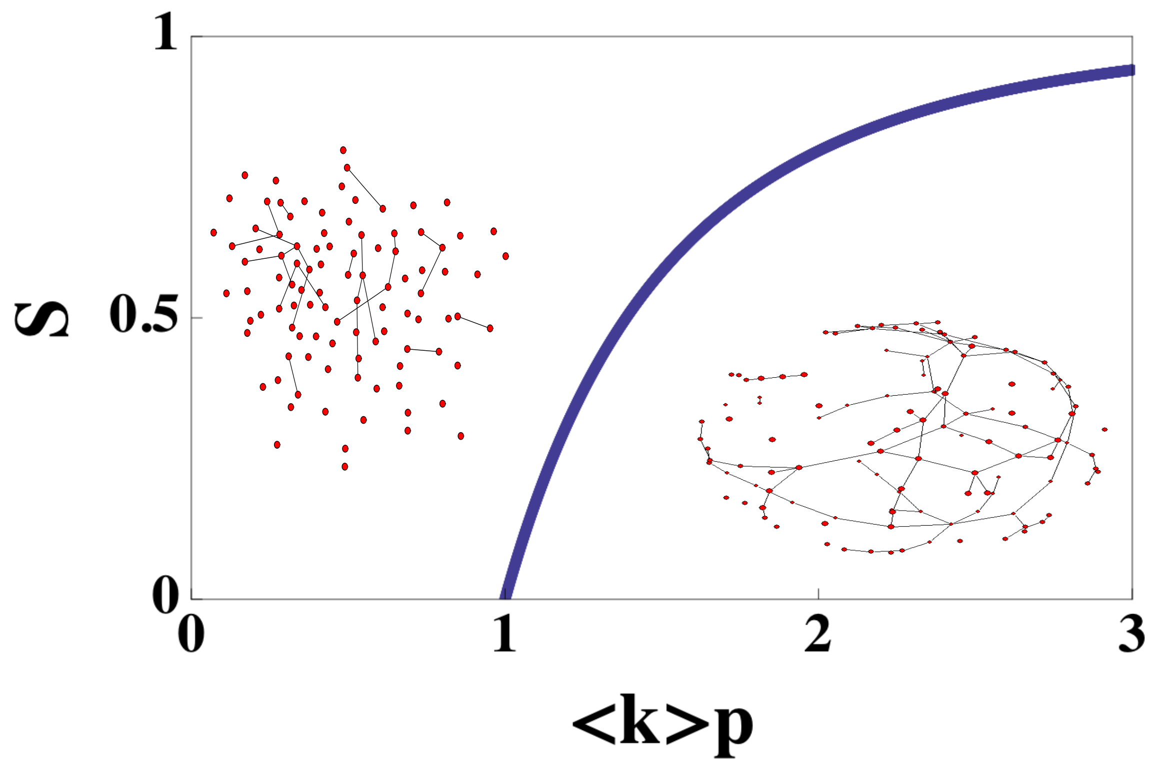
Robustness of

complex networks



We assume that a fraction 1-p of nodes is damaged.
We evaluate the robustness of the network by calculating the fraction S of nodes in the giant component after this inflicted damage.

Percolation transition in Poisson networks



S is the fraction of nodes in the giant component

$$S = \begin{cases} (p - p_c)^{\beta} & for\ p \geq p_c \\ 0 & for\ p < p_c \end{cases}$$

$$p_c = 1/\langle k \rangle$$

$$\beta = 1$$

Generalized percolation in multiplex networks

Interdependent multiplex networks

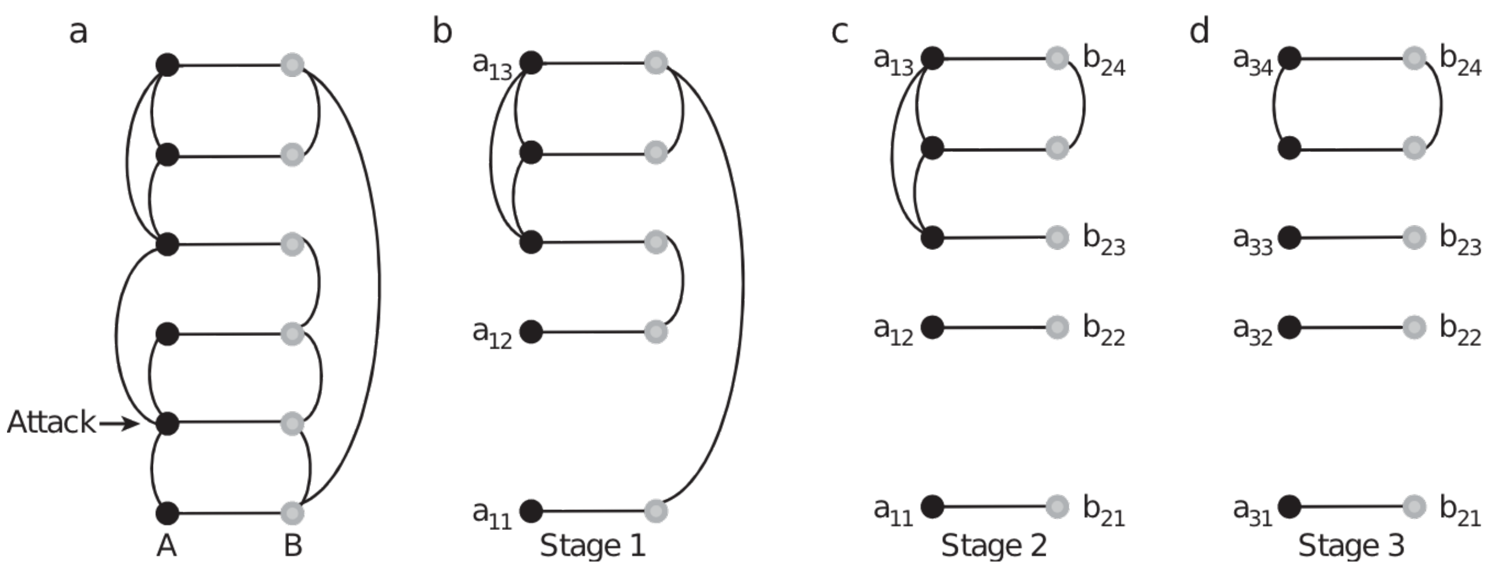
A multiplex network is **interdependent**

if all the interlinks imply the interdependence of the connected replica nodes.

Two nodes are interdependent if the damage of one node implies the damage of the other interdependent node, independently on the rest of the network.

Mutually connected giant component

Any two nodes of the mutually connected giant component are connected by at least one path in each layer of the multiplex network



Buldyrev et al. 2010

Case of a Poisson multiplex network with M Layers

Nodes are damaged with probability 1-p

Fraction of nodes in the GC of single Poisson layer with average degree c:

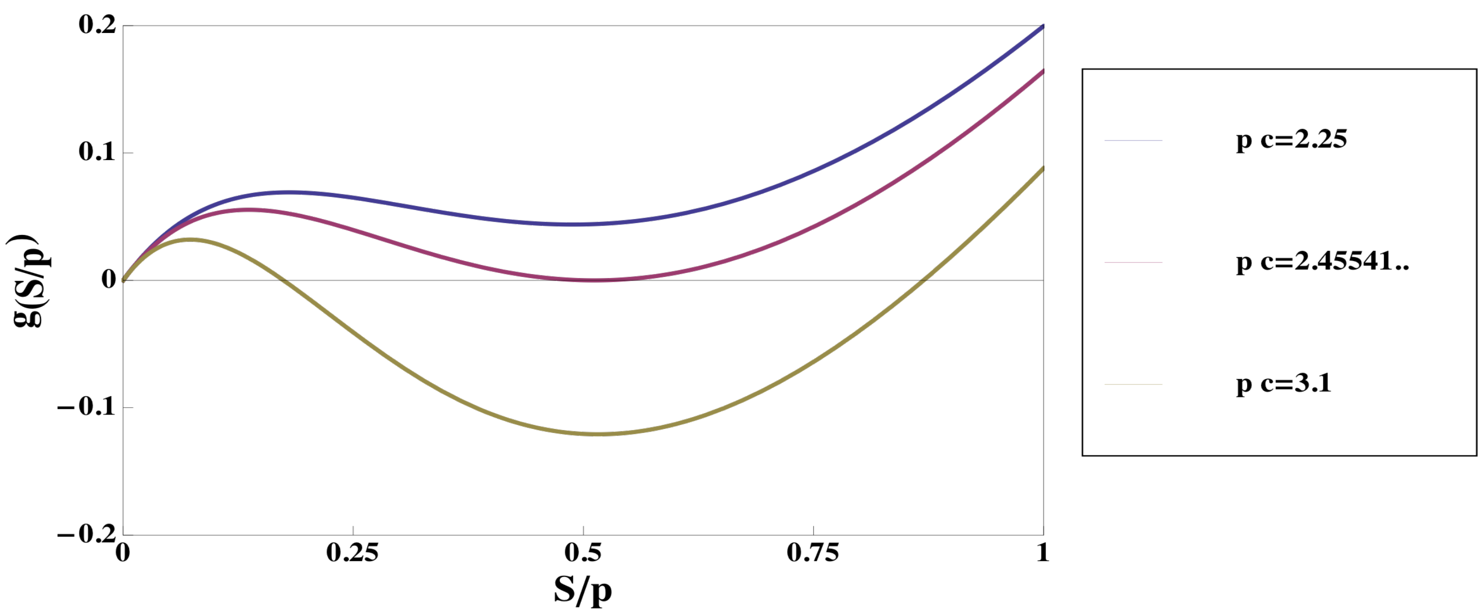
$$\boxed{S = p\left(1 - e^{-cS}\right)}$$

Fraction of nodes in the MCGC of multiplex network with M Poisson layers of average degree c:

$$\boxed{S = p\left(1 - e^{-cS}\right)^{M}}$$

Percolation on two interdependent Poisson networks with average degree c

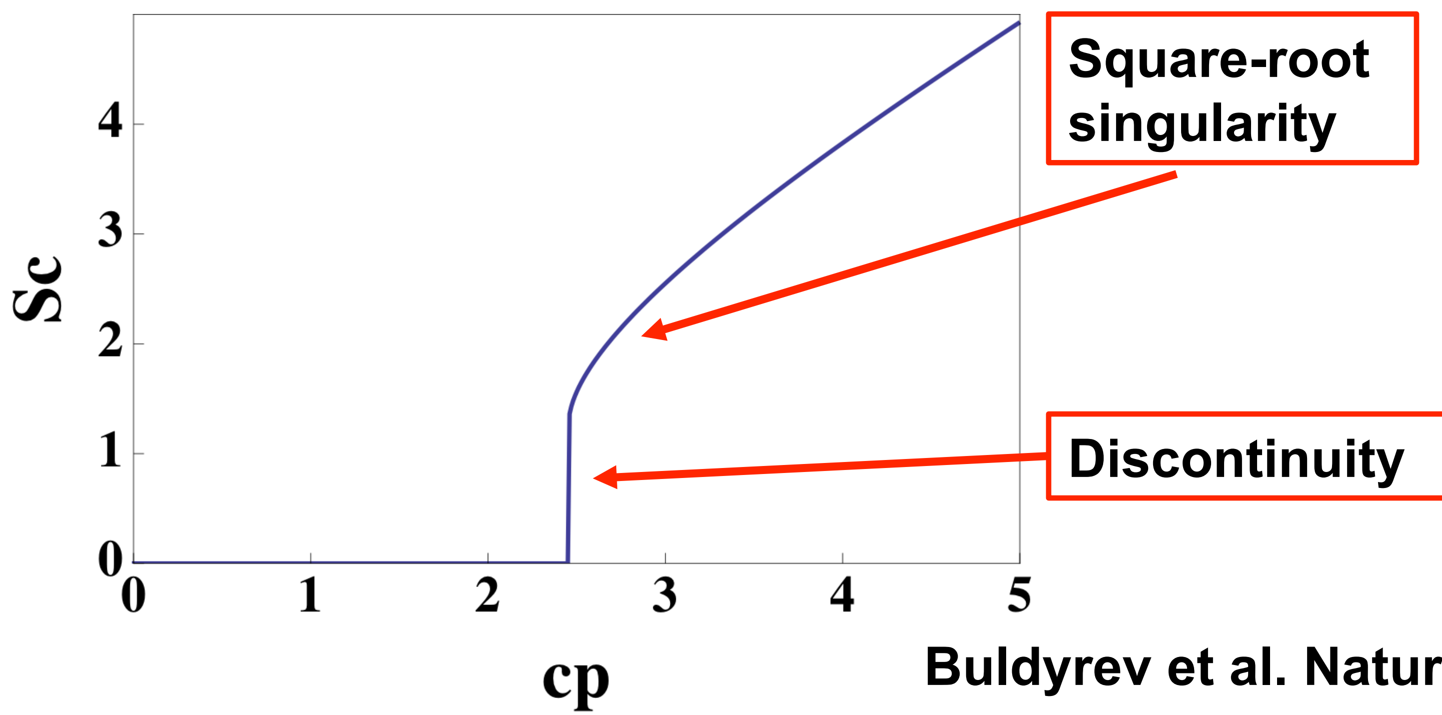
$$g(x = S/p) = x - \left(1 - e^{-cpx}\right)^2 = 0$$



The percolation transition at cp=2.455…
is discontinuous!

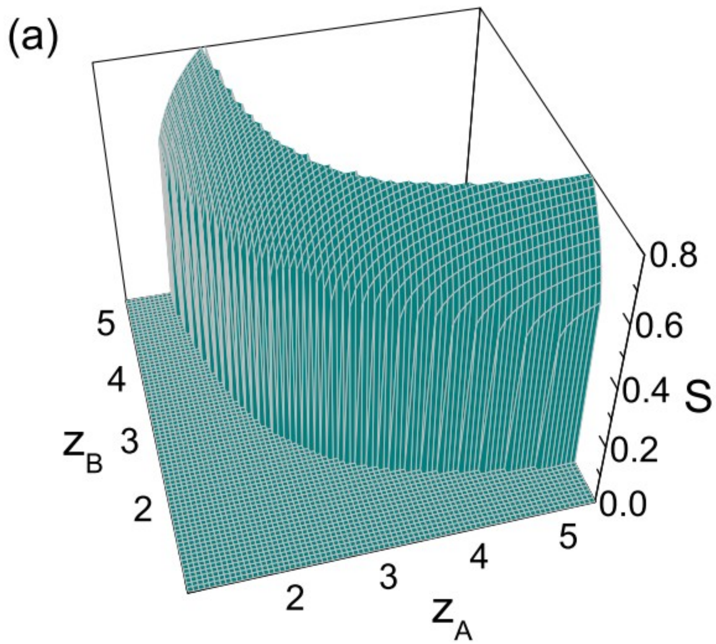
Discontinuous hybrid transition

Mutually connected giant component
in a muplex network with M=2 Poisson layers
of average degree c

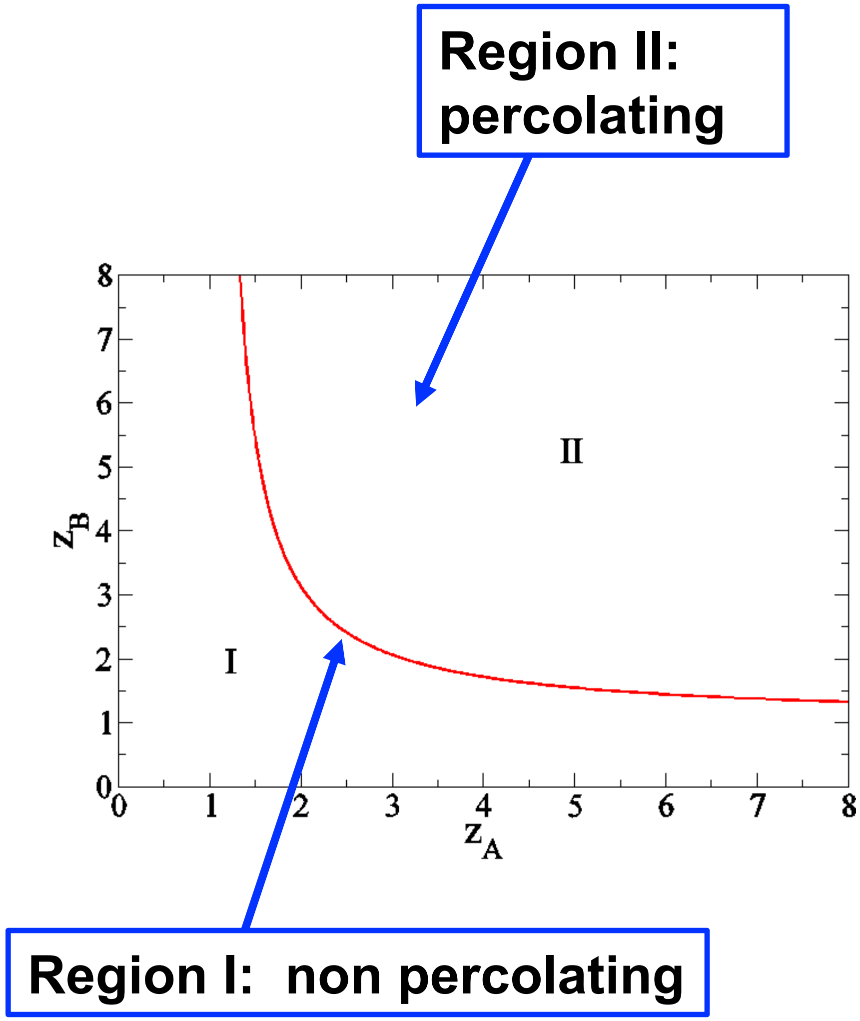


Buldyrev et al. Nature

Phase diagram of ER-ER interdepedent networks With average degree z_A and z_B

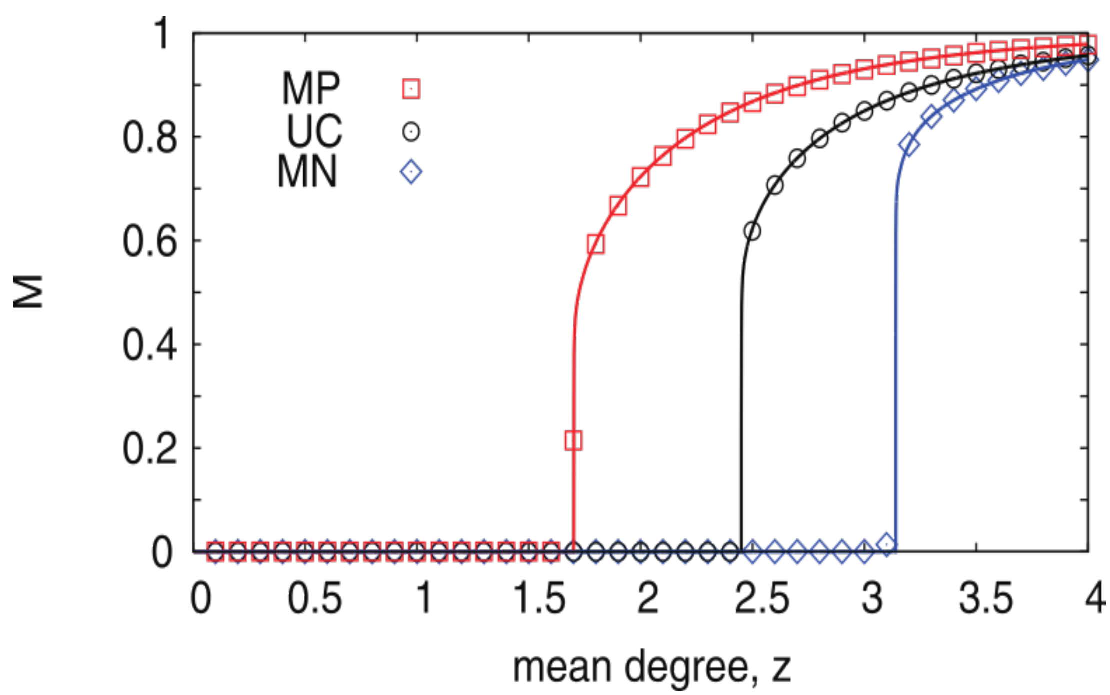


Son S.-W., et al. EPL(2012)



Effects of degree correlations

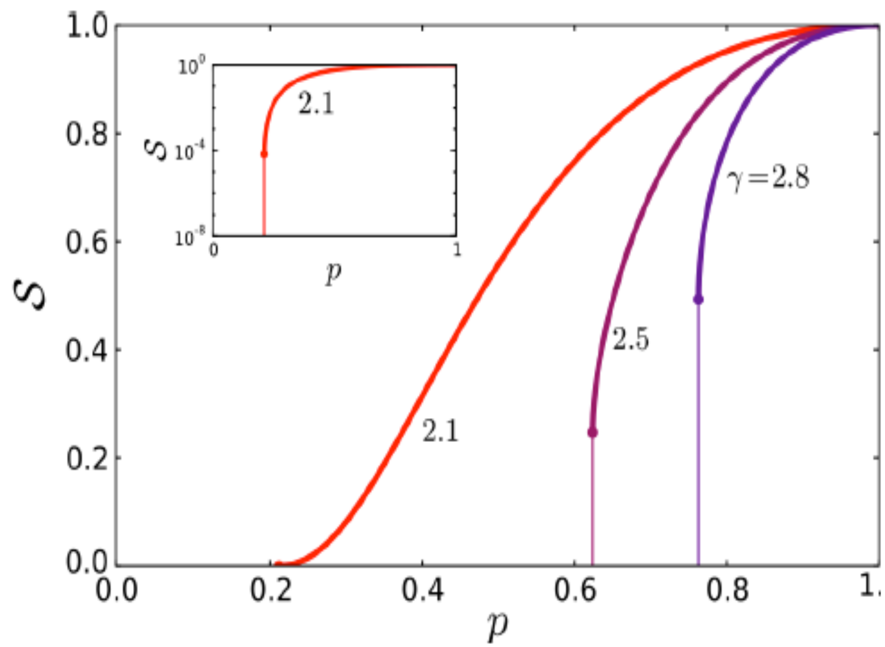
Positive degree correlations improve the robustness of a multiplex network.
MP maximally positive Degree correlations



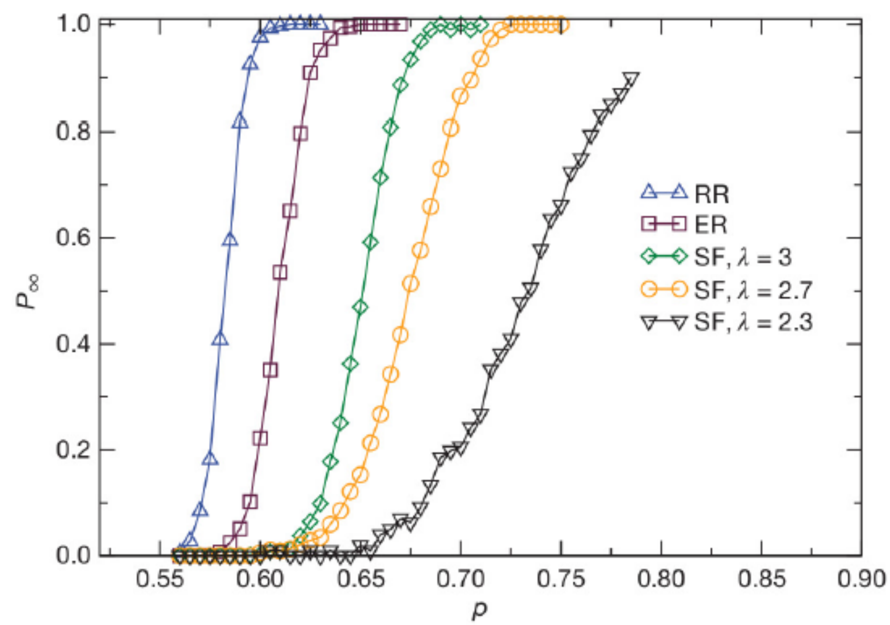
Negative degree correlations reduce the robustness of a multiplex network.
MN maximally negative degree correlations

Mutually connected component in scale free multiplex network

Fixed minimal degree (Baxter et al.)

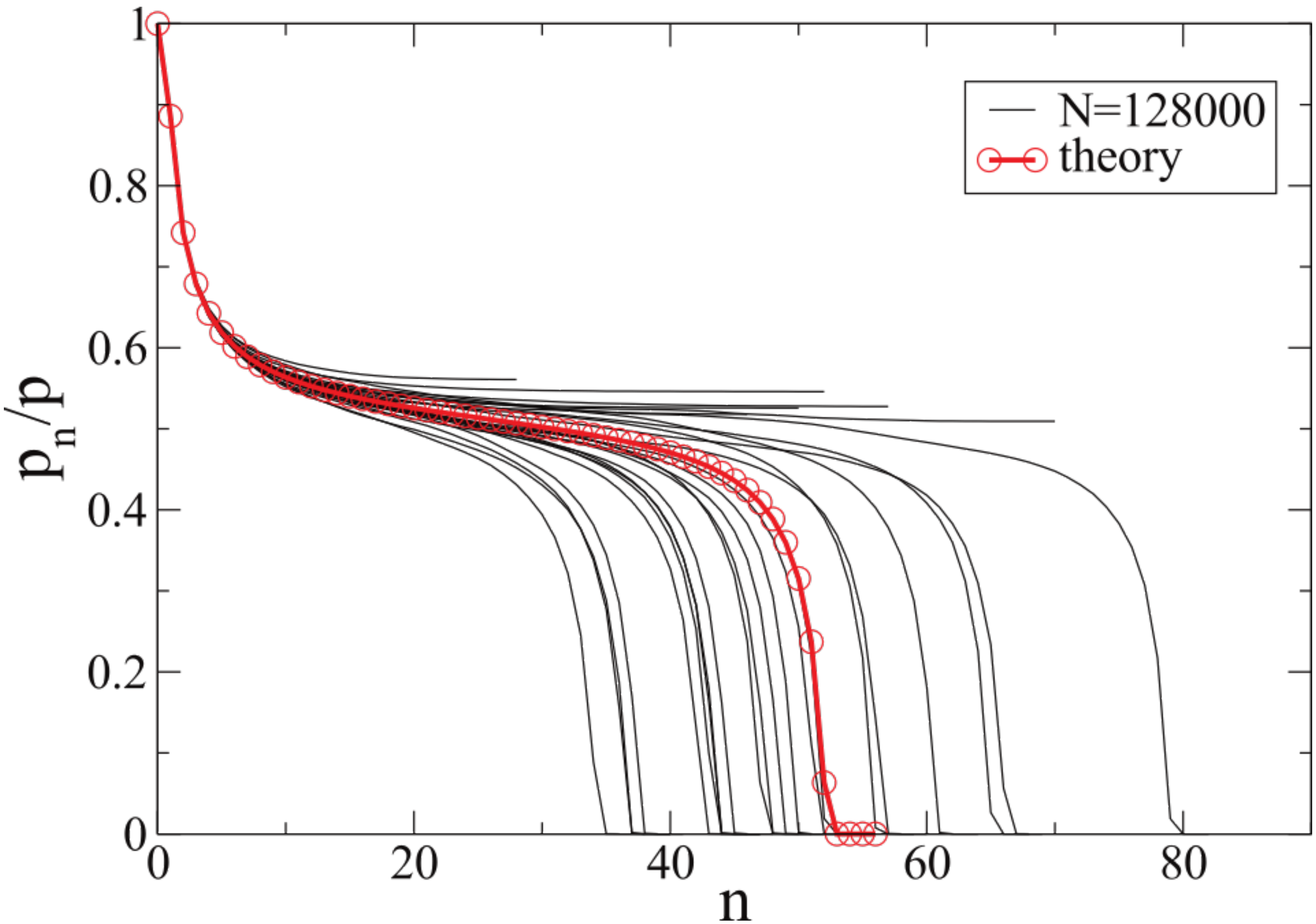


Fixed average degree (Parshani et al.)



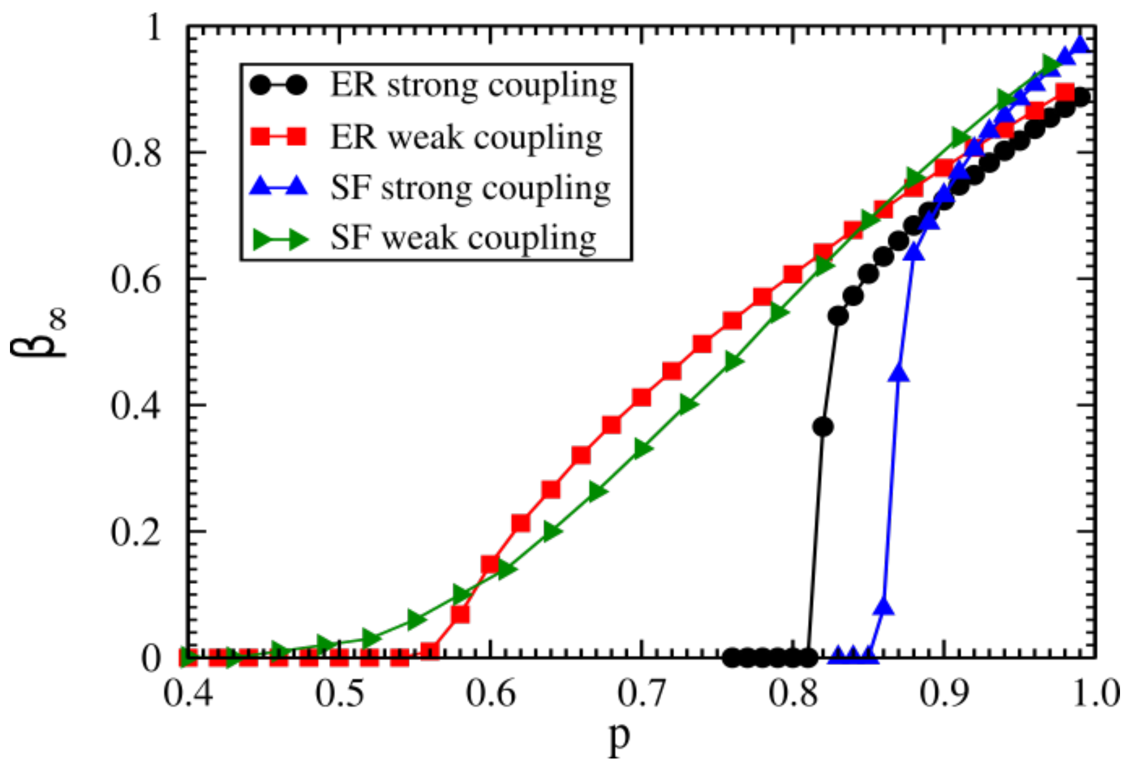
The discontinuity decreases, p_c increases with decreasing γ exponent

Cascade of failure events at the percolation transition



Buldyrev et al. Nature

Partial interdependence changes the nature of the percolation transition



Allowing for partial interdependence can change the nature of the transition from discontinuous to continuous.

Duplex network with Poisson Layers and Link Overlap

Duplex networks with Poisson multidegree distribution with

$$\langle k^{01}\rangle = \langle k^{10}\rangle = c_1$$

$$\langle k^{11}\rangle = c_2$$

MCGC

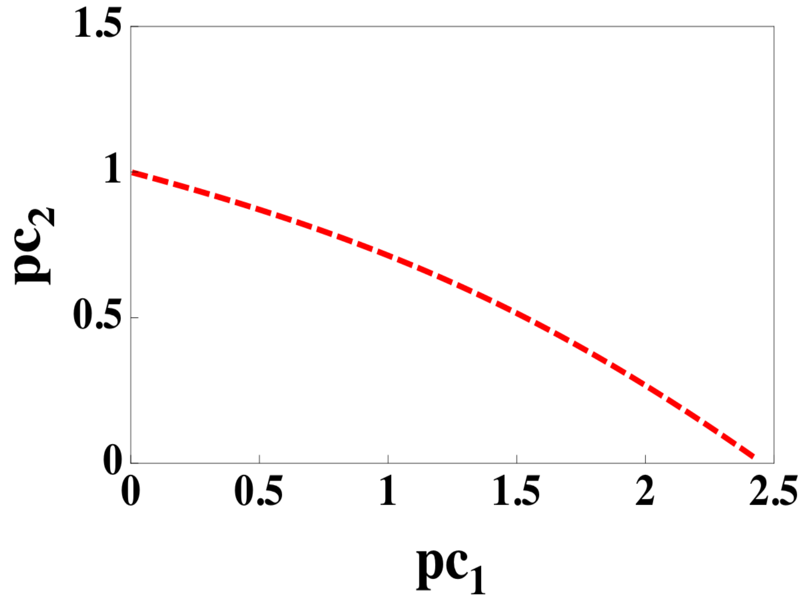
$$\boxed{\begin{aligned} S &= p\left(1-2e^{-c_1 S - c_2(S+S_{2,1})} + e^{-2c_1 S - c_2(S+S_{2,1})}\right) \\ S_{(1,1),(1,0)} = S_{2,1} &= p\left(e^{-c_1 S - c_2(S+S_{2,1})} - e^{-2c_1 S - c_2(S+2S_{2,1})}\right) \end{aligned}}$$

Phase diagram for the MCGC in a duplex network

Duplex networks with Poisson multidegree distribution with

$$\langle k^{01} \rangle = \langle k^{10} \rangle = c_1$$

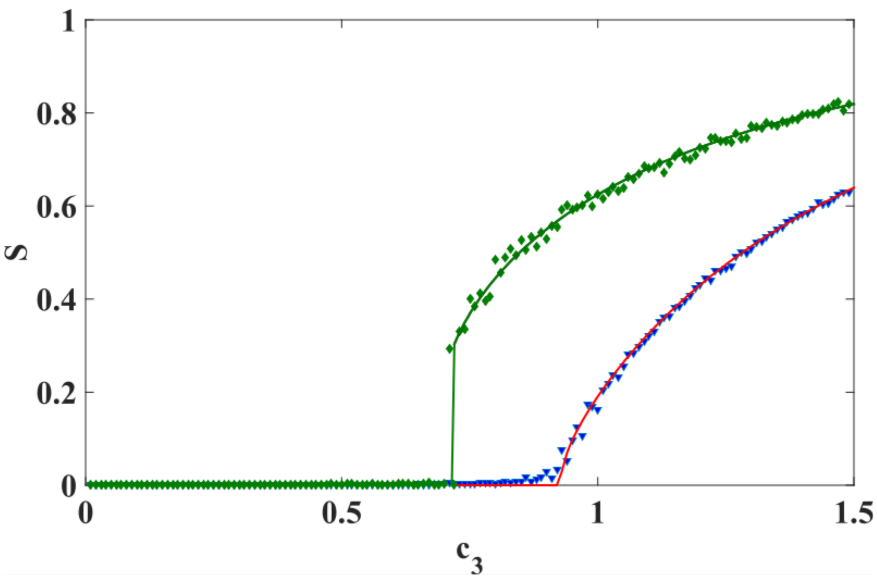
$$\langle k^{11} \rangle = c_2$$



Cellai et al (2016)

Multiplex network with three Poisson layers and link overlap

Multiplex networks with three layers with Poisson multidegree distribution



$$\langle k^{001}\rangle = \langle k^{010}\rangle = \langle k^{100}\rangle = c_1$$

$$\langle k^{110}\rangle = \langle k^{101}\rangle = \langle k^{011}\rangle = c_2$$

$$\langle k^{111}\rangle = c_3$$

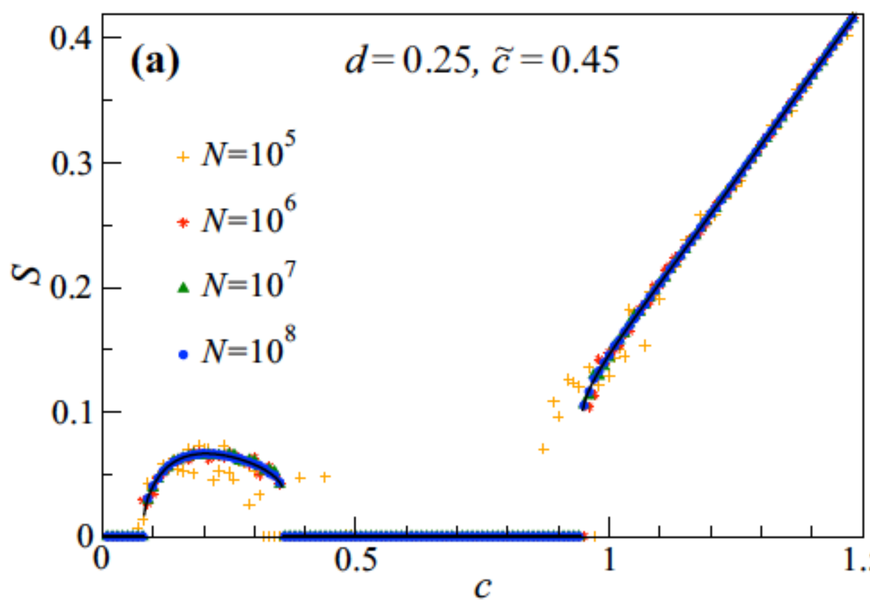
The determination of the MCGC involve solving a non-linear system of three variables

The network has a continuous phase transition only for complete overlap of the links

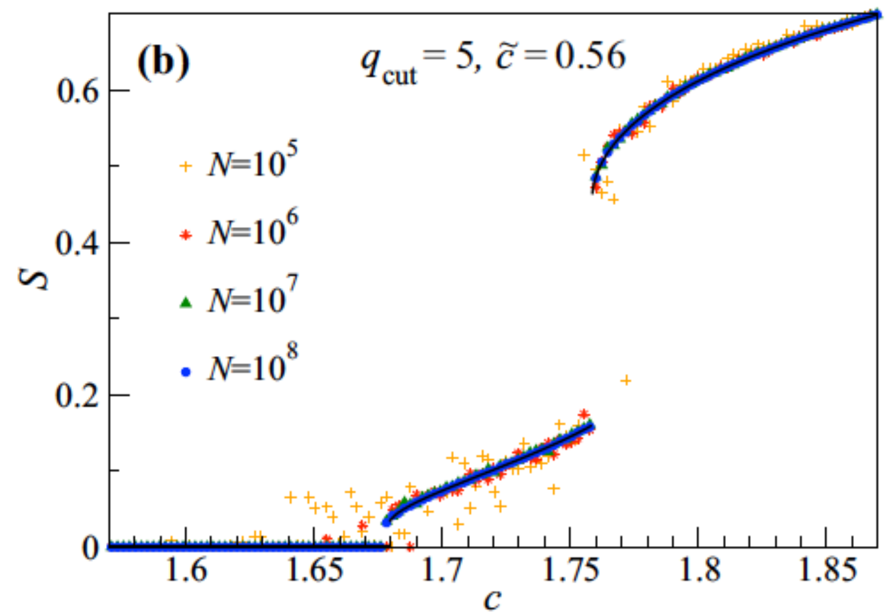
Multiplex networks with correlated multidegrees

This theory can be extended to multiplex networks with correlated multidegrees

Dissasortative correlations



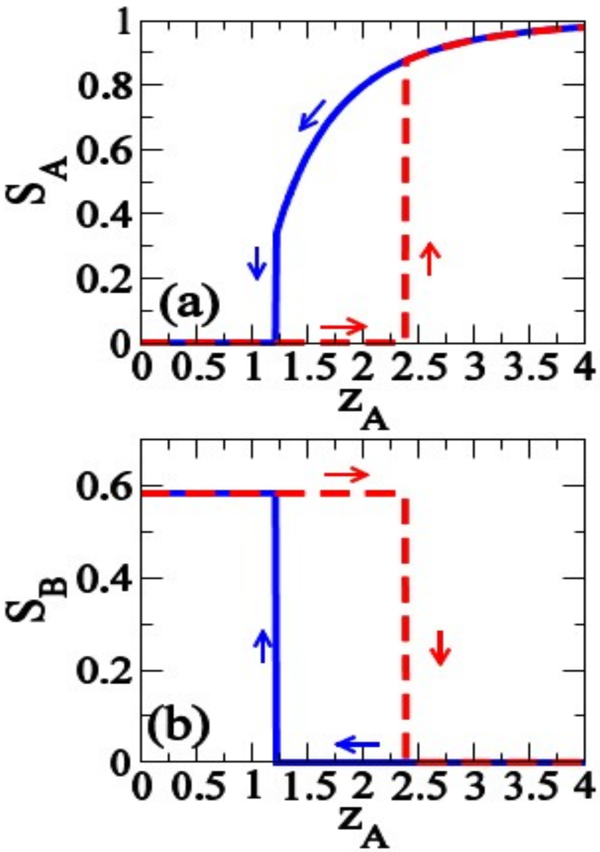
Assortative correlations

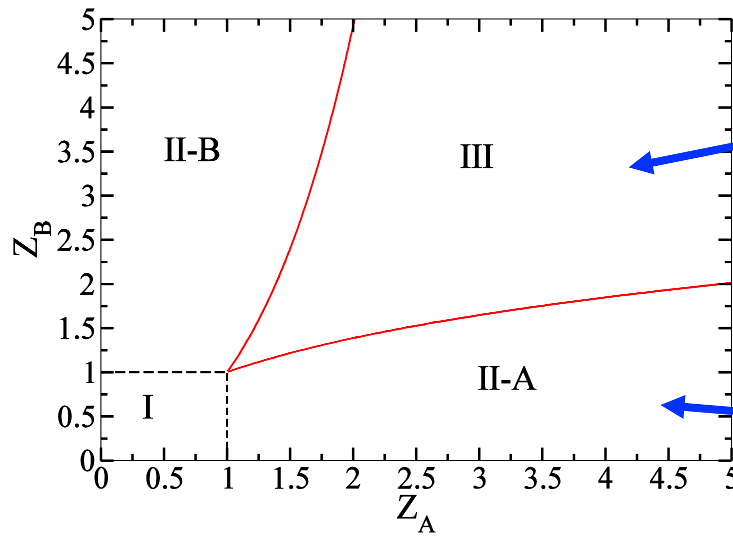


Baxter et al. 2016

Competing networks

The function of a node in a network is incompatible with the function of the same node in the other network network





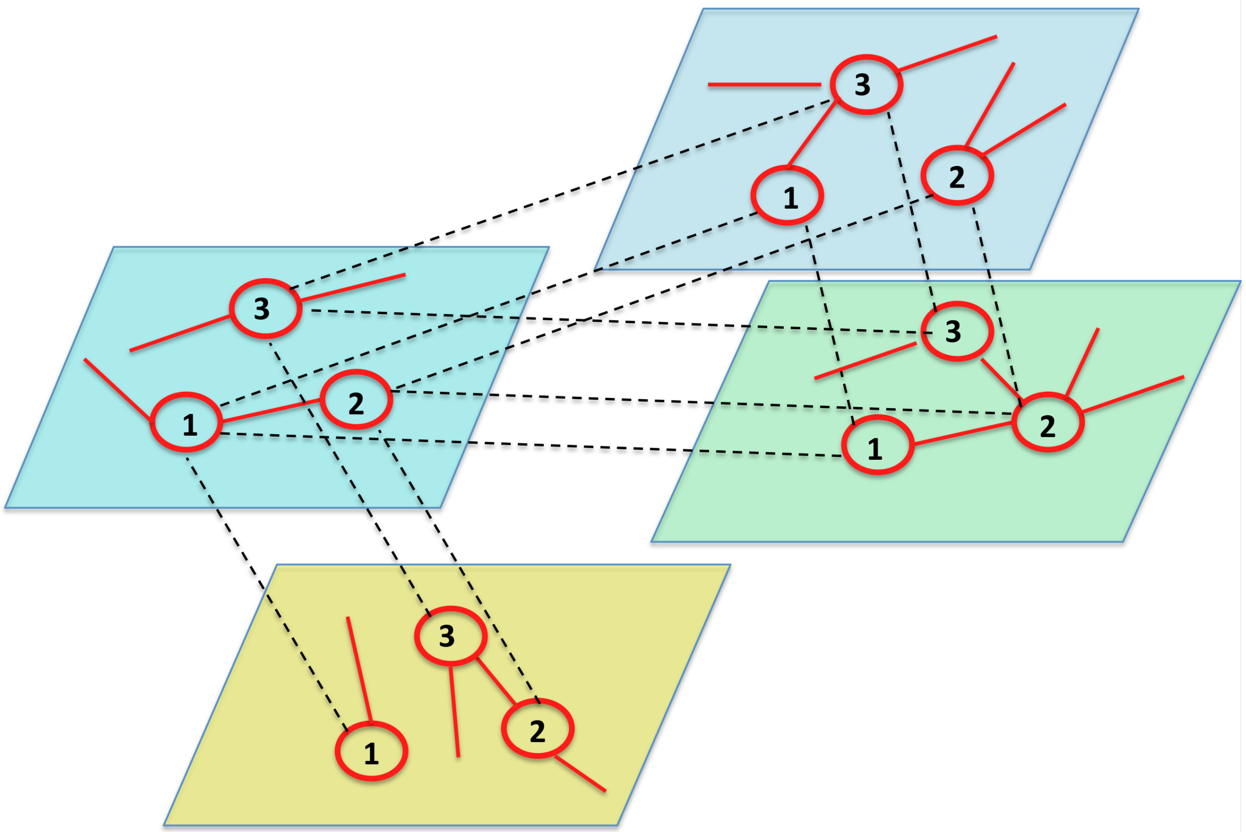
Region III: Bistable region, either one of the networks percolates

Region II: only one network can percolate

K. Zhao et al. JSTAT (2013)

Percolation in network of networks

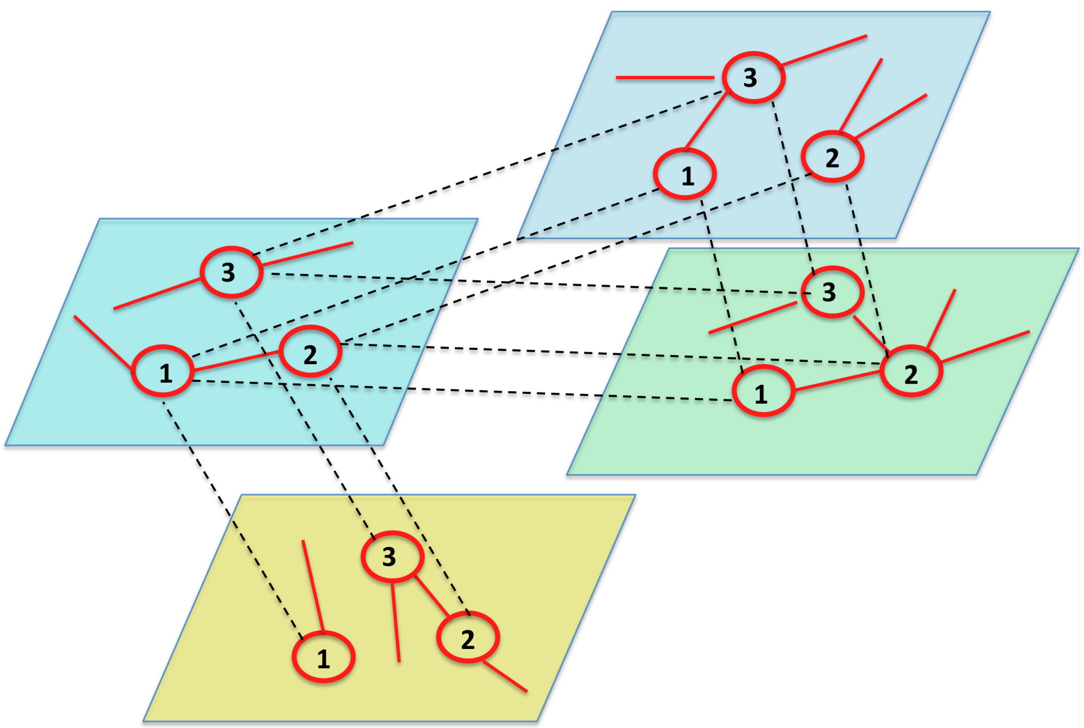
Network of Networks Case I



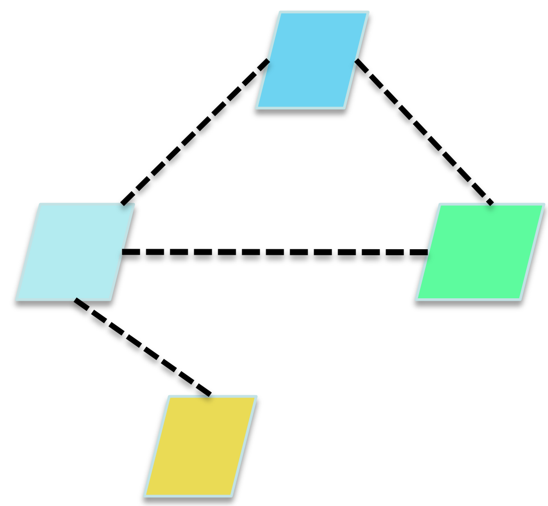
If a network is interacting with another network all the nodes of the network are interdependent with their "replica nodes" on the other network and vice versa.

The network of networks

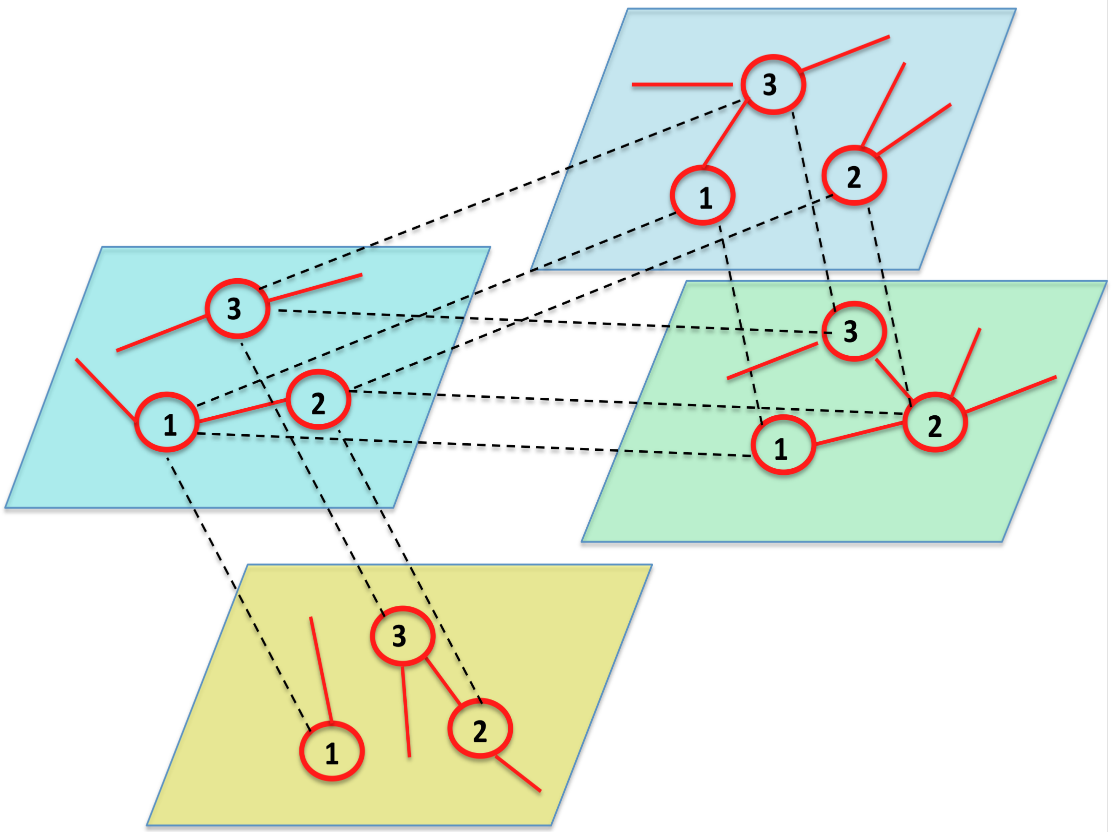
Interacting networks



Supernetwork



Network of Networks Case I



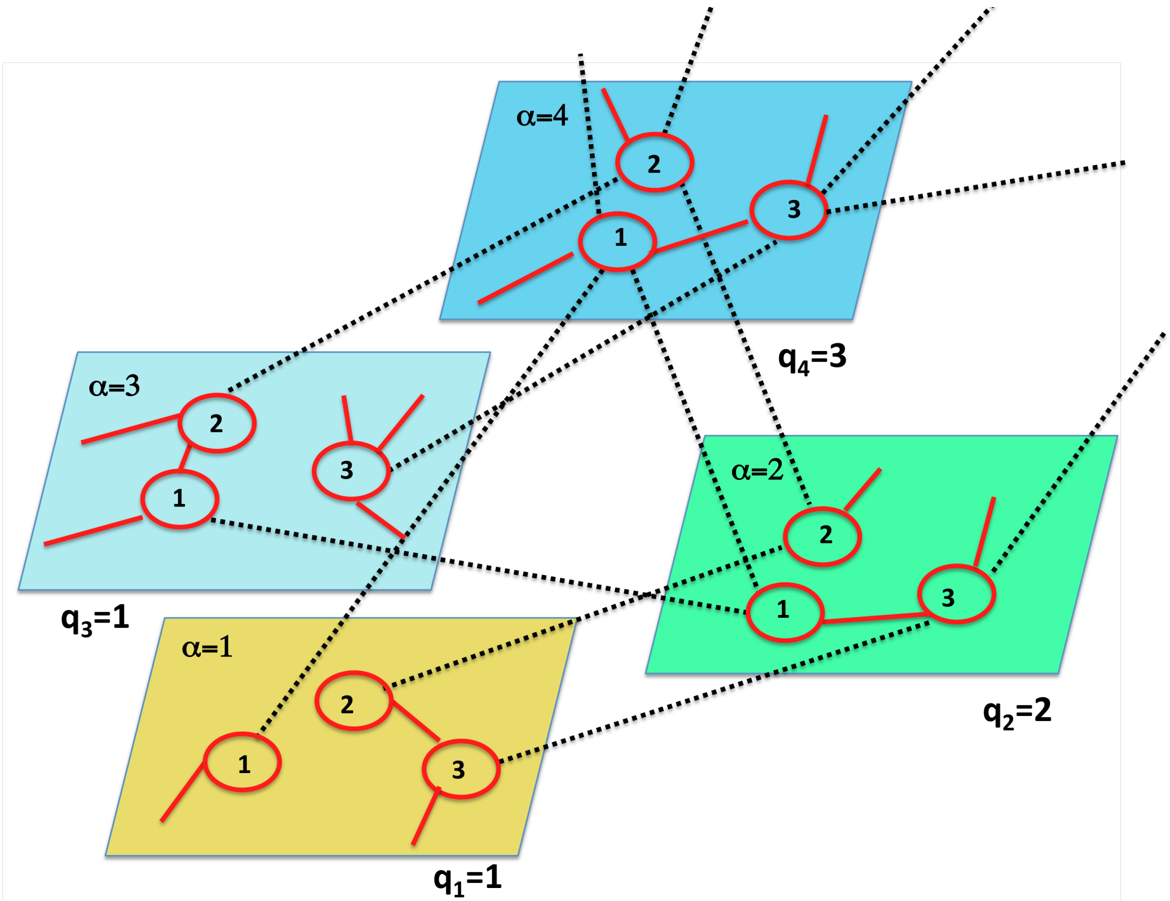
A node is in the mutually connected giant component if all the nodes that can be reached by interlinks have at least one neighbor in their layer that is in the percolation cluster.

Robustness of the network of networks

- **The robustness of a network of networks belonging to the case I is independent on the structure of the network of networks as long as the network of networks is connected.**
- All the layers start to percolate when the fraction of non-damaged nodes $p>p_c$
- The transition is discontinuous as long as $M>1$ if the layers are not correlated.

G. Bianconi, S.N. Dorogovtsev and J. F. F. Mendes 2014

Network of networks Case II



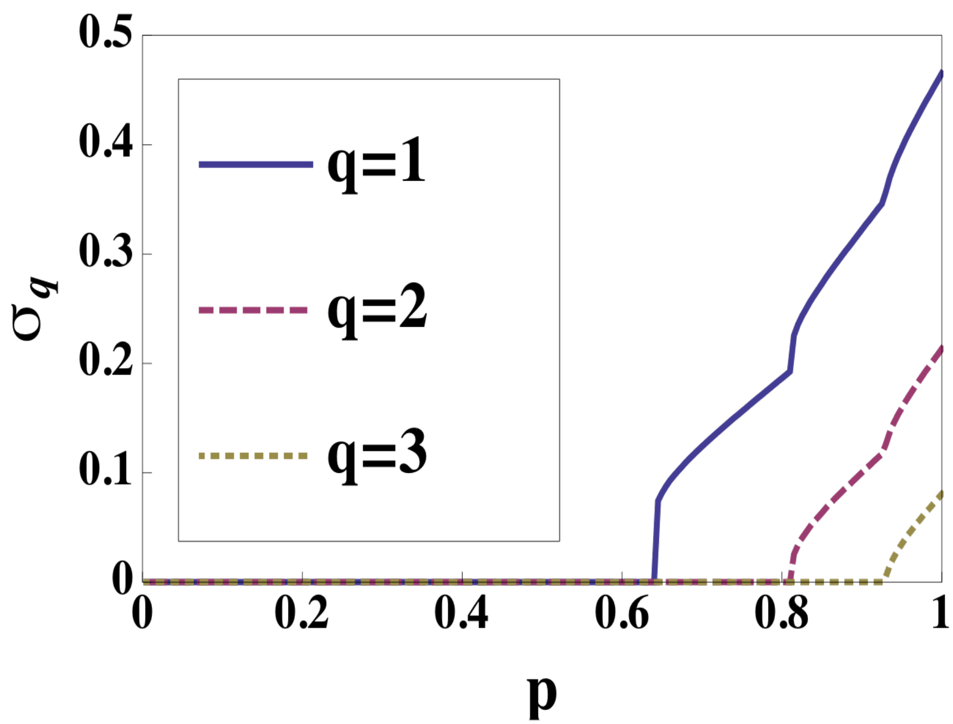
Every layer α has a supradegree $q_{,\alpha.}$

Therefore every node of layer α has q_α links to q_α replica nodes in some other layer chosen randomly

Main results for case II

- **The layers with higher superdegree are more fragile than layers with low superdegree.**
- **In the networks there are multiple percolation transitions corresponding to the activation of layers with increasing value of the superdegree.**
- Each of these transitions is discontinuous is the networks in the different layers are not correlated for r=1
- **If r<1 some of these transitions can become continuous**

Percolation in layers with superdegree q



Case γ=2.8
c=20

Multiple phase transitions!
Layers with larger superdegree are more vulnerable!

G. Bianconi and S.N Dorogovstev (2014)

Nature Physics News & Views

news & views

MULTILAYER NETWORKS

Dangerous liaisons?

Many networks interact with one another by forming multilayer networks, but these structures can lead to large cascading failures. The secret that guarantees the robustness of multilayer networks seems to be in their correlations.

Ginestra Bianconi

Natural complex systems evolve according to chance and necessity — trial and error — because they are driven by biological evolution. The expectation is that networks describing natural complex systems, such as the brain and biological networks within the cell, should be robust to random failure. Otherwise, they would have not survived under evolutionary pressure. But many natural networks do not live in isolation; instead they interact with one another to form multilayer networks — and evidence is mounting that random networks of networks are acutely susceptible to failure. Writing in *Nature Physics*, Saulo Reis and colleagues[1] have now identified the key correlations responsible for maintaining robustness within these multilayer networks.

In the past fifteen years, network theory[2,3] has granted solid ground to the expectation that natural networks resist failure. It has also extended the realm of robust systems to man-made self-organized networks that do not obey a centralized design, such as the Internet or the World Wide Web. In fact, it has been shown that many isolated complex biological, technological and social networks are scale free, meaning that their nodes

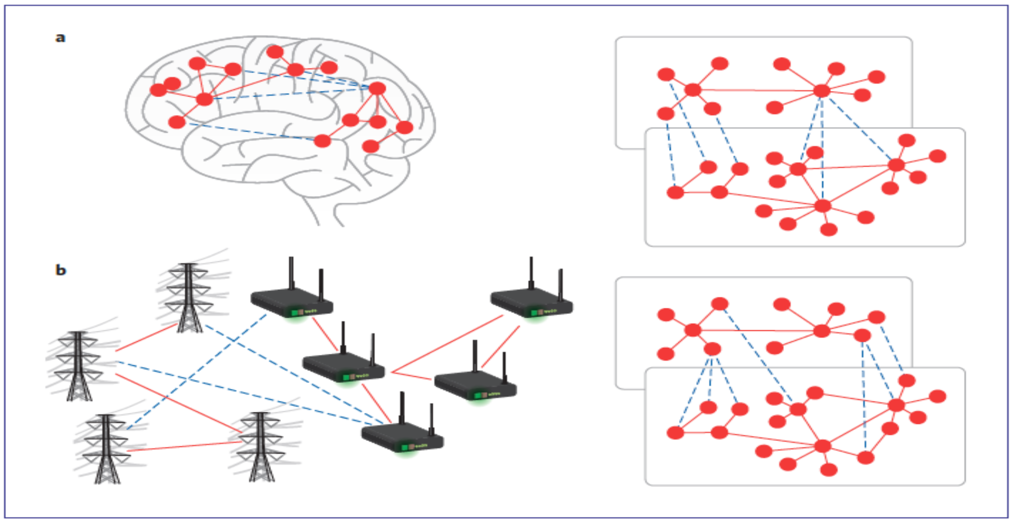


Figure 1 | Reis *et al.*[1] have shown that correlations between intra- (red) and interlayer (blue dotted) interactions influence the robustness of multilayer networks. **a**, In the brain, each network layer has multilayer assortativity and the hubs in each layer are also the nodes with more interlinks, so liaisons between layers are trustworthy. **b**, In complex infrastructures (such as power grids and the Internet), if the interlinks are random, the resulting multilayer network is affected by large cascades of failures[6], and liaisons can be considered dangerous.

Redundant interdependencies

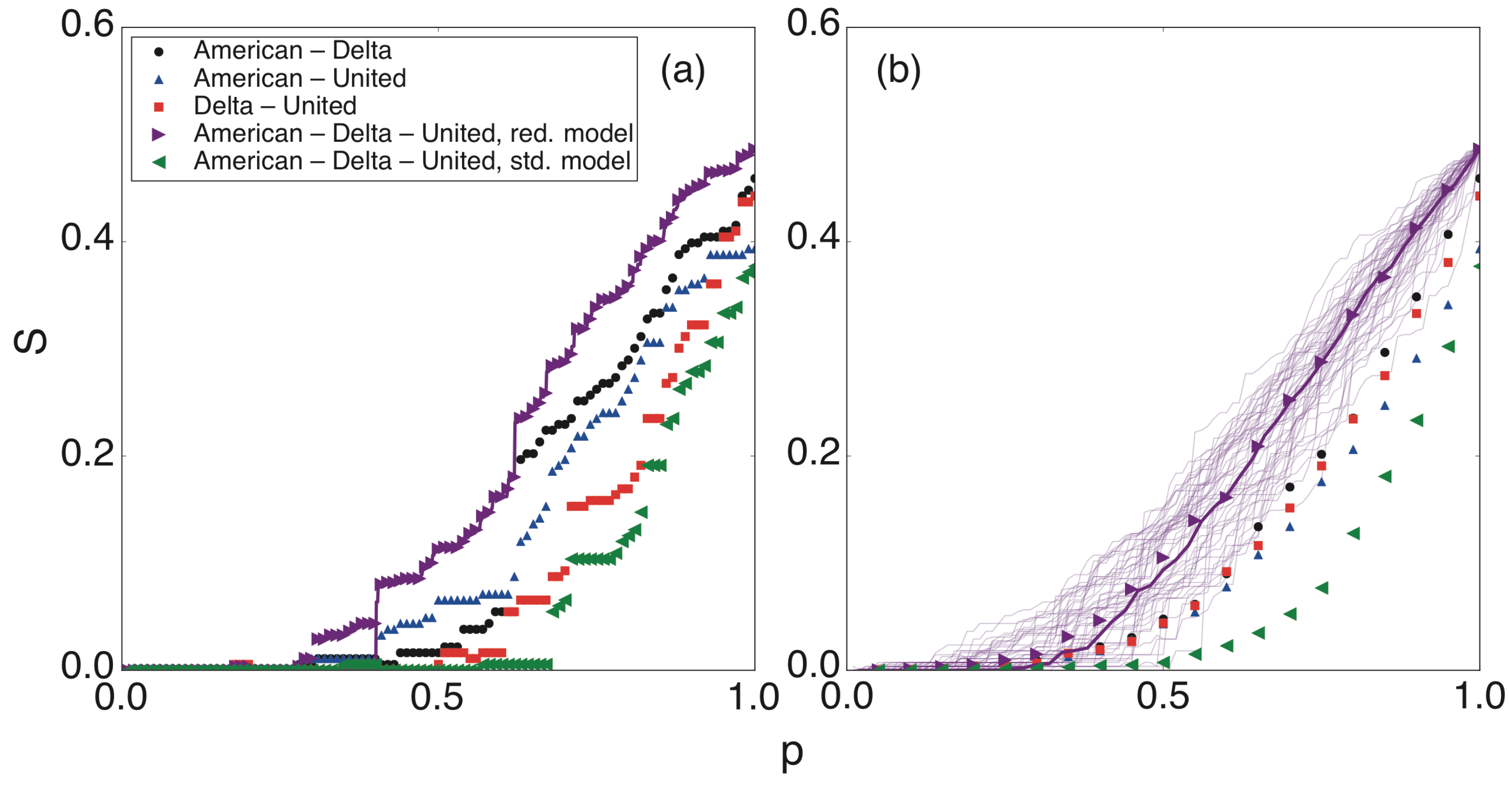
If a node is interdependent on each one of its replica nodes, the more layers we add the more fragile is a network.

If, instead interdependencies are redundant

and a node is the Redundant MCGC if at least one replica node is active,

then more layers we add to the network the more robust it becomes

Redundant interdependencies boost the robustness of multilayer networks



Radicchi Bianconi 2016

Conclusions

Percolation on interdependent networks captures the possible mechanisms yielding fragile multilayer networks

Percolation in multilayer interdependent networks display surprising novel phenomena

- In presence of interdependencies, the percolation transition becomes discontinuous and hybrid and is characterized by large avalanches of failure events.
- In presence of partial interdependencies it can become continuous.
- In network of networks it is possible to observe multiple phase transition.
- Redundant interdependencies might explain why many natural made networks have many layers as in this framework the robustness increases with the number of layers.

References

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- Parshani et al. *PRL* 105, 048701 (2010).
- Min et al. PRE 89, 042811(2014).
- Son et al. EPL 97, 16006 (2012).
- Baxter et al. PRL109, 248701 (2012).
- Bianconi and Dorogovtsev 89, 062814 (2014).
- Bianconi et al. PRE 91, 012804 (2015).
- Bianconi and Radicchi, *arXiv:1610.05378 (2016)*.
- K. Zhao et al. JSTAT P05005 (2013).

Message passing algorithm for percolation

Message passing algorithms are widely used for characterizing critical phenomena and dynamical systems in complex networks

- **Percolation on single networks**

(Karrer, Newman, Zdeborova PRL 2014)

- **Network control**

(Liu, Slotine & Barabasi Nature 2011)

- **Epidemic spreading in multi-slice networks**

(Valdano et al. PRX 2015)

Message passage algorithm for the Giant Component of a single network

The initial node damage is indicated by the variables s_i associated to the nodes of the network:
$s_i=0$ if node i is damaged and $s_i=1$ otherwise.

The message going from node i to not j follows

$$\boxed{\sigma_{i\to j} = s_i\left(1-\prod_{l\in N(i)\setminus j}(1-\sigma_{l\to i})\right)}$$

The node i is in the giant component if $\sigma_i=1$ otherwise $\sigma_i=0$ where

$$\boxed{\sigma_i = s_i\left(1-\prod_{l\in N(i)}(1-\sigma_{l\to i})\right)}$$

Message passage algorithm for the Mutually Connected Giant Component in absence of link overlap

The initial node damage is indicated by putting
$s_i=0$ if node i is damaged and $s_i=1$ otherwise.

The generic message going from node i to node j is updated according to

$$\boxed{\sigma_{i\to j} = s_i \prod_{\alpha=1,M}\left(1-\prod_{l\in N_\alpha(i)\setminus j}(1-\sigma_{l\to i})\right)}$$

A node i is in the MCGC if $\sigma_i=1$ where

$$\boxed{\sigma_i = s_i \prod_{\alpha=1,M}\left(1-\prod_{l\in N_\alpha(i)}(1-\sigma_{l\to i})\right)}$$

Case of a Poisson multiplex network with M Layers

Nodes are damaged with probability 1-p

Fraction of nodes in the GC of single Poisson layer with average degree c:

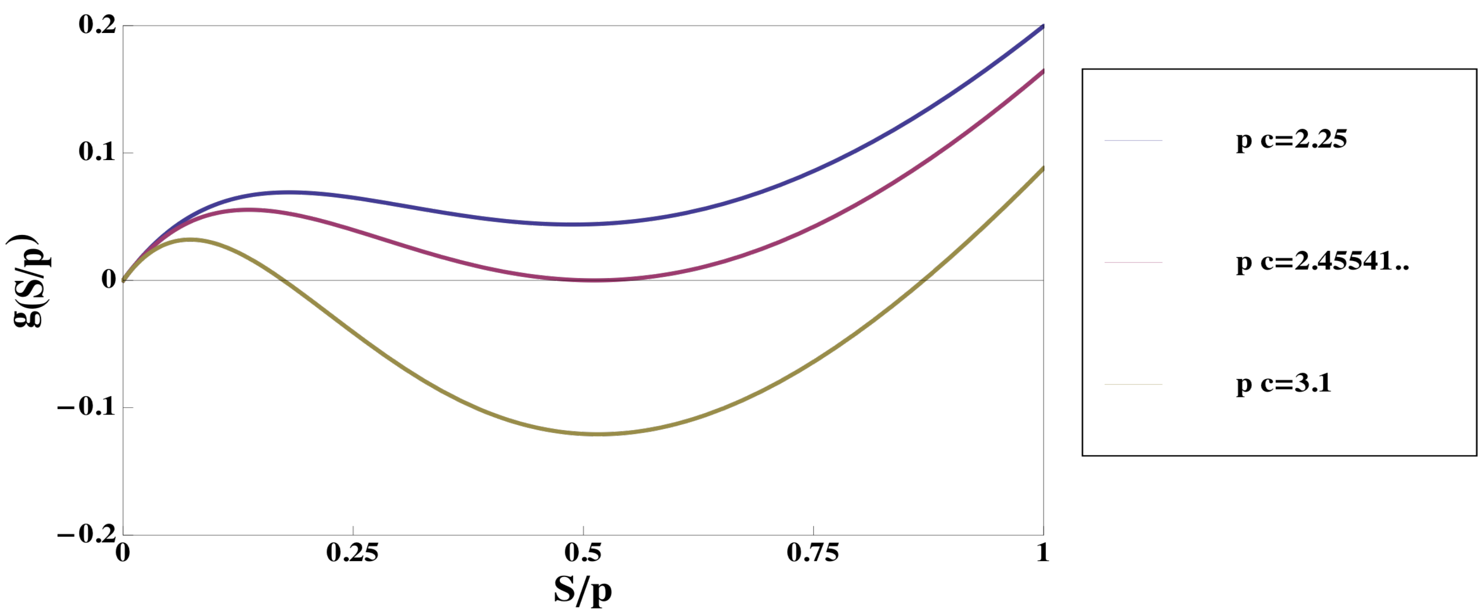
$$\boxed{S = p\left(1 - e^{-cS}\right)}$$

Fraction of nodes in the MCGC of multiplex network with M Poisson layers of average degree c:

$$\boxed{S = p\left(1 - e^{-cS}\right)^{M}}$$

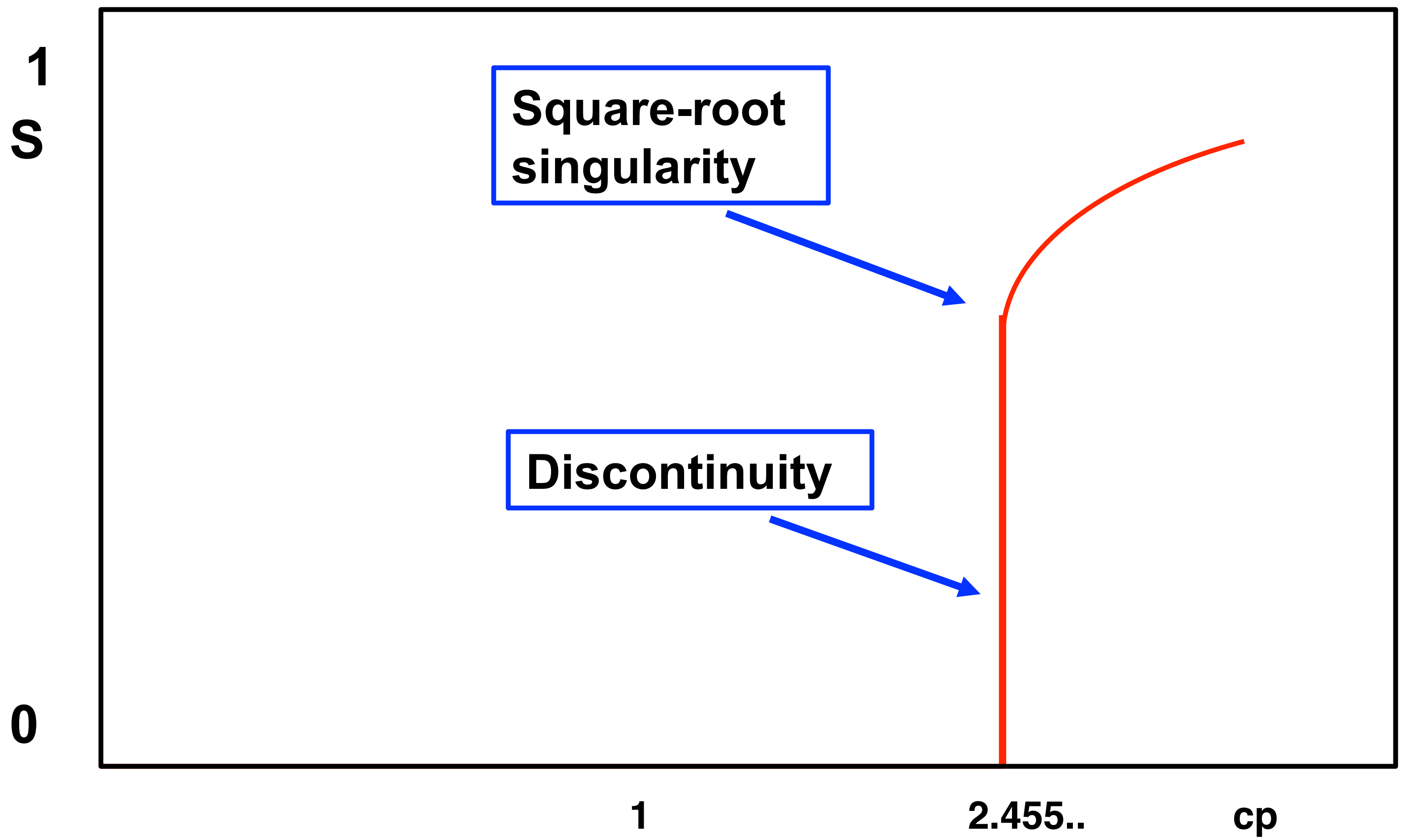
Percolation on two interdependent Poisson networks with average degree c

$$g(x = S/p) = x - \left(1 - e^{-cpx}\right)^2 = 0$$



The percolation transition at cp=2.455… is discontinuous!

Discontinuous Emergence of the mutually connected giant component in a duplex of Poisson network



Buldyrev et al Nature 2010, Baxter et al. PRL 2012

Percolation in multiplex networks with overlap of the links: the message passing approach

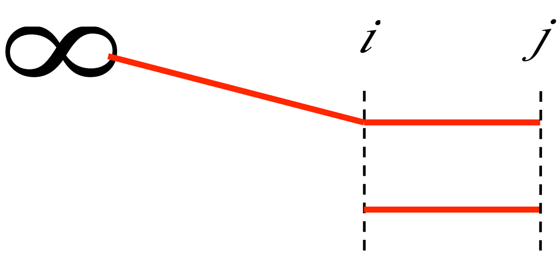
Directed percolation problem

Nodes in the directed mutually connected giant component (DMCGC) can be found by using the same algorithm used in absence of overlap of the links

In absence of overlap of the links

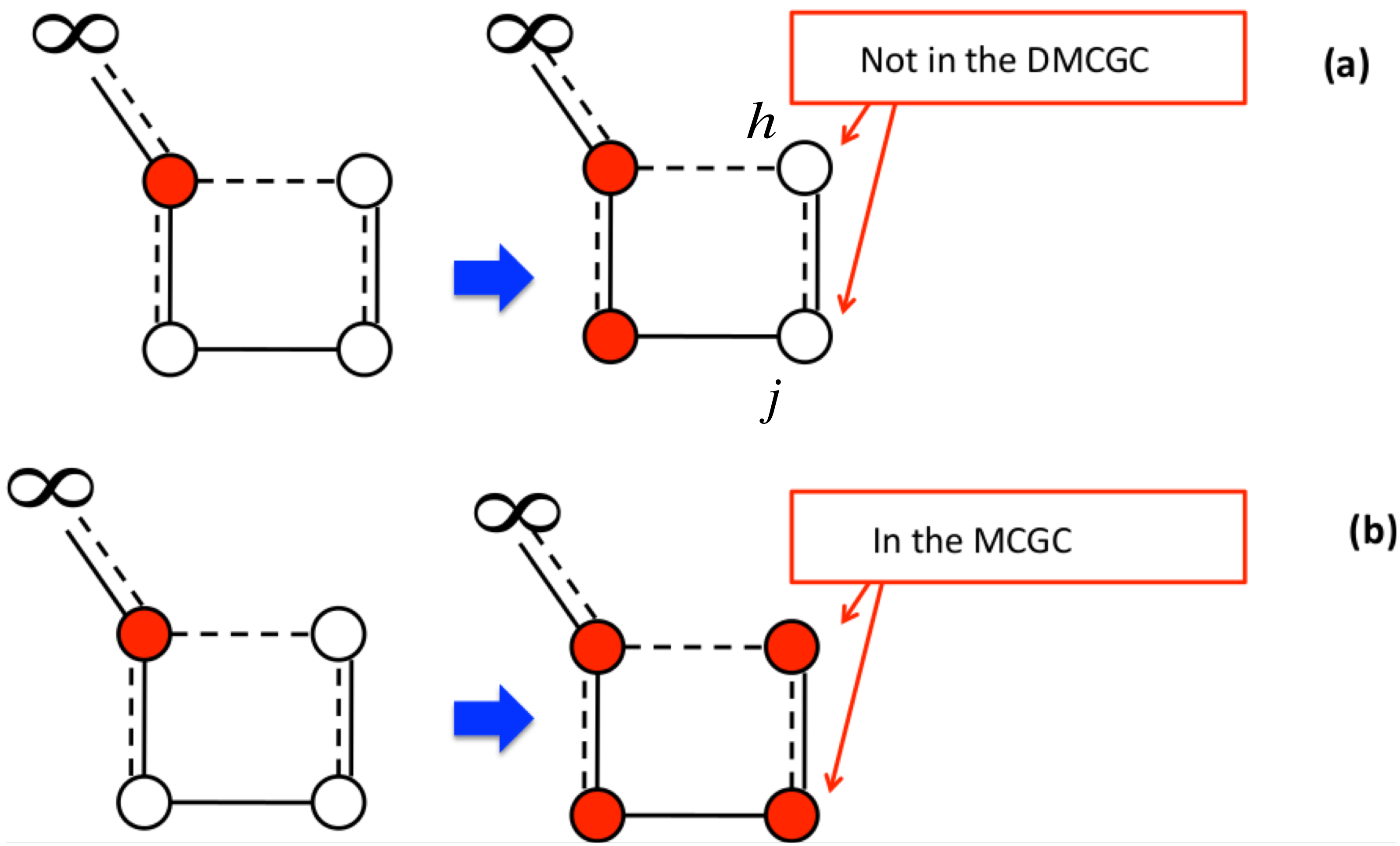
DMCGC=MCGC

Specifically we will have



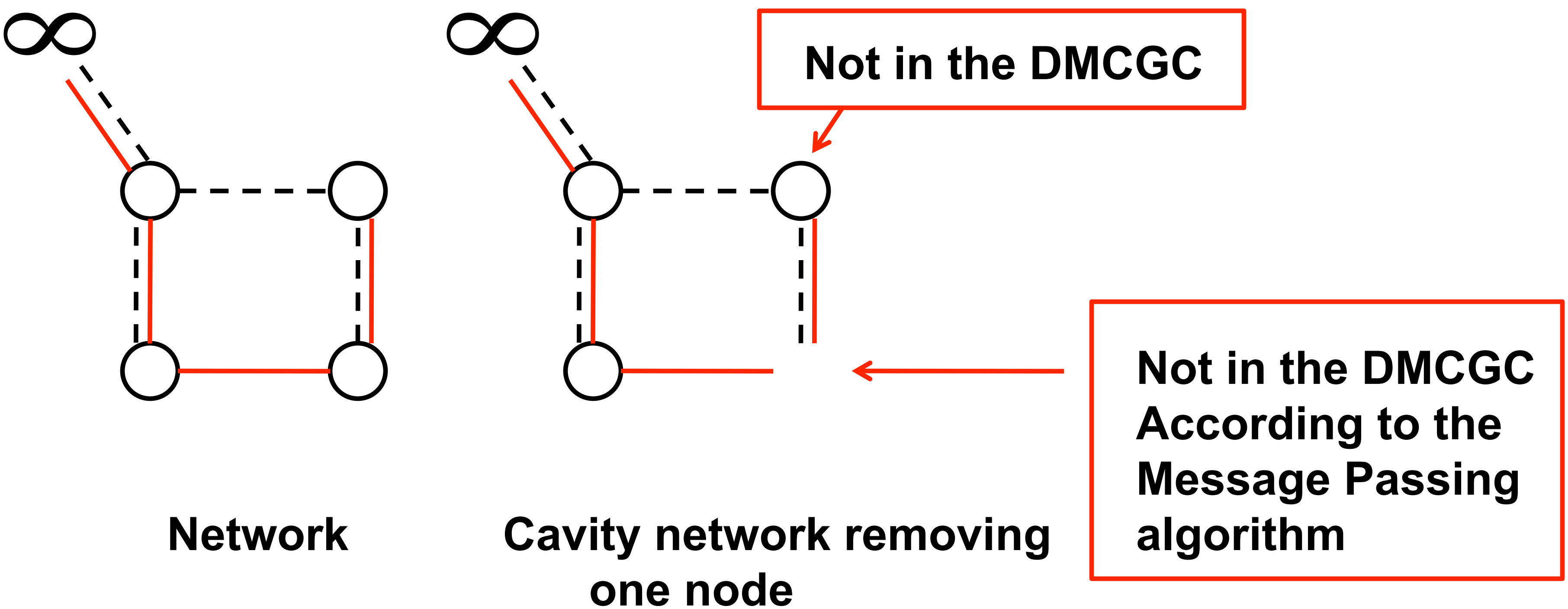
$$\sigma_{i\to j} = 0$$

Difference between the DMCGC and the MCGC



Min et al. (2015) Cellai et al. (2016)

Why in presence of overlap the DMCGC is not equal to the MCGC



Required properties of the message passing algorithm for the MCGC

- The MCGC must be of maximum size:
  - the messages are polarized
  - the sender node must assume that the target node is in the MCGC.
- The messages must indicate the set of layers $\vec{n} = (n_1, n_2 \ldots, n_M)$ that connect the sender node to the MCGC.

The algorithm

The message

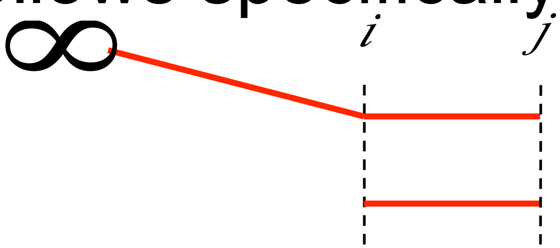
$$\vec{n}_{i\to j}=(n^{[1]}_{i\to j},n^{[2]}_{i\to j},\ldots n^{[M]}_{i\to j}),\qquad n^{[\alpha]}_{i\to j}=0,1$$

indicates that

assuming that j belongs to the MCGC

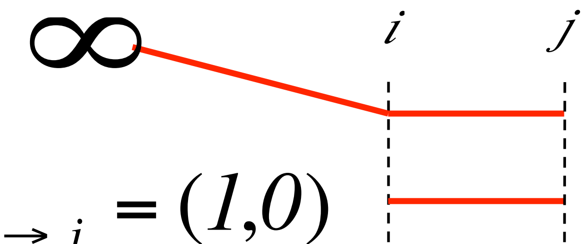
- node i must be in the MCGC
- node i connects node j to the MCGC *exclusively* through the layers α with $n^{[\alpha]}_{i\to j}=1$

It follows specifically that we have

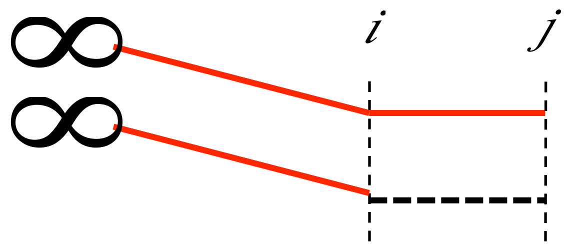


$$\vec{n}_{i\to j}=(1,0)$$

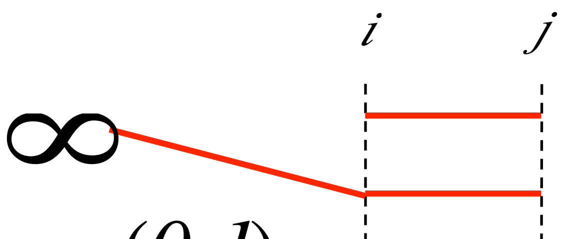
Non-trivial cases for M=2



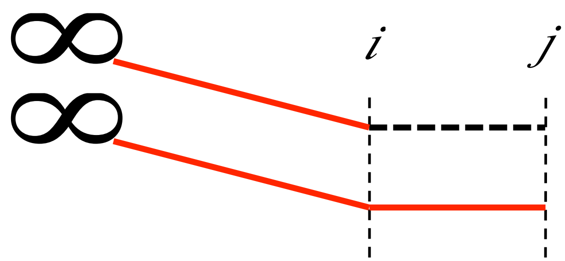
$$\vec{n}_{i\to j} = (1,0)$$



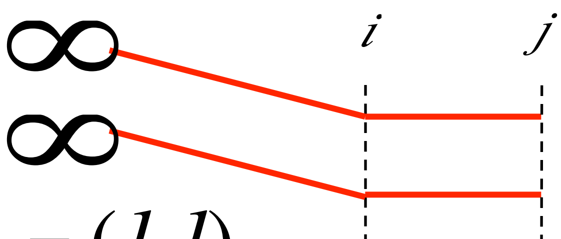
$$\vec{n}_{i\to j} = (1,0)$$



$$\vec{n}_{i\to j} = (0,1)$$

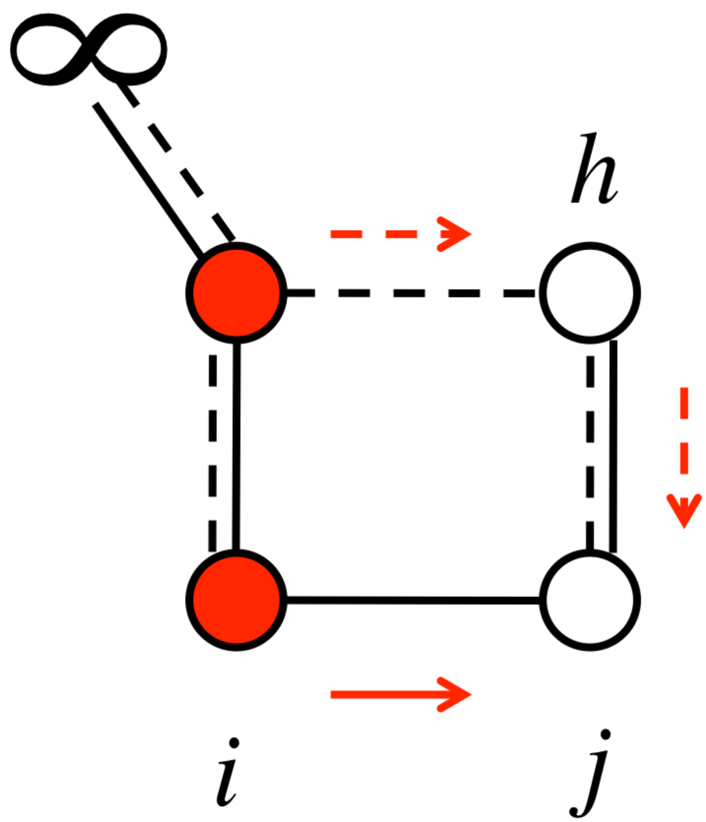


$$\vec{n}_{i\to j} = (0,1)$$



$$\vec{n}_{i\to j} = (1,1)$$

How this algorithm can predict that node j and h are in the MCGC



Cellai et al. (2016)

Duplex network with Poisson Layers and Link Overlap

Duplex networks with Poisson multidegree distribution with

$$\langle k^{01}\rangle = \langle k^{10}\rangle = c_1$$
$$\langle k^{11}\rangle = c_2$$

MCGC

$$\boxed{\begin{aligned} S &= p\left(1 - 2e^{-c_1 S - c_2(S+S_{2,1})} + e^{-2c_1 S - c_2(S+S_{2,1})}\right) \\ S_{(1,1),(1,0)} &= S_{2,1} = p\left(e^{-c_1 S - c_2(S+S_{2,1})} - e^{-2c_1 S - c_2(S+2S_{2,1})}\right) \end{aligned}}$$

DMGC

$$\boxed{S = p\left(1 - 2e^{-(c_1+c_2)S} + e^{-(2c_1+c_2)S}\right)}$$

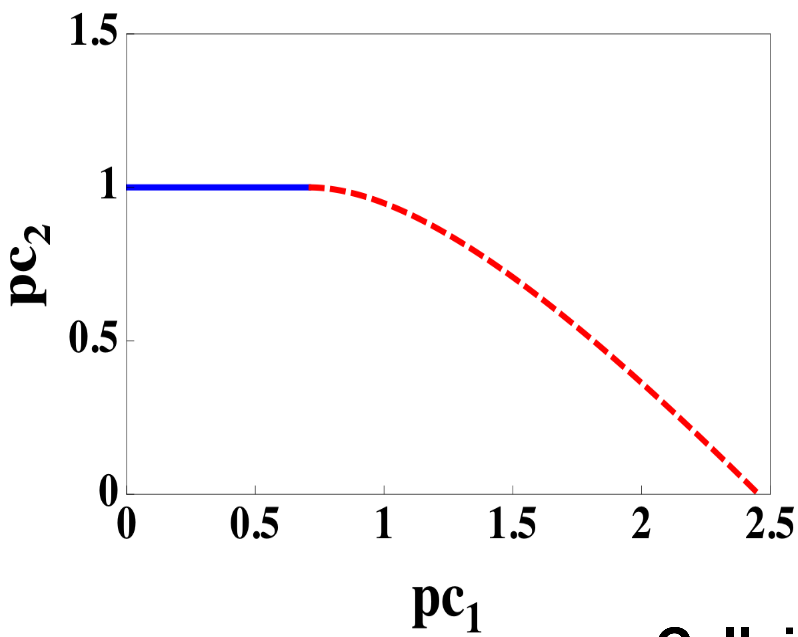
Phase diagram for DMCGC and MCGC

Duplex networks with Poisson multidegree distribution with

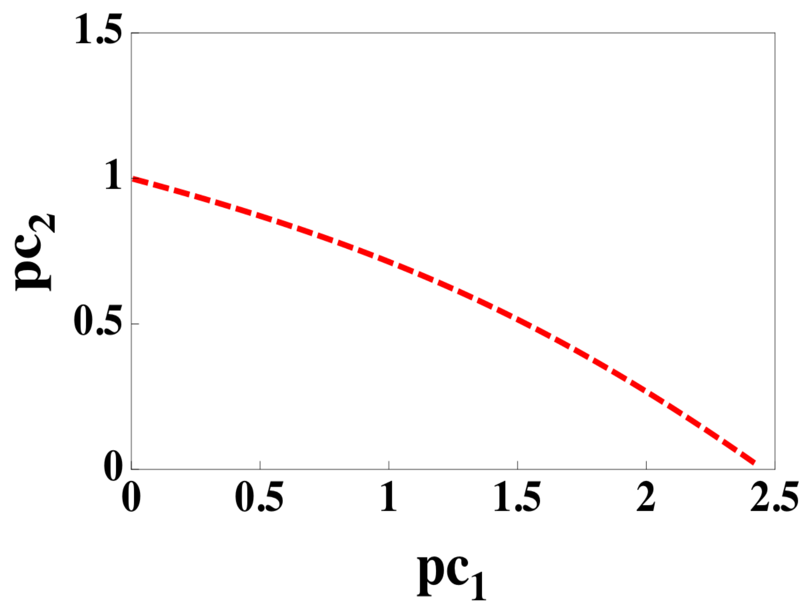
$$\langle k^{01}\rangle = \langle k^{10}\rangle = c_1$$

$$\langle k^{11}\rangle = c_2$$

DMCGC



MCGC



Cellai et al. PRE (2013); Cellai et al (2016)

Conclusions

We have formulated a message passing theory for percolation and directed percolation in multiplex network with link overlap.

- *Both algorithms reduce to percolation in multiplex network in absence of overlap and to percolation on single network in presence of complete overlap.*
- *The algorithm for directed percolation has an epidemic spreading interpretation. The algorithm for percolation does not have a feed-forward character.*
- *The two critical phenomena have different phase diagrams.*

The algorithm for the MCGC can be used to study

1. *the robustness of real multiplex networks and*
2. *to study the percolation phase diagram of multiplex networks with link overlap and arbitrary number of layers.*

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