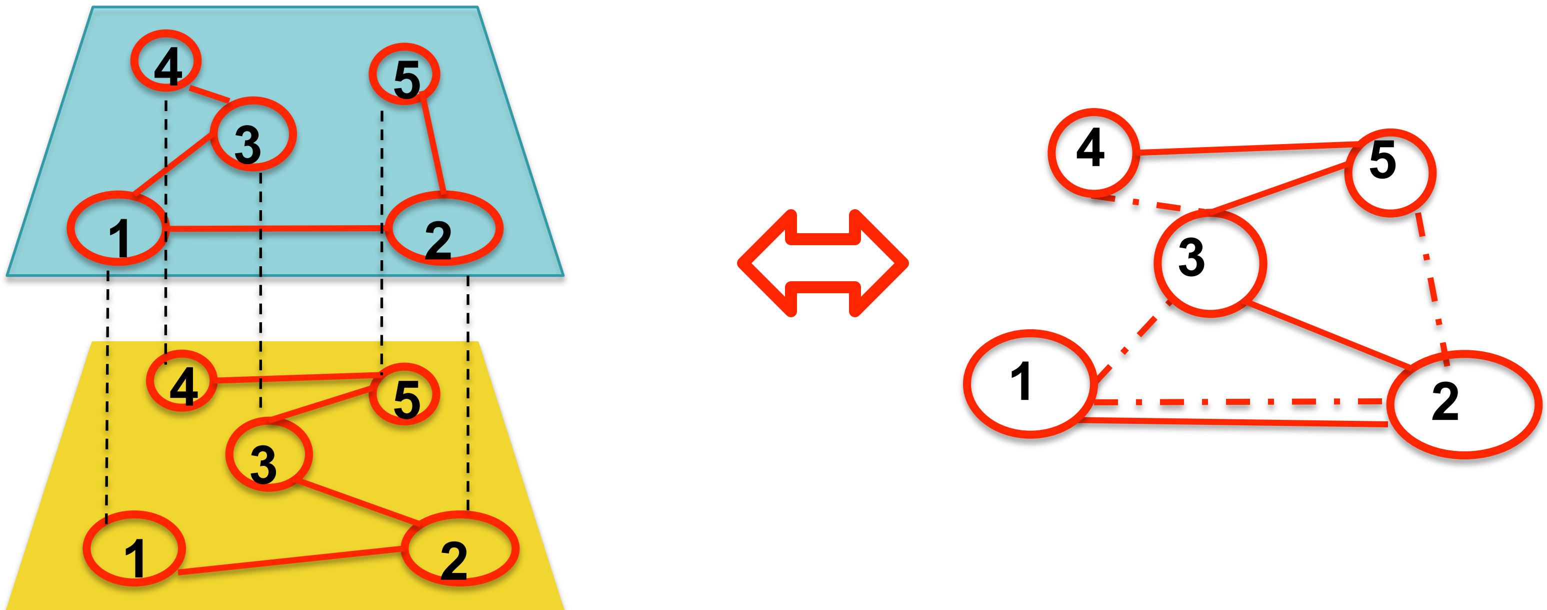

Multilayer Networks:

diffusion, epidemic spreading and centrality measures

LTCC Course Multilayer Networks

23-24 November 2016

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Representation of a multiplex

A multiplex network of N nodes formed by M layers

is fully specified by

M adjacency matrices

$$\boxed{a^{[\alpha]}}$$

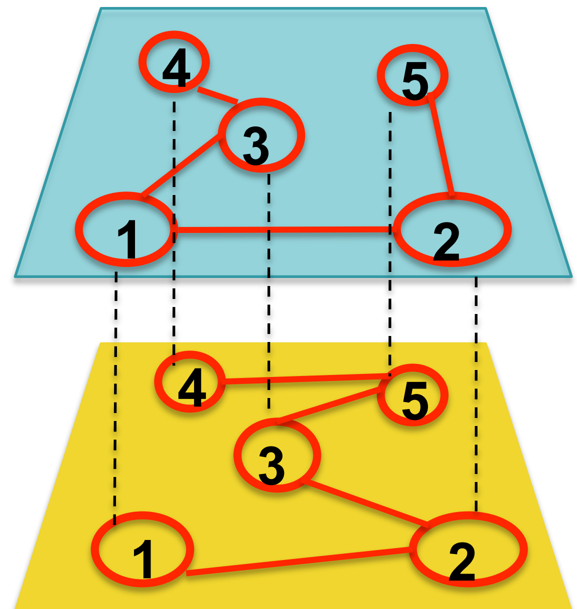
with α=1, 2, … M

of matrix elements

$$\boxed{a_{ij}^{[\alpha]} = \begin{cases} 1 \ \ if \ \ node \ i \ and \ node \ \ j \ are \ linked \ in \ layer \ \alpha \\ 0 \qquad\qquad\qquad\qquad otherwise \end{cases}}$$

Supra-adjacency matrix

The supra-adjacency matrix includes all the links in each layers and the interlinks

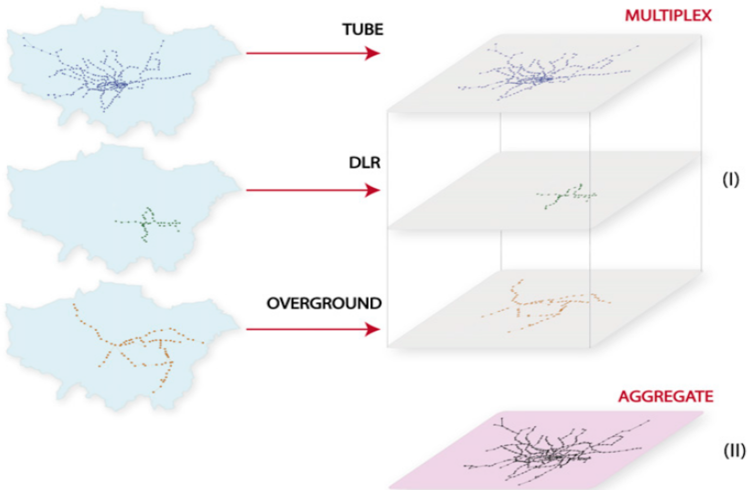


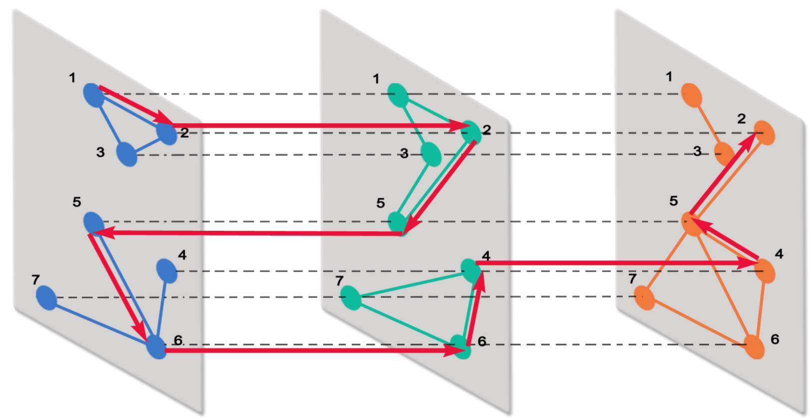
It indicates if a node i in layer α is connected to a node j in layer β

$$A_{i\alpha,\, j\beta} = \begin{cases} a_{ij}^{[\alpha]} & if \quad \alpha = \beta \\ \delta_{ij} & if \quad \alpha \neq \beta \end{cases}$$

$$\mathbf{A} = \left(\begin{array}{c|c|c|c} \mathbf{a}^{[1]} & \mathbf{I} & \cdots & \mathbf{I} \\ \hline \mathbf{I} & \mathbf{a}^{[2]} & \cdots & \mathbf{I} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{I} & \mathbf{I} & \cdots & \mathbf{a}^{[M]} \end{array} \right),$$

Diffusion in multiplex networks





Interlinks are essential for diffusion across the layers of multiplex networks

-ex: transportation networks, social online networks

S. Gomez et al., Phys. Rev. Lett.(2013)
M. De Domenico PNAS (2014)

Diffusion

The diffusion equation on a multiplex network where it is possible to diffuse along interlinks is given by

$$\boxed{\frac{d}{dt} x_{\mathbf{i}}^{[\alpha]} = -D^{[\alpha]} \sum_{j=1...N} a_{ij}^{[\alpha]} (x_j^{[\alpha]} - x_i^{[\alpha]}) + \sum_{\beta=1..,M} D^{[\alpha,\beta]} (x_i^{[\beta]} - x_i^{[\alpha]})}$$

$x_i^{[\alpha]}$ **Dynamical state of replica node (i,α)**

$D^{[\alpha]}$ **Intra-layer Diffusion constant**

$D^{[\alpha,\beta]}$ **Inter-layer diffusion constant**

Gomez et al. (2012)

Diffusion in a single layer

$$\boxed{\frac{d}{dt} x_{\mathbf{i}}^{[\alpha]} = -D^{[\alpha]} \sum_{j=1...N} a_{ij}^{[\alpha]} (x_j^{[\alpha]} - x_i^{[\alpha]})}$$

In matrix form the above equation reads

$$\frac{d}{dt}\mathbf{x}^{[\alpha]} = -\mathbf{L}^{[\alpha]}\mathbf{x}^{[\alpha]}$$

with

$$\mathbf{x}^{[\alpha]} = \begin{pmatrix} x_1^{[\alpha]} \\ x_2^{[\alpha]} \\ \vdots \\ x_M^{[\alpha]} \end{pmatrix}$$

and the Laplacian matrix given by

$$L_{ij}^{[\alpha]} = D^{[\alpha]}\left(k_i^{[\alpha]}\delta_{ij} - a_{ij}^{[\alpha]}\right)$$

Diffusion in a single layer

For a network with a single connected component, the eigenvalues of the Laplacian can be ordered as

$$0 = \lambda_1^{[\alpha]} < \lambda_2^{[\alpha]} \leq \lambda_3^{[\alpha]} \leq \ldots \lambda_M^{[\alpha]}$$

In networks with a spectral gap, the typical timescale for relaxation of the dynamics to the stationary state, is given by

$$\tau = \frac{1}{\lambda_2^{[\alpha]}}$$

General multilayer networks

The diffusion equation reads in matrix form

$$\frac{d\mathbf{X}}{dt} = -\mathcal{L}\mathbf{X}, \qquad \mathbf{X} = \begin{pmatrix} \mathbf{x}^{[1]} \\ \hline \mathbf{x}^{[2]} \\ \hline \vdots \\ \hline \mathbf{x}^{[M]} \end{pmatrix}$$

where the Supra-Laplacian is given by

$$\mathcal{L} = \left(\begin{array}{c|c|c|c} D^{[1]}\mathbf{L}^{[1]} & 0 & \dots & 0 \\ \hline 0 & D^{[2]}\mathbf{L}^{[2]} & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & D^{[M]}\mathbf{L}^{[M]} \end{array}\right) + \left(\begin{array}{c|c|c|c} \sum_{\beta\neq 1} D^{[1,\beta]}\mathbf{I} & -D^{[1,2]}\mathbf{I} & \dots & -D^{[1,M]}\mathbf{I} \\ \hline -D^{[2,1]}\mathbf{I} & \sum_{\beta\neq 2} D^{[2,\beta]}\mathbf{I} & \dots & -D^{[2,M]}\mathbf{I} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -D^{[M,1]}\mathbf{I} & -D^{[M,2]}\mathbf{I} & \dots & \sum_{\beta\neq M} D^{[M,\beta]}\mathbf{I} \end{array}\right)$$

The typical timescale determining the relaxation dynamics is

$$\boxed{\tau = \frac{1}{\lambda_2}}$$

Gomez et al. (2012)

Case of two layers (M=2)

Assuming

$$D^{[1]} = D^{[2]} = 1$$

$$D^{[1,2]} = D^{[2,1]} = D_x$$

we have

$$\mathcal{L} = \left(\begin{array}{c|c} \mathbf{L}^{[1]} + D_x \mathbf{I} & -D_x \mathbf{I} \\ \hline -D_x \mathbf{I} & \mathbf{L}^{[2]} + D_x \mathbf{I} \end{array} \right)$$

Limit cases

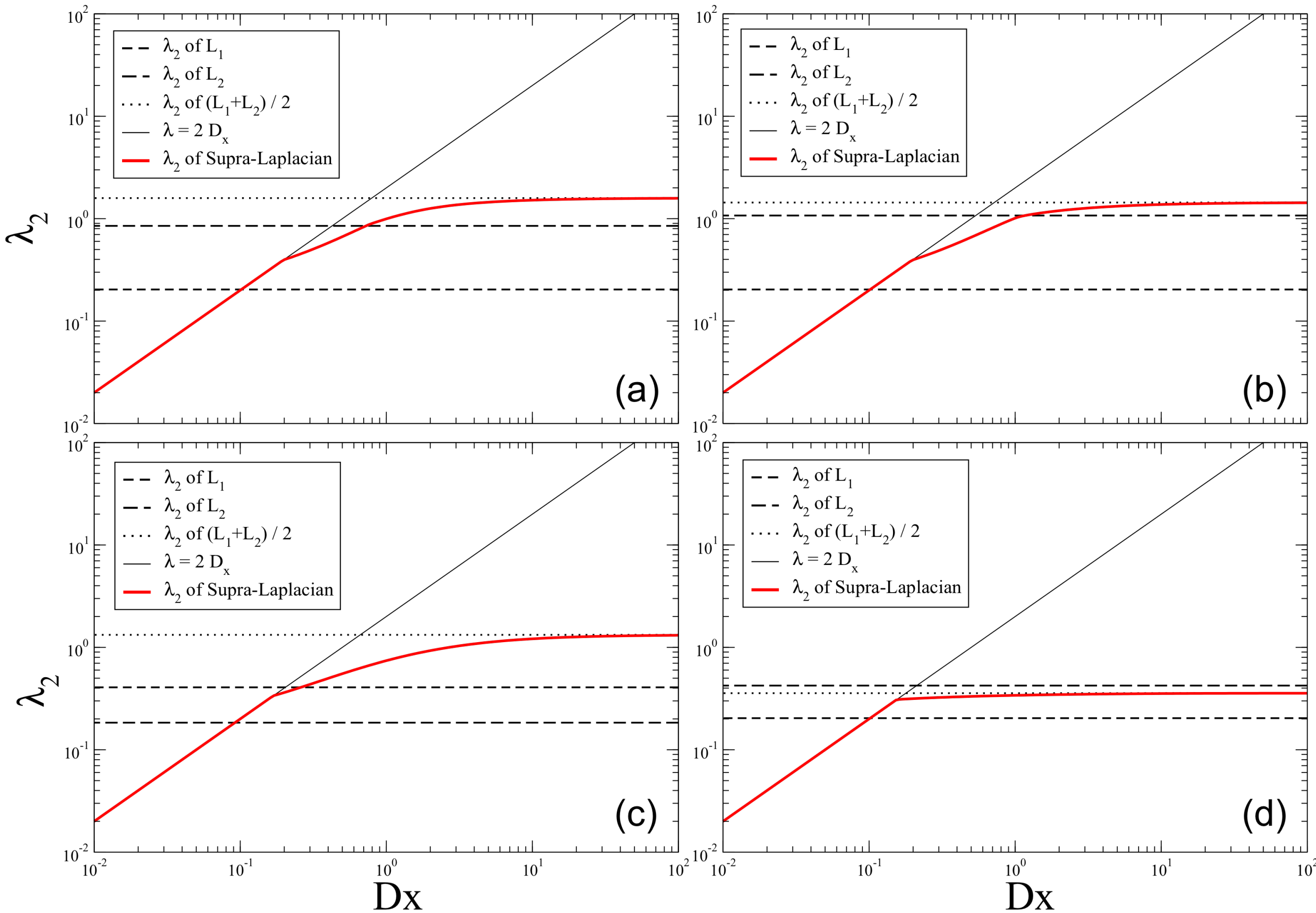
$$D_x << 1 \quad \lambda_2 = 2D_x$$

$$D_x >> 1 \quad \lambda_2 = \frac{\lambda_s}{2} \geq \frac{\lambda_2^{[1]} + \lambda_2^{[2]}}{2} \geq \min(\lambda_2^{[1]}, \lambda_2^{[2]})$$

For small D_x, τ is controlled by the interlayer diffusion constant,
For large D_x, the diffusion is faster than the diffusion
on the slower layer and we can observe also superdiffusion

Gomez et al. (2012)

Smallest non-zero eigenvalue of the Supra-Laplacian



Gomez et al. (2012)

Epidemic spreading

SIS model

S_i Susceptible Individual on node i

I_i Infected Individual on node i

SIS Model on a network

$$S_i + I_j \xrightarrow{\beta} I_i + I_j$$

With node i nearest neighbour of node j

$$I_i \xrightarrow{\mu} S_i$$

SIR model

S_i Susceptible Individual on node i

I_i Infected Individual on node I

R_i Recovered Individual on node i

SIR Model on a network

$$S_i + I_j \xrightarrow{\beta} I_i + I_j$$

With node i nearest neighbour of node j

$$I_i \xrightarrow{\mu} R_i$$

Phase transition

The control parameter of the transition is

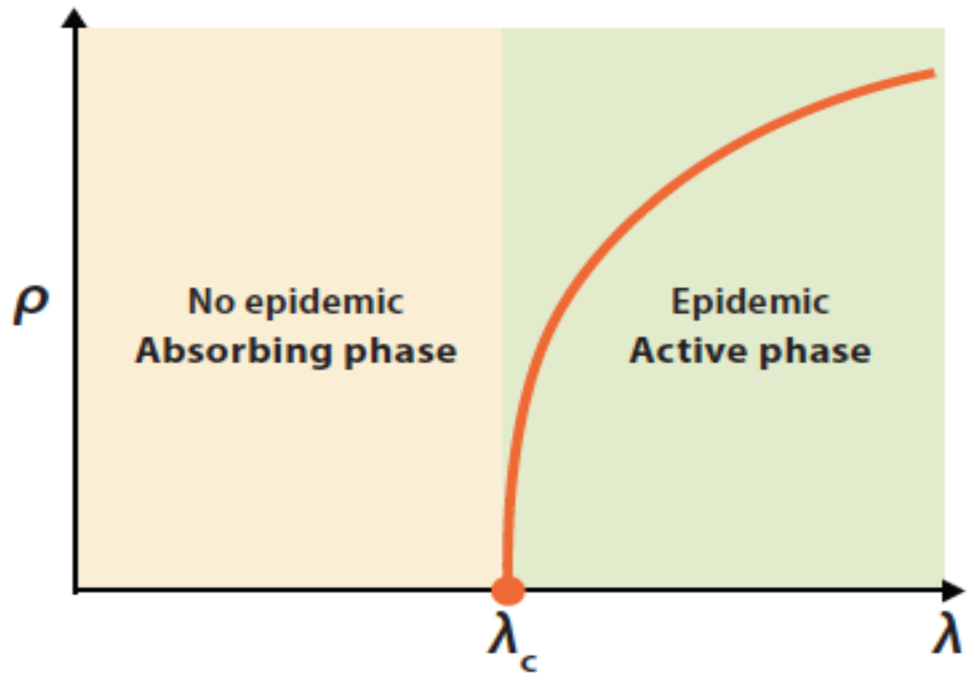
$$\boxed{\lambda = \frac{\beta}{\mu}}$$

The order parameter ρ is the total fraction of nodes that are infected in an outbreak started from a single node, that has the critical behavior

$$\boxed{\rho \propto \begin{cases} (\lambda - \lambda_c)^{\beta} & for\ \lambda > \lambda_c \\ 0 & for\ \ \lambda \leq \lambda_c \end{cases}}$$

Phase transition

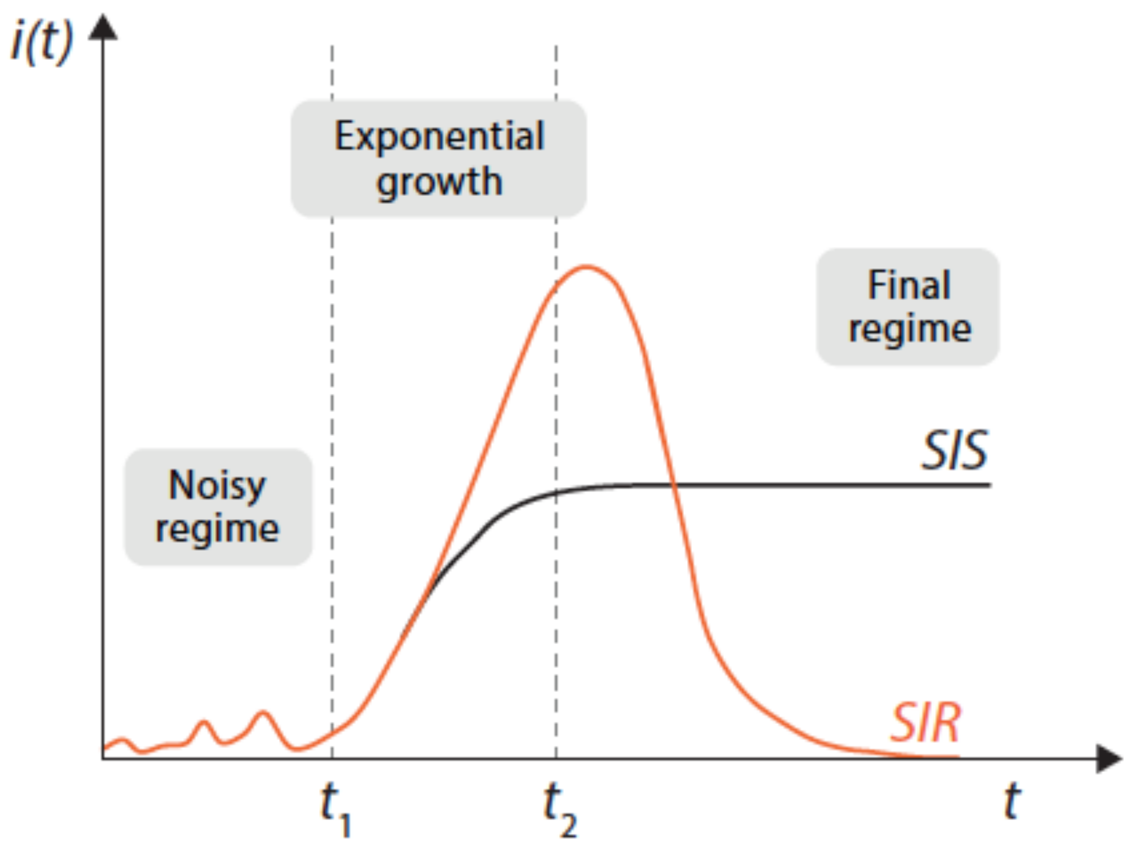
A typical epidemic spreading transition in the SIS



A similar behavior is followed by the order parameter as a function of the control parameter in the SIR model

Temporal evolution of the number of infected individuals

A typical temporal profile of the fraction *i(t)* of infected individuals in the SIS and in the SIR model



Mean-field approximation for SIS dynamics in single layer

The mean-field equation determining the probability ρ_i that node i is infected is given by

$$\boxed{\rho_i(t+\Delta t) = \rho_i(t) + [1-\rho_i(t)]\lambda(\Delta t)\sum_j a_{ij}\rho_j(t) - (\Delta t)\rho_i(t)}$$

Linearizing the steady state solution for ρ_i<<1

$$\rho_i^* \approx \lambda \sum_j a_{ij}\rho_j^*$$

The epidemic threshold λ_c satisfies

$$\boxed{\lambda_c \Lambda = 1}$$

Epidemic spreading in a multilayer network

The infectivity can depend on the layers

α and β

of the two node in contact,

we have then

$$\lambda^{[\alpha,\beta]}$$

Mean-field approximation for SIS model in multilayer networks

The mean-field equation determining the probability $\rho_{i\alpha}$ that node (i,α) is infected is given by

$$\boxed{\rho_{i\alpha}(t+\Delta t) = \rho_{i\alpha}(t) + [1-\rho_{i\alpha}(t)](\Delta t)\sum_{j,\beta}\lambda^{[\alpha,\beta]}A_{i\alpha,j\beta}\rho_{j\beta}(t) - (\Delta t)\rho_{i\alpha}(t)}$$

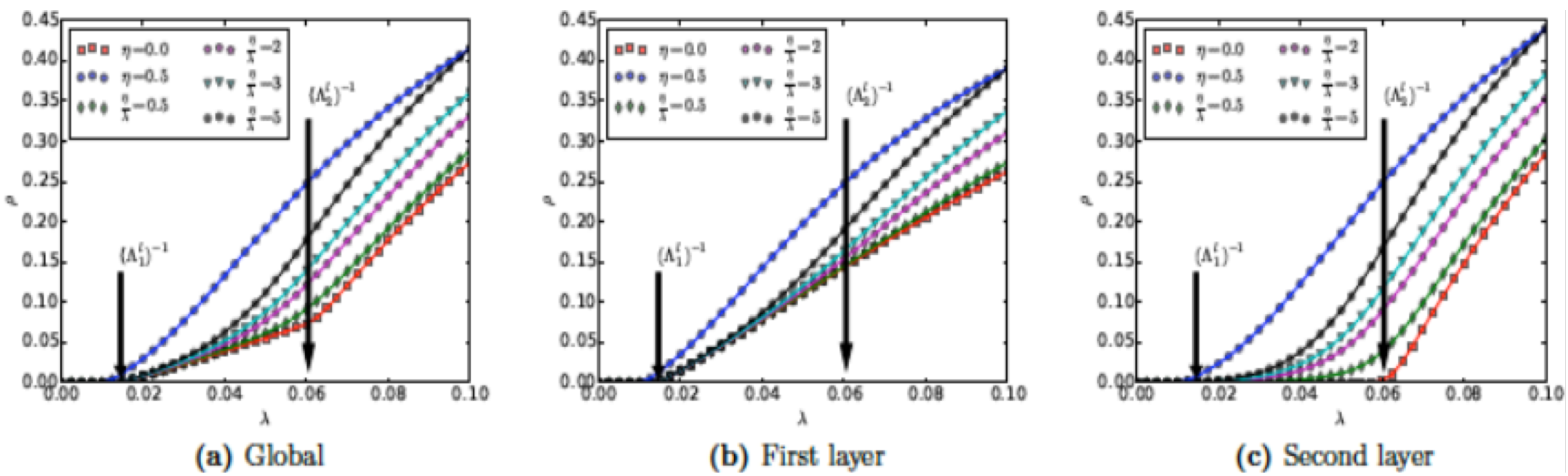
Linearizing the steady state solution for $\rho_{i\alpha} \ll 1$

$$\rho^{*}_{i\alpha} \approx \sum_{j,\beta}\lambda^{[\alpha,\beta]}A_{i\alpha,j\beta}\rho^{*}_{j\beta}$$

$$\boxed{\hat{\Lambda} = 1}$$

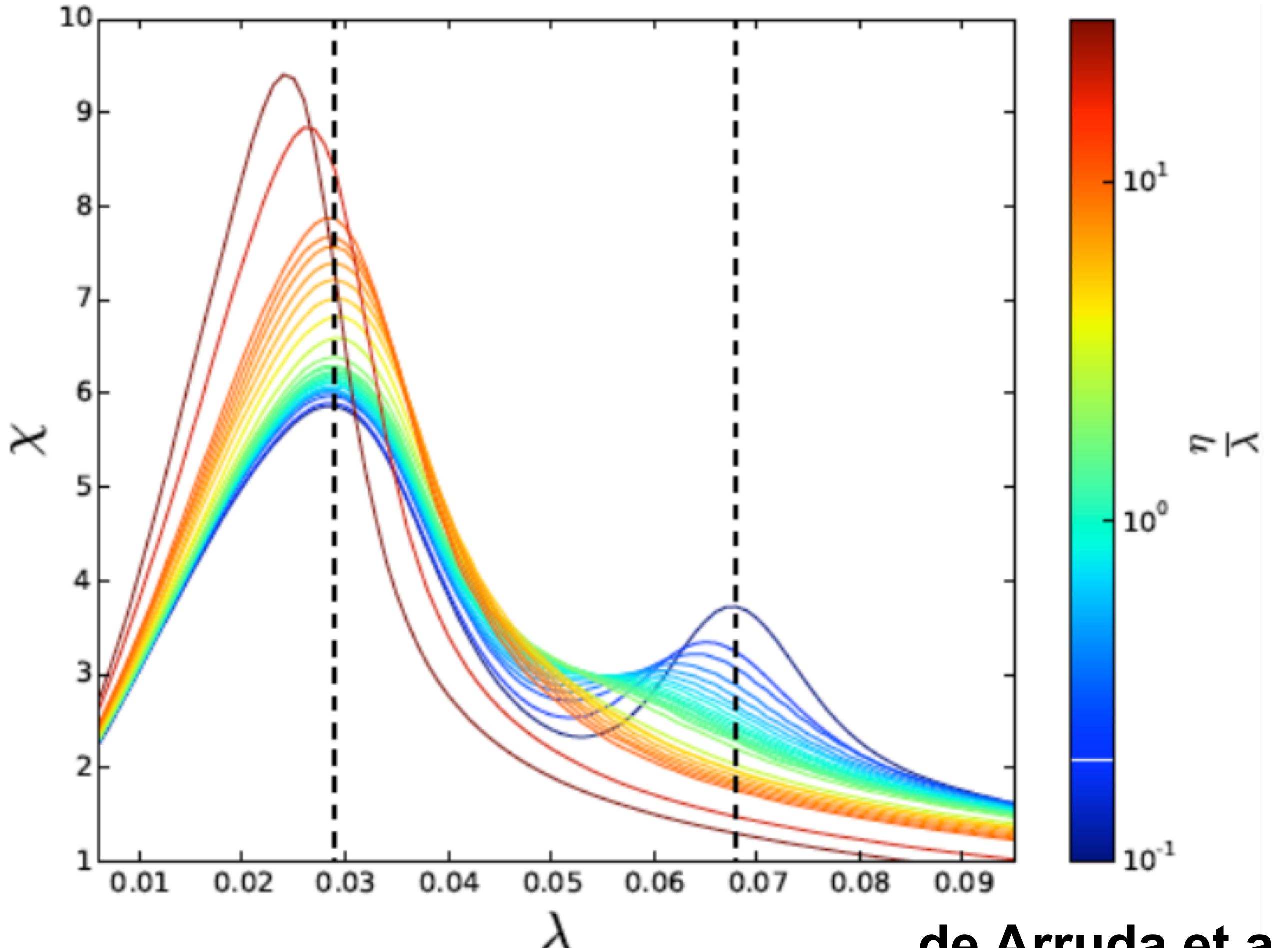
Cozzo (2013)

Epidemics can spread in a multilayer network also if it cannot spread in the single layer taken in isolation



de Arruda et al. (2013)

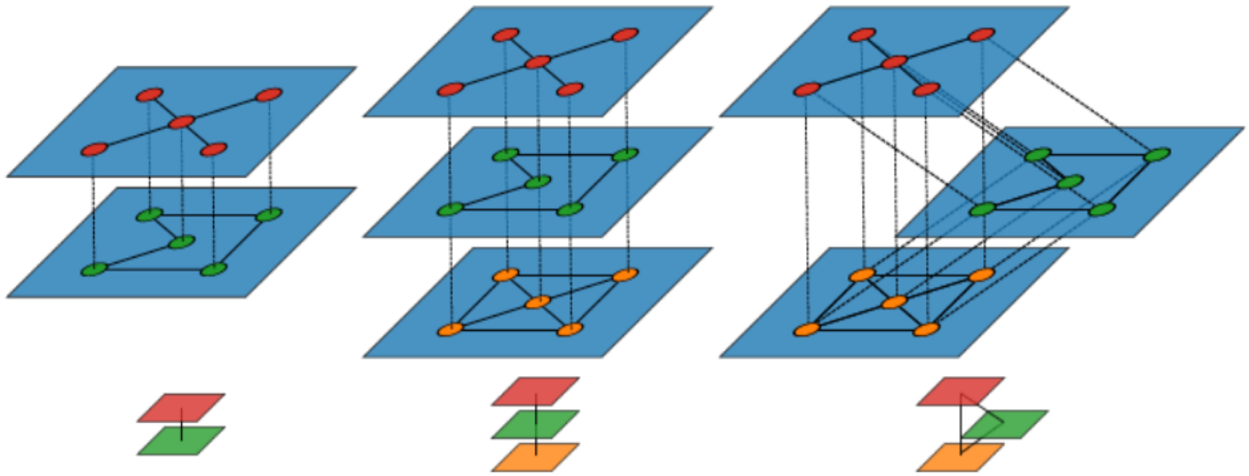
Interplay between structure and dynamics (M=2)



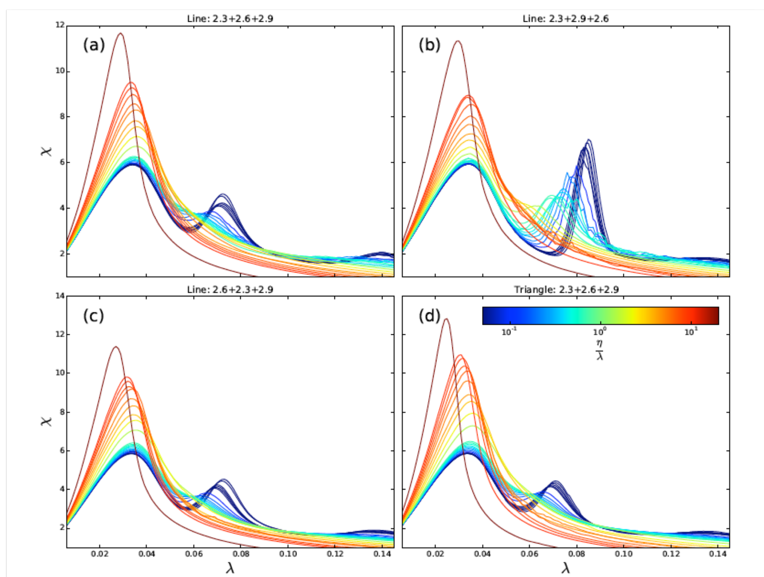
de Arruda et al. (2013)

Interplay between structure and dynamics

Multilayer network structure



Susceptibility



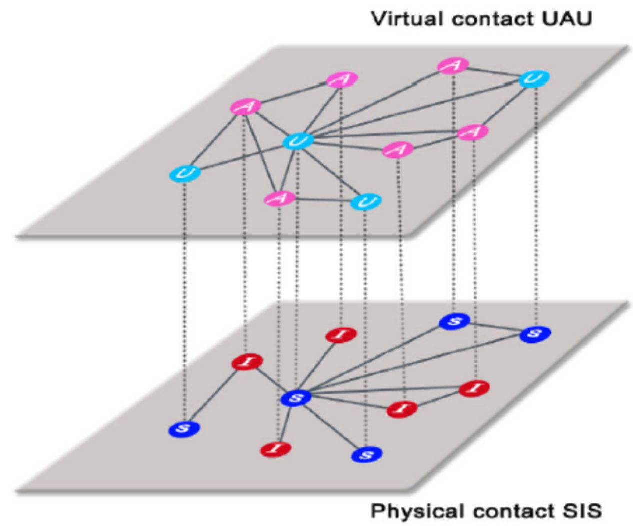
de Arruda et al. (2013)

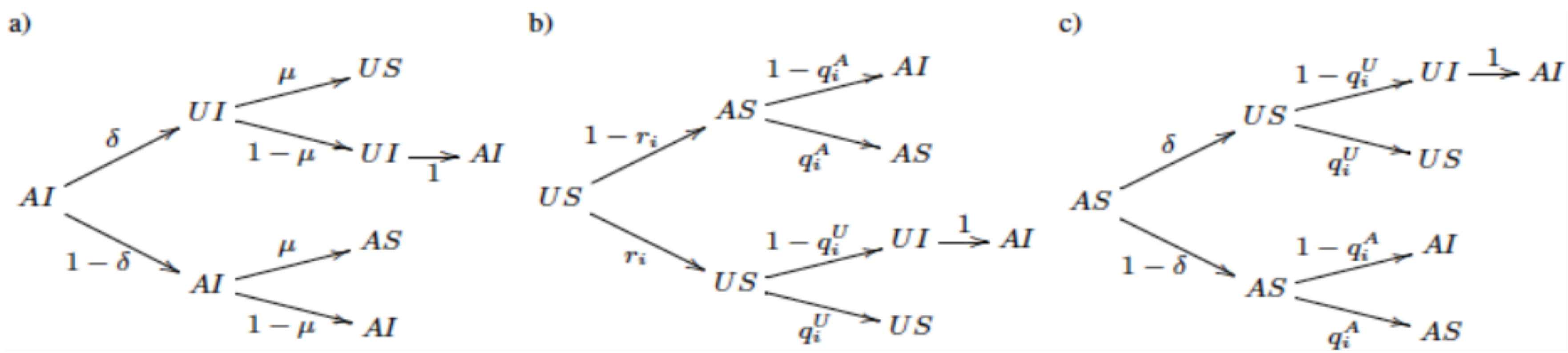
Spreading of awareness and viral epidemic spreading

Granell et al. (2013)

Two coupled SIS model on the virtual contact and physical contact layer

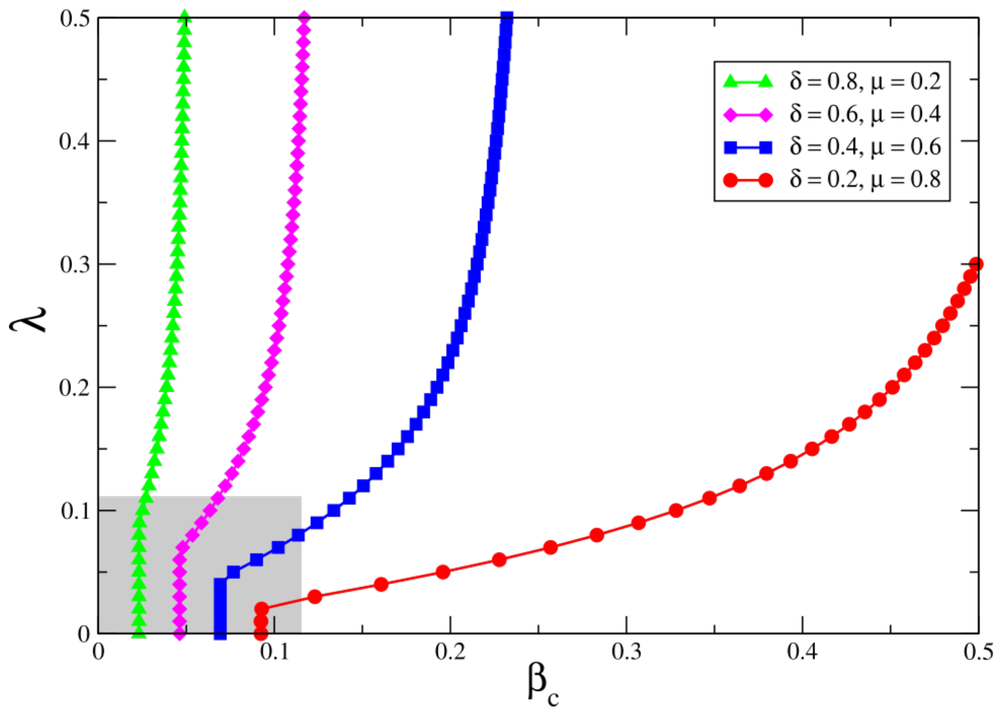
Unaware-Aware-Unaware
Susceptible-Infected-Susceptible





Phase diagram of the model

β_c critical infection rate of the epidemic spreading
λ infection rate of the awareness behavior

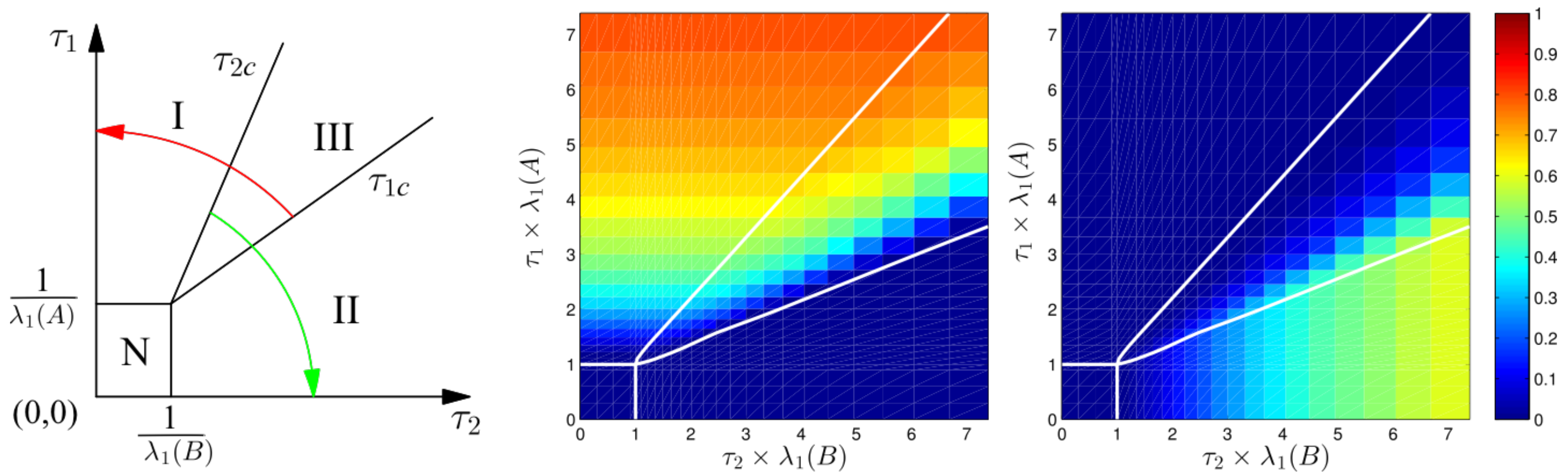


Granell et al. (2013)

For low λ the dynamics on the virtual layer does not affect the epidemic
For high λ the dynamics on the virtual layer retards the epidemics

Competing epidemics

Epidemics I spreads in layer I
Epidemics II spreads in layer II
A single node can be infected at most with one disease

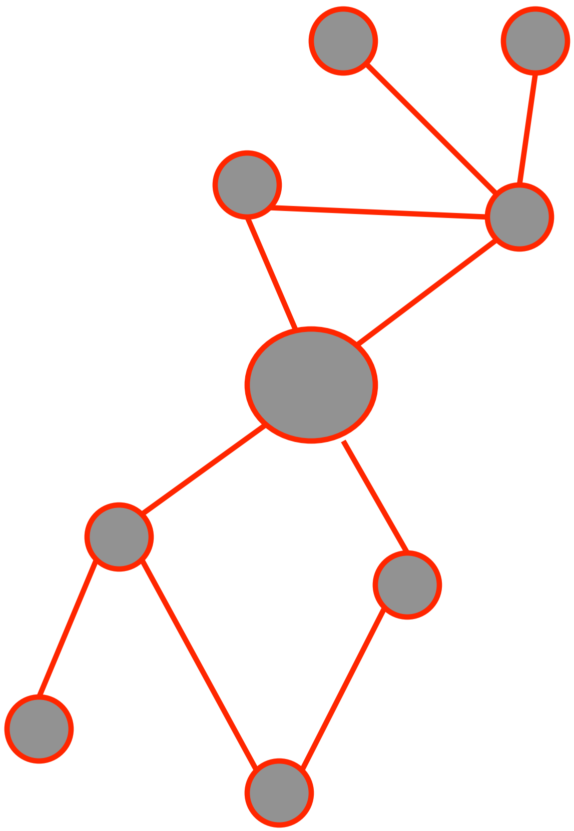


Sahneh and Scoglio (2014)

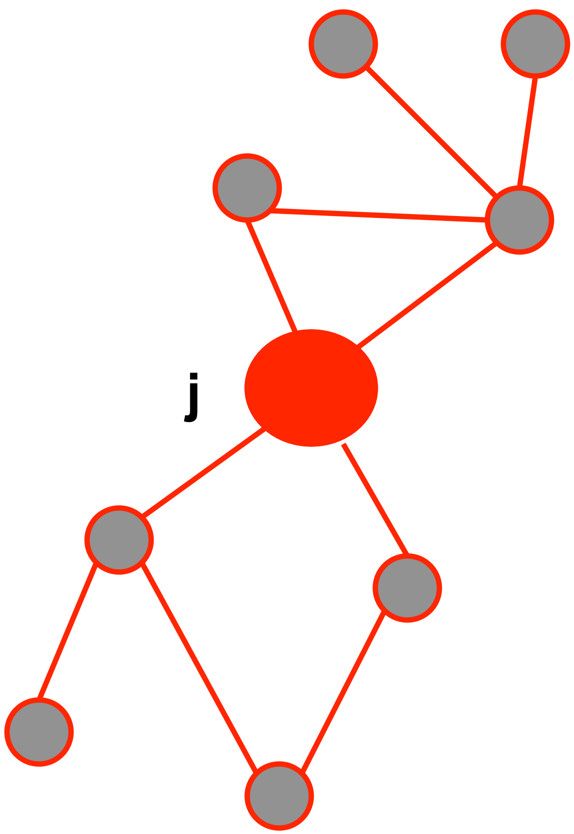
N neither of the epidemic spreads
I (II) only the epidemics in the I (II) layer spreads
III the epidemics spreads in both layers

PageRank

The Page Rank is based on a random walk

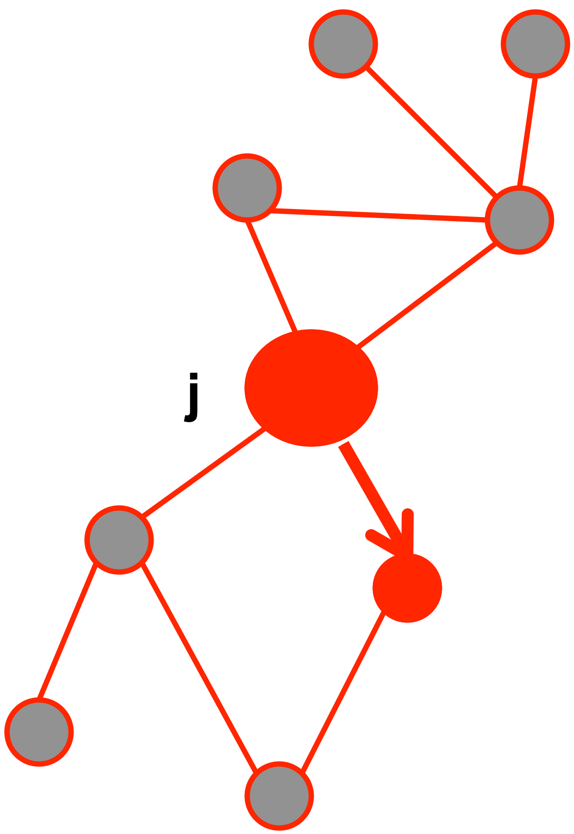


The Page Rank is based on a random walk



- We assume to have a random walker on the node j of the network

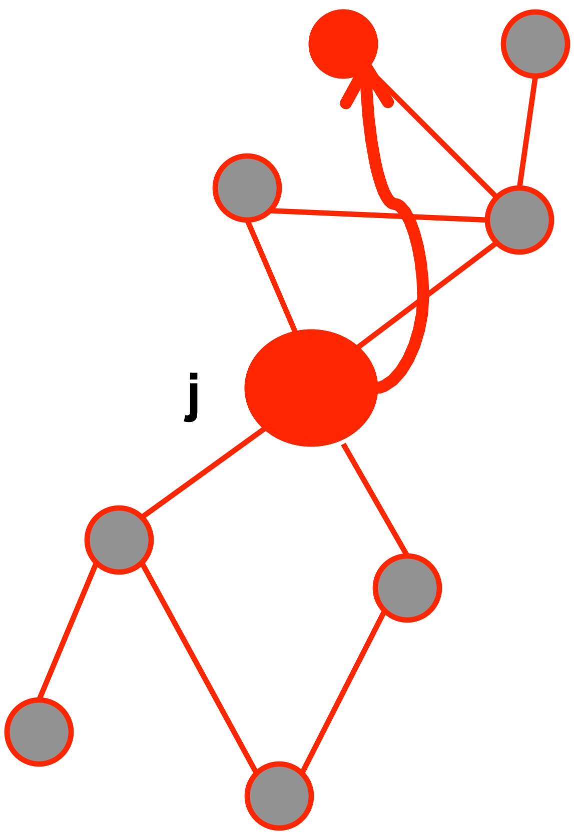
The Page Rank is based on a random walk



- We assume to have a random walker on the node j of the network

- With probability $\tilde{\alpha}$ the random walker hops to a neighbor node

The Page Rank is based on a random walk



- We assume to have a random walker on the node j of the network
- With probability $\tilde{\alpha}$ the random walker hops to a neighbor node
- With a probability $1-\tilde{\alpha}$ it jumps to a random node

PageRank

The PageRank x_i of node i is the probability that in the stationary state we find the random walker on node i

$$\boxed{x_i = \tilde{\alpha} \sum_j \frac{A_{ij}}{g_j} x_j + \beta}$$

with

$g_j = \max(k_j, 1),\quad k_i$ indicating the degree of node i

$$A_{ij} = \begin{cases} 1 \text{ if node } j \text{ links to node } i \\ 0 \quad \text{otherwise} \end{cases}, \quad \tilde{\alpha} = 0.85$$

Quantifying the

centrality of the nodes

with the

Functional Multiplex PageRank

Influences of multilinks

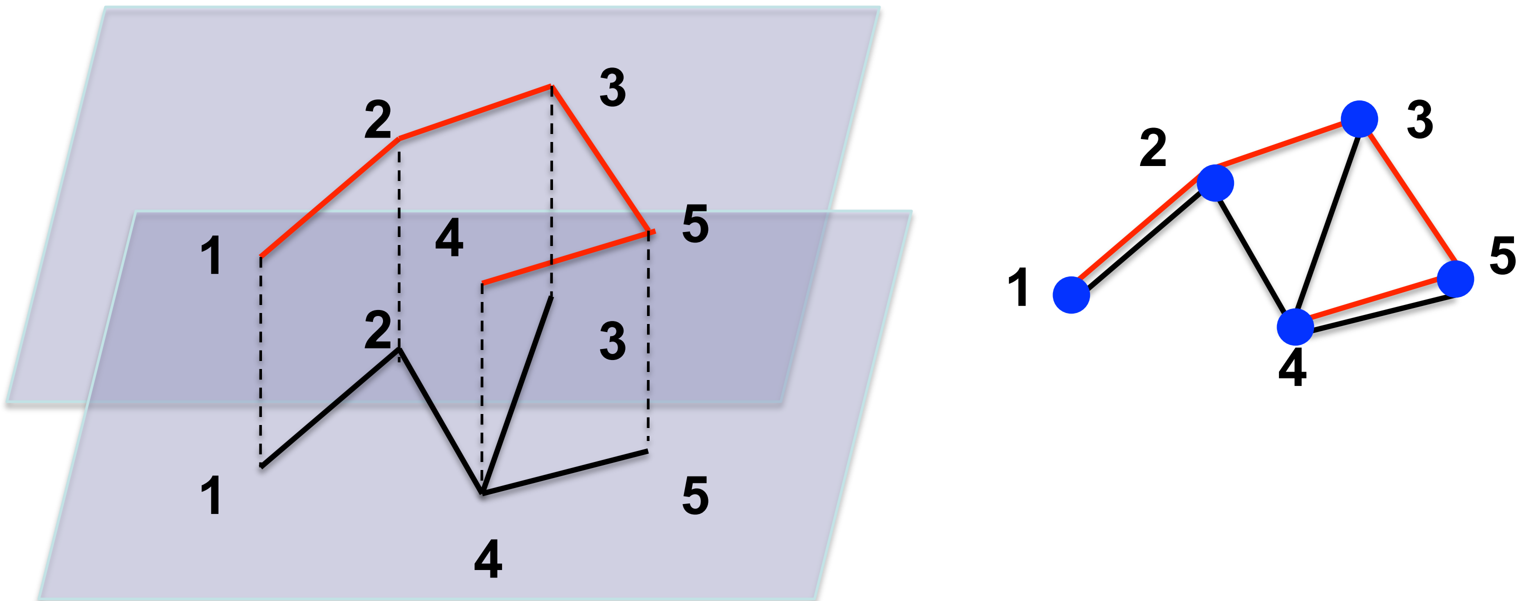
In a multiplex network different pattern of connections might contribute differently to the centrality of a node

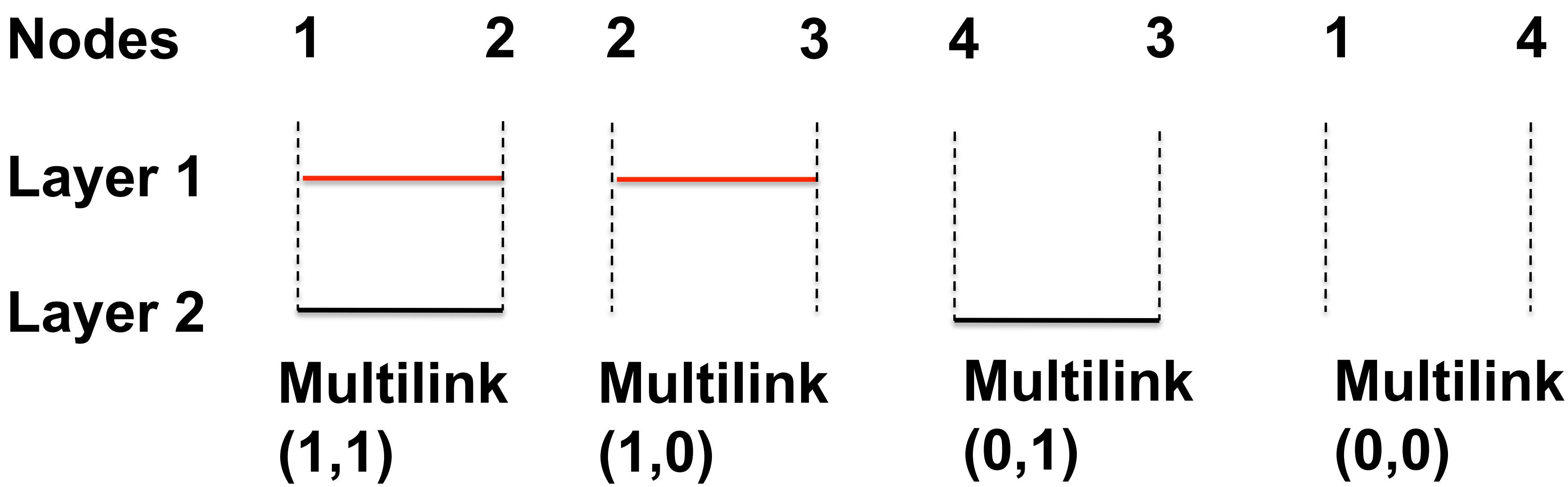
The influence of a multilink $\vec{m}$
is indicated by $z^{\vec{m}}$

J. Iacovacci et al. 2016

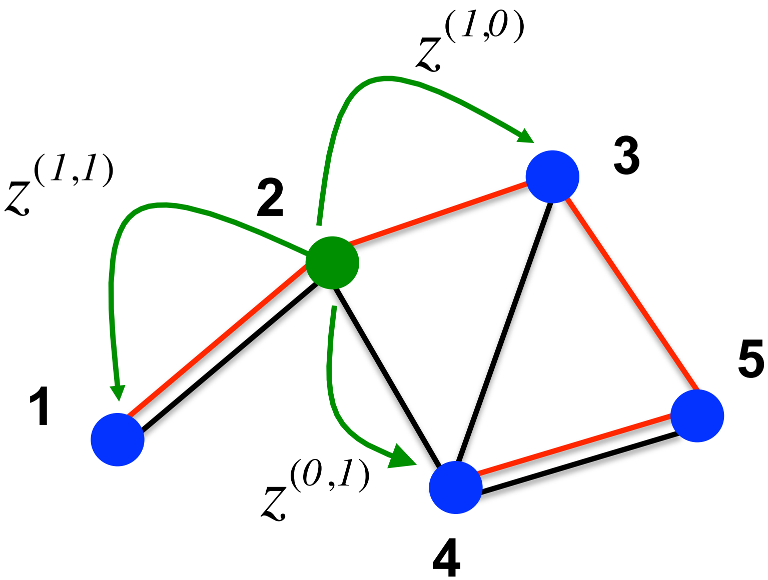
Multilinks

G. Bianconi
PRE (2013)



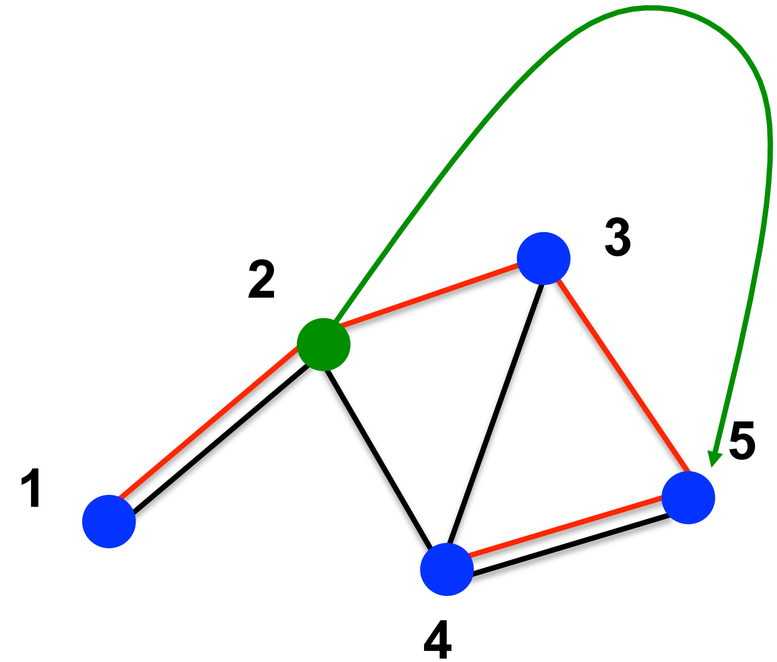


Functional Multiplex PageRank



(a)

The random walker can jump to a neighbor node, with a probability proportional to the influence of the corresponding multilink



(b)

The random walker can jump to a random node (teleportation)

Functional Multiplex PageRank

The Functional Multiplex PageRank assigns to each node a function indicating the centrality of the nodes when multilinks of different types have different influences $z^{\vec{m}}$

It is given by

$$\boxed{X_i(\mathbf{z}) = \tilde{\alpha}\sum_{j=1}^{N} \frac{A_{ij}^{\vec{m}} z^{\vec{m}}}{g_j} X_j(\mathbf{z}) + \beta v_i}$$

where

$$\begin{aligned}
g_j &= \max\left(1, \sum_{i=1..N} A_{ij}^{\vec{m}} z^{\vec{m}}\right)\\
v_i &= \theta\left(\sum_{j=1..N} A_{ij}^{\vec{m}} z^{\vec{m}} + \sum_{j=1..N} A_{ji}^{\vec{m}} z^{\vec{m}}\right)\\
z^{\vec{0}} &= 0, \quad \tilde{\alpha} > 0
\end{aligned}$$

and $\beta > 0$ fixed by the normalization condition

$$\sum_{i=1}^{N} X_i(\mathbf{z}) = 1$$

Functional Multiplex PageRank

The centrality of a node i

is a function

$$\boxed{X_i(\mathbf{z})}$$

depending on the values of the influences **z** attributed to multilinks

For a duplex network

$$\mathbf{z} = (z^{(1,0)}, z^{(0,1)}, z^{(1,1)})$$

Non-linear effects due to the overlap of the links

The Functional Multiplex PageRank allows for the inclusion of strong non-linear effects due to the overlap of the links.

For example, in a duplex network we can have

$$z^{(1,1)} \neq z^{(0,1)} + z^{(1,0)}$$

and we can weight multilinks (1,1) much more or much less than the sum of the weight of multilinks (0,1) and (1,0).

Absolute Multiplex PageRank

From the
Functional Multiplex PageRank
we can extract the
Absolute Multiplex PageRank
given by

$$\boxed{X_i^* = \max_{\mathbf{z}} X_i(\mathbf{z})}$$

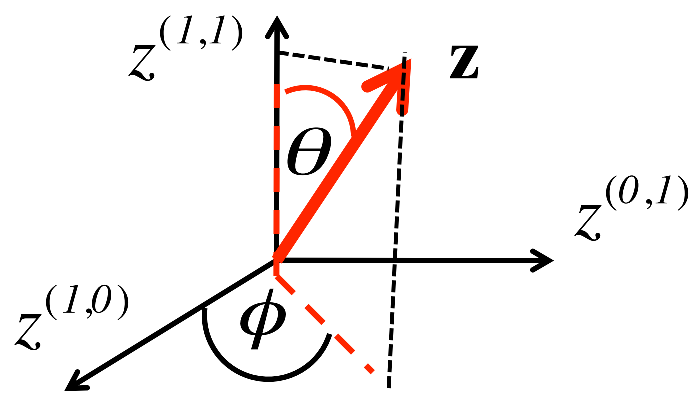
which can provide an overall ranking of the nodes
of the multiplex network

The case of a duplex network (M=2)

The Functional Multiplex PageRank,

depends only of the direction of the vector of influences **z,** therefore we take

$$\boxed{\begin{aligned} z^{(1,0)} &= \sin\theta\cos\phi \\ z^{(0,1)} &= \sin\theta\sin\phi \\ z^{(1,1)} &= \cos\theta \end{aligned}}$$



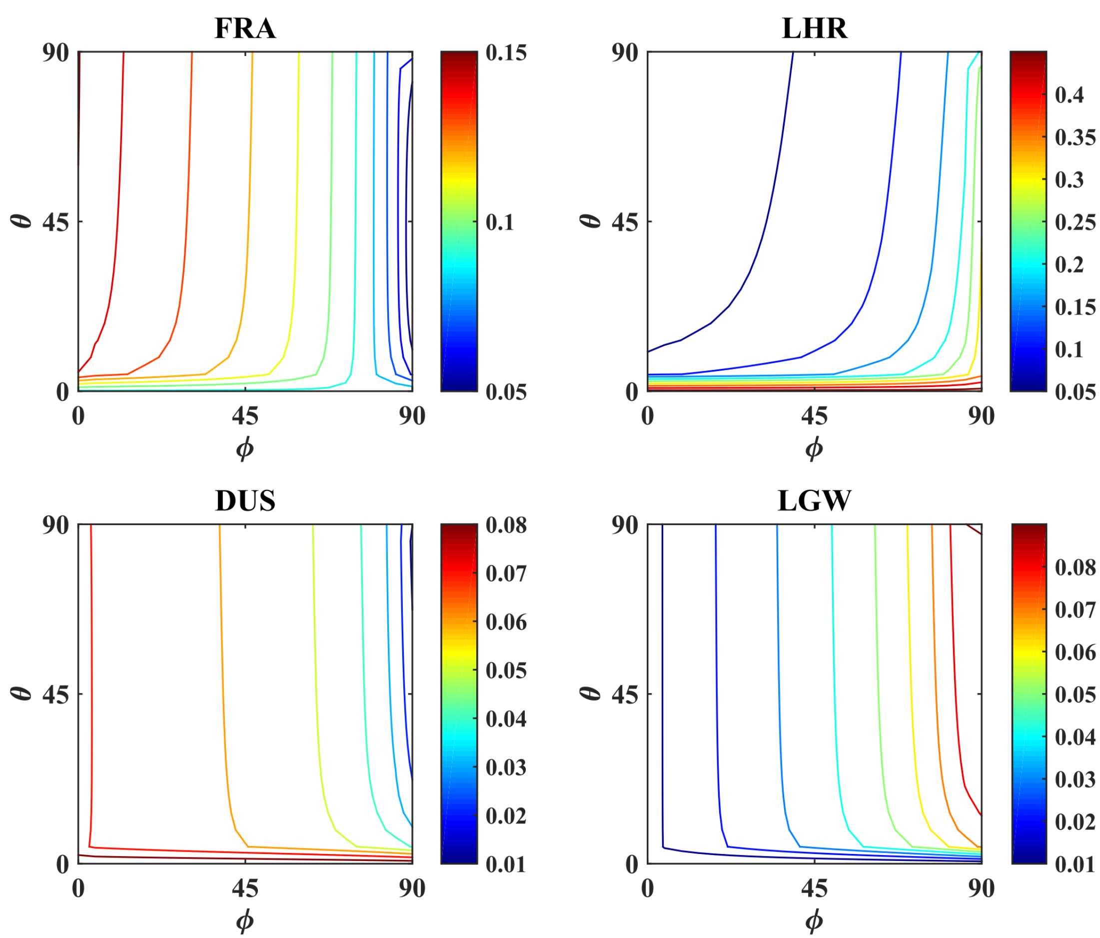
with

$$\boxed{\theta,\phi \in [0,\pi/2]}$$

Top ranked airports in the duplex Lufthansa/British Airways network according to the Absolute Multiplex PageRank

Rank	Airport
1	Heathrow Airport (LHR)
2	Munich Airport (MUC)
3	Frankfurt Airport (FRA)
4	Gatwick Airport (LGW)

Different pattern to success of major airports



For $\phi=0^o$ $\theta=90^o$ multilinks (1,0) have major influence

For $\phi=90^o$ $\theta=90^o$ multilinks (0,1) have major influence

For $\theta=0^o$ multilinks (1,1) have major influence

Correlations between the *pattern to success*

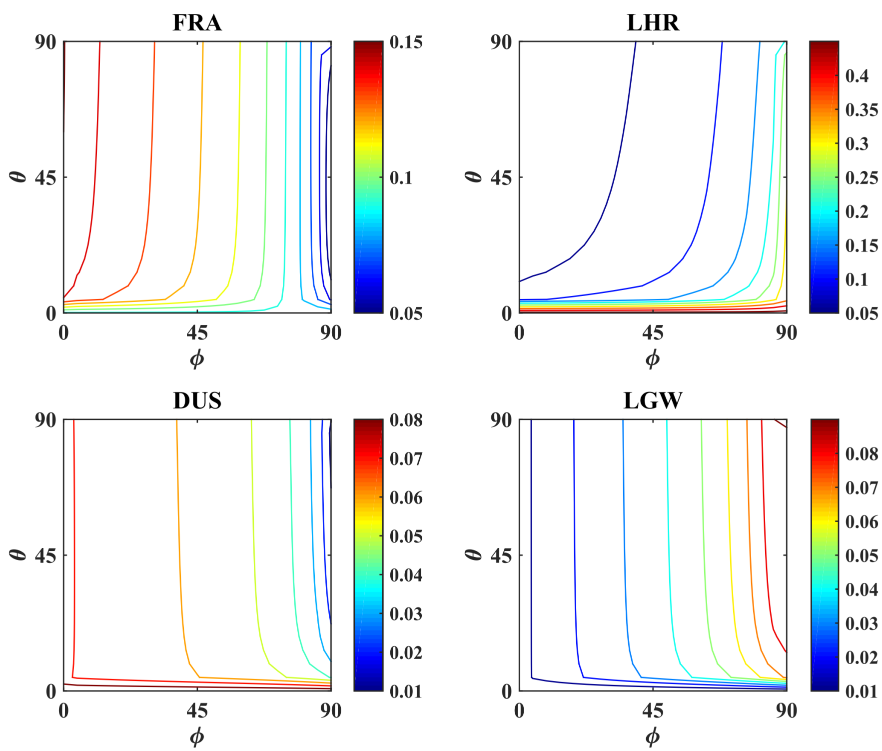
$$\rho = \frac{\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle}{\sigma_i \sigma_j}$$

where the average and the standard deviation are calculated on a grid (ϕ_r ,θ_s) with r=1,2,..,N_ϕ and s=1,2,…,N_θ

$$\langle Y \rangle = \frac{1}{N_\phi N_\theta} \sum_{r=1..N_\phi} \sum_{s=1..N_\theta} Y(\phi_r, \theta_s)$$

$$\sigma_i = \sqrt{\langle X_i^2 \rangle - \langle X_i \rangle^2}$$

Correlations between the pattern to success between major airports



ρ	LHR	FRA	LGW	DUS
LHR	1	-0.797	0.484	0.351
FRA	-0.797	1	-0.983	0.275
LGW	0.484	-0.983	1	-0.729
DUS	0.351	0.2758	-0.729	1

Duplex connectome network of C.elegans

Top ranked neurons

Rank	Neuron	Rank	Neuron
1	AVAR	6	PVCR
2	AVAL	7	AVDR
3	AVBL	8	AVER
4	AVBR	9	AVEL
5	PVCL	10	DVA

Similar neurons types have correlated *pattern to success*

Pearson correlations

Conclusions

Diffusion and epidemic spreading
are fundamental dynamical processes which
display a rich interplay between structure and dynamics

Multilayer networks display properties that are not observed in single networks taken in isolation

- For strong diffusion constant between the layers, diffusion is faster than in the slowest layer and one can also observe superdifusion
- Epidemics can spread in the multilayer network also when it cannot spread in the single layers taken in isolation. Epidemic spreading is strongly affected by the strcuture of the multilayer network
- The Functional Multiplex PageRank is based on the random walk on multiplex network. It assigns a function to a node *the pattern of success* and detects nodes with similar role

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