Modular Group and Modular Forms: Exercises 2, 1 Dec. 2008

These problems are also on basic material, involving facts about group actions, function theory on complex tori and other surfaces.

1. Show that any automorphism (bijective biholomorphic self mapping) of the complex plane is a complex affine map

$$T(z) = az + b, \quad a \neq 0.$$

[Hint: such a mapping has the property that $f(z) = 1/T(z)$ has a removable singularity at the isolated point ∞.]

If in addition T commutes with the complex conjugation operator $J : z \mapsto \overline{z}$ then what can be said about T?

2. Let T be a hyperbolic element of $PSL(2, \mathbb{R})$ fixing two points a and b in $\partial\mathcal{U}$, and let ℓ be the h-axis joining a and b. If $P \in \ell$ show that the hyperbolic distance $d_h(P, T(P))$ is a constant $l(T)$: it is the *translation length* of T.

Can you relate $l(T)$ to the matrix of T?

3. Look up the definition of *Dirichlet fundamental domain* in either [Jones-Singerman, *Complex Functions*] or [Beardon, *Geometry of Discrete Groups*] if you don't know it. Verify that for $\Gamma(1)$ with centre $z_0 = it$ and $t > 1$ this domain is the standard triangular region defined in lectures.

4. Let $N > 1$ be a positive integer. The *principal congruence subgroup* $\Gamma(N)$ *with level* N is defined to be the kernel of the mapping '*reduction* mod N' from $SL(2, \mathbb{Z})$ to $SL(2, \mathbb{Z}/N\mathbb{Z})$. Show that this is an epimorphism. Show that the kernel has no torsion elements if $N > 2$.
If $N = 2$, show that the subgroup $\Gamma(2)$ has index 12. Can you calculate the index for N prime? for general N? (see [J-S] chapter 6).

5. (a) Write $\mathbb{C}^* = \mathbb{C} - \{0\}$ and let Λ denote the multiplicative subgroup $\{\lambda^n \,|\, n \in \mathbb{Z}\}$ where $\lambda \neq 0$ is fixed. Show that the coset space $\mathbb{C}^*/\Lambda = X$ is a compact Riemann surface. What is its universal covering space?
(b) Define a function f by the rule

$$f(z) = z \sum_{n=-\infty}^{\infty} (\lambda^{n/2} z - \lambda^{-n/2})^{-2}.$$

Prove that this sum converges uniformly on compact subsets of $\mathbb{C}^* - \Lambda$ and that the function satisfies $f(\lambda z) = f(z)$ for all z. What is the singularity of f at $z = 1$? Deduce that f determines a meromorphic function on the Riemann surface X.

Weekly course summary:

Modular group/ automorphic forms (Bill Harvey, KCL).

Week 1. The upper half plane, Moebius maps and hyperbolic plane geometry.

Summary notes are posted on the LTCC website.

Week 2. Action of SL(2,Z) as hyperbolic isometries.

Notes on this are in preparation.

Week 3. Lattices, elliptic functions and Eisenstein series.

Week 4. Automorphic forms in general; projective embedding of modular quotient spaces.

Week 5. Related topics, chosen from quadratic forms, discussion of higher dimensional modular varieties or the Grothendieck-Belyi theory of arithmetically defined algebraic curves.