Modular Group and Modular Forms: Exercises 1, Week 1. Nov. 2008

These problems are on basic material, involving facts about group actions, function theory on complex tori and other surfaces.

1. Show that the action of $G = SL(2, \mathbb{R})$ on the upper half plane is transitive: given any two points $z \neq w$, there is an element $\gamma \in G$ with $w = \gamma(z)$. Show also that the action is transitive on the tangent space at each point: for any two unit tangent vectors u, v based at z, say, there is a γ with $\gamma'(z).u = v$.

2. Show that the hyperbolic line element ds_h^2 is G-invariant.

3. Show that the set of matrices $\gamma \in G$ which fix the point i is the compact subgroup $K = SO(2, \mathbb{R}) \cong S^1$ of rotation matrices. Deduce that the space $\mathcal{U}$ is isomorphic to the (right) K-coset space $K\backslash G$.

4. Find all the compact subgroups of G. [In other words, what are the compact subgroups of K?]

5. Show that the modular group $SL(2, \mathbb{Z})$ acts transitively on the set $\mathbb{Q} \cup \infty$. Hence find parabolic elements of the modular group which fix a given rational point p/q.

Weekly course summary:

In preparation, to appear next week.