Measure and Category

Marianna Csörnyei

mari@math.ucl.ac.uk
http:/www.ucl.ac.uk/∼ucahmcs

A (very short) Introduction to Cardinals

- The cardinality of a set A is equal to the cardinality of a set B, denoted $|A| = |B|$, if there exists a bijection from A to B.
- A countable set A is an infinite set that has the same cardinality as the set of natural numbers $\mathbb{N}$. That is, the elements of the set can be listed in a sequence
  $A = \{a_1, a_2, a_3, \dots\}$.
  If an infinite set is not countable, we say it is uncountable.
- The cardinality of the set of real numbers $\mathbb{R}$ is called continuum.

Examples of Countable Sets

- The set of integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ is countable.
- The set of rationals $\mathbb{Q}$ is countable.
  For each positive integer k there are only a finite number of rational numbers $\frac{p}{q}$ in reduced form for which $|p| + q = k$. List those for which $k = 1$, then those for which $k = 2$, and so on:

$$\mathbb{Q} = \{\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{3}, \ldots\}$$

- Countable union of countable sets is countable.
  This follows from the fact that $\mathbb{N}$ can be decomposed as the union of countable many sequences:

  $1, 2, 4, 8, 16, \ldots$
  $3, 6, 12, 24, \ldots$
  $5, 10, 20, 40, \ldots$
  $7, 14, 28, 56, \ldots$

Cantor Theorem

Theorem (Cantor) *For any sequence of real numbers $x_1, x_2, x_3, \ldots$ there is an $x \in \mathbb{R}$ such that $x \neq x_n$ for every n. That is, $\mathbb{R}$ is uncountable.*

Proof.

- Let I_1 be a closed interval such that $x_1 \notin I_1$.
- Let I_2 be a closed subinterval of I_1 such that $x_2 \notin I_2$.
- Proceeding inductively, let I_n be a closed subinterval of I_{n-1} such that $x_n \notin I_n$.
- The nested sequence of intervals has a non-empty intersection. If $x \in \bigcap I_n$, then $x \neq x_n$ for any n. □

Homeworks

1. Show that every (open or closed) interval has continuum many points.
2. Show that $\mathbb{N}$ has countably many finite subsets and continuum many infinite subsets.
3. Show that there are continuum many irrational numbers.
4. Show that there are continuum many infinite sequences of 0's and 1's.
5. The Cantor set is created by repeatedly deleting the open middle thirds of a set of line segments. One starts by deleting the open middle third $(\frac{1}{3}, \frac{2}{3})$ from the interval $[0, 1]$, leaving two line segments: $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$. And so on. The Cantor set contains all points in the interval $[0, 1]$ that are not deleted at any step in this infinite process.
   Show that the Cantor set has continuum many points.

Baire Category Theorem and the Banach-Mazur Game

- A set $A \subset \mathbb{R}$ is dense in the interval I, if A has a non-empty intersection with every subinterval of I. It is dense, if it is dense in every interval.
- More generally: a subset A of a topological space X is dense if it meets each non-empty open subset of X.
- A set $A \subset \mathbb{R}$ is nowhere dense if it is not dense in any interval, i.e. every interval has a subinterval contained in the complement of A.
- More generally: a subset A of a topological space X is nowhere dense if every non-empty open subset of X has a non-empty open subset contained in the complement of A.

Example. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is countable and dense. The Cantor set is not countable and it is nowhere dense.

Remark. Any subset of a nowhere dense set is nowhere dense. The union of finitely many nowhere dense sets is nowhere dense. The closure of a nowhere dense set is nowhere dense.

Baire Category Theorem

Definition.

- A set is said to be of first category if it can be represented as a countable union of nowhere dense sets.
- A set is of second category if it is not of first category.
- The complement of a first category set is called residual.

Theorem (Baire Category Theorem) *Every non-empty open subset of $\mathbb{R}$ is of second category, i.e. it cannot be represented as a countable union of nowhere dense sets.*

Proof.

- Suppose $G = \bigcup_n A_n$, where G is non-empty open and each A_n is nowhere dense. Choose an interval $I_0 \subset G$.
- Choose a closed subinterval $I_1 \subset I_0$ disjoint from A_1. Choose a closed subinterval $I_2 \subset I_1$ disjoint from A_2. And so on.
- The intersection $\bigcap I_n$ is non-empty, and it is disjoint from each A_n. This is a contradiction since $\bigcap I_n \subset I_0 \subset G$. □

Duality on $\mathbb{R}$

Sets of measure zero and sets of first category are "small" in one sense or another:

Both are σ-ideals.
Both include all countable subsets.
Both include some uncountable subsets, e.g. Cantor set.
Neither class includes intervals.
The complement of any set of either class is dense.
Etc...

Duality Principle

- sets of measure zero $\leftrightarrow$ sets of first category
- sets of positive measure $\leftrightarrow$ sets of second category
- sets of full measure $\leftrightarrow$ residual sets
  (We will make this explicit later.)

However...

Theorem. $\mathbb{R}$ *can be decomposed into the union of two sets A and B such that A is of first category and B is of measure zero.*

Proof.

- Let $q_1, q_2 \dots$ be an enumeration of $\mathbb{Q}$.
- Let I_{ij} denote the open interval $(q_i - \frac{1}{2^{i+j}}, q_i + \frac{1}{2^{i+j}})$.
- Let $G_j = \bigcup_{i=1}^{\infty} I_{ij}$ and $B = \bigcap_{j=1}^{\infty} G_j$.
- Then $B \subset G_j$ for each j, hence the measure of B is at most $\sum_{i=1}^{\infty} |I_{ij}| = \sum_{i=1}^{\infty} \frac{2}{2^{i+j}} = \frac{1}{2^{j-1}} \to 0$ as $j \to \infty$. Hence B is a null set.
- On the other hand, G_j is dense and open, therefore its complement is nowhere dense.
- $A = \mathbb{R} \setminus B = \bigcup_{j=1}^{\infty} (\mathbb{R} \setminus G_j)$ is of first category. □

The Banach-Mazur Game

Player (I) is "dealt" an arbitrary subset $A \subset \mathbb{R}$. The game is played as follows:

- Player (I) chooses arbitrarily a closed interval I_1.
- Player (II) chooses a closed interval $I_2 \subset I_1$.
- Player (I) chooses a closed subinterval $I_3 \subset I_2$.
- Etc... Together the players determine a nested sequence of closed intervals $I_1 \supset I_2 \supset \dots$, (I) choosing those with odd index, (II) those with even index.
- If $\bigcap I_n$ has a point common with A then (I) wins; otherwise (II) wins.

Question. Which player can ensure, by choosing his intervals cleverly, that he will win, no matter how well his opponent plays? That is, which player has a "winning strategy"?

Winning Strategy

A strategy for either player is a rule that specifies what move he will make in every possible situation:

- At his nth move, (II) knows which intervals $I_1, I_2, \ldots, I_{2n-1}$ have been chosen before (and he knows the set A). From this information, his strategy must tell him which closed interval to choose for I_{2n}.
- Thus, a strategy for (II) is a sequence of interval-valued functions $f_n(I_1, I_2, \ldots, I_{2n-1})$. The rules of the game demand that $f_n(I_1, I_2, \ldots, I_{2n-1}) \subset I_{2n-1}$. The function f_n must be defined for all intervals that satisfy

$$I_1 \supset I_2 \supset \cdots \supset I_{2n-1} \quad \textit{and} \quad I_{2k} = f_k(I_1, \ldots, I_{2k-1}). \qquad (*)$$

- This is a winning strategy of (II), if $\bigcap I_n$ is disjoint from A for any sequence of intervals that satisfy $(*)$. The winning strategy of (I) is defined analogously.

Main Theorem

Theorem. *Player (II) has a winning strategy if and only if A is of first category.*

Remark. Player (II) cannot have a winning strategy if A contains an interval. So this theorem implies Baire Category Theorem: sets containing intervals, in particular, non-empty open sets are of second category.

Proof of: A is of first category $\Longrightarrow$ (II) has a winning strategy.

- Write $A = \bigcup A_n$, where each A_n is nowhere dense.
- In his nth step, player (II) can choose an interval I_{2n} disjoint from A_n.
- Then $\bigcap I_n$ is disjoint from each A_n, hence it is disjoint from A. □

Proof of: (II) has a winning strategy $\Longrightarrow$ A is of first category

- Let $f_1, f_2, \ldots$ be a winning strategy for (II).
- Choose closed intervals $I_1, I_2, \ldots$ such that the intervals $J_i = f_1(I_i)$ are pairwise disjoint and their union is dense. Then $A_1 = \mathbb{R} \setminus \bigcup_i J_i$ is nowhere dense.
- Choose closed intervals $I_{i1}, I_{i2}, I_{i3}, \ldots$ inside each interval J_i, such that the intervals $J_{ij} = f_2(I_i, J_i, I_{ij})$ are disjoint, and their union is dense in J_i. Then $A_2 = \mathbb{R} \setminus \bigcup_{ij} I_{ij}$ is nowhere dense.
- Choose a dense set of closed intervals $I_{ij1}, I_{ij2}, I_{ij3}, \ldots$ inside each interval J_{ij}, such that $J_{ijk} = f_3(I_i, J_i, I_{ij}, J_{ij}, I_{ijk})$ are disjoint, and their union is dense in J_{ij}. And so on.
- Claim: $A \subset \bigcup_n A_n$. Suppose there is $x \in A \setminus \bigcup_n A_n$.
- Since $x \notin A_1$, therefore there is an index i such that $x \in J_i$.
- Since $x \notin A_2$, there is an index j such that $x \in J_{ij}$.
- And so on. This defines a nested sequence of intervals so that x is in their intersection. Since $x \in A$, this contradicts the fact that $I_i, J_i, I_{ij}, J_{ij}, I_{ijk}, \ldots$ is a winning game for (II). $\square$

Complete Metric Spaces

- A metric space is a set X with a distance function $d(x, y)$ defined for all pairs of points of X and satisfying:
  - $d(x, y) > 0$ for all $x \neq y$, and $d(x, x) = 0$
  - $d(x, y) = d(y, x)$
  - $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)
- A sequence $x_1, x_2, \ldots$ of points of X converges to a point $x \in X$ if $d(x_n, x) \to 0$. A sequence is convergent, if it converges to some point.
- A sequence of points $x_1, x_2, \ldots$ is called a Cauchy sequence, if for each $\varepsilon > 0$ there is a positive integer n such that $d(x_i, x_j) < \varepsilon$ for all $i, j \geq n$.
- Every convergent sequence is Cauchy, but the converse is not generally true. However, there is an important class of metric spaces in which every Cauchy sequence is convergent. Such a metric space is called complete.

Baire Category Theorem in Complete Metric Spaces

- We denote $B(x, r) = \{y : d(x, y) < r\}$ the ball of centre x and radius r.
- A set $G \subset X$ is open, if for each $x \in G$, G contains some ball with centre x. The balls $B(x, r)$ are open sets, and arbitrary unions and finite intersections of open sets are open. The complement of an open set is closed. A set $F \subset X$ is closed if and only if $x_1, x_2, \cdots \in F$, $x_n \to x$ imply $x \in F$.
- The smallest closed set that contains a set A is the closure of A. It is denoted by $\text{cl}(A)$. The largest open set contained in A is called the interior of A. It is denoted by $\text{int}(A)$.
- The notion of open and closed sets allow us to define dense sets, sets of first and second category, etc.

Definition. A topological space X is called a Baire space if every non-empty open set in X is of second category.

Theorem. *Every complete metric space is a Baire space.*

Homeworks

1. Prove the theorem that every complete metric space is a Baire space. (Where did your proof use that the metric space is complete? It should.)
2. A topological space is called separable, if it contains a countable dense set. A separable complete metric space is called a Polish space.
   Formulate an analogue of Banach-Mazur Game in Polish spaces, and prove that (II) has winning strategy if and only if A is of first category.
3. The Choquet Game is played as follows. Player (I) chooses a non-empty open set $G_1 \subset X$. Player (II) chooses a non-empty open subset $G_2 \subset G_1$. Player (I) chooses a non-empty open subset $G_3 \subset G_2$. And so on. Player (I) wins if $\bigcap G_n = \emptyset$.
   Find a winning strategy for $X = \mathbb{R}$.
   Find a winning strategy if X is a Polish space.
   Prove that Player (I) has no winning strategy if and only if X is a Baire space.

Concluding Remarks

In Homework 3, "Player (I) has no winning strategy" is not the same as "Player (II) has a winning strategy". For infinite games it may happen that none of the two players have a winning strategy.

Even in the Banach-Mazur Game: we proved that Player (II) has winning strategy if and only if A is of first category. Denote $B = \mathbb{R} \setminus A$. It is not difficult to see that Player (I) has a winning strategy if and only if there is an interval I_1 so that $I_1 \cap B$ is of first category. Then Player (I) can choose this interval I_1 as his first move, then he plays so that $\bigcap_n I_n$ is disjoint from B.

Fact. *There is a set $A \subset \mathbb{R}$ such that A is of second category, and $I \cap B$ is of second category for each interval I. Therefore none of the players has a winning strategy in the corresponding Banach-Mazur Game.*

Baire Category Theorem as a Proof of Existence

Baire Category Theorem is an important tool in analysis for "proving existence". An illustrating example is the proof of the existence of nowhere differentiable functions.

Many examples of nowhere differentiable continuous functions are known, the first having been constructed by Weierstrass:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + \frac{3}{2}\pi$.

It is quite hard to prove that Weierstrass' function is nowhere differentiable. But Weierstrass' function is far from being an isolated example: Banach gave a simple proof that, in the sense of category, almost all continuous functions are nowhere differentiable. It turns out that, in fact, it is exceptional for a continuous function to have a derivative anywhere in $[0, 1]$:

Typical Continuous Functions

Theorem (Banach) *A typical continuous function is nowhere differentiable.*

Definition. By typical we mean that all continuous functions, except for those in a first category subset of $C[0,1]$, exhibit the behaviour we describe. That is, a property T is typical, if $\{f \in C[0,1] : f \text{ has property } T\}$ is residual in $C[0,1]$.

Note that if $T_1, T_2, \ldots$ are typical properties, then "T_1 and T_2 and..." is also typical.

As usual, $C[0,1]$ denotes the space of all continuous functions defined on the interval $[0,1]$, with the so-called uniform metric: $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. The metric is called uniform because $f_n \to f$ if and only if f_n converges to f uniformly. This is a Polish space, so Baire Category Theorem can be applied for $C[0,1]$.

Proof of Banach Theorem

A function f is said to be locally increasing at a point x, if there is a small neighbourhood of x in which $f(y) \leq f(x)$ for all $y \leq x$ and $f(y) \geq f(x)$ for all $y \geq x$. The proof of Banach Theorem is based on the following lemma:

Lemma. *A typical continuous function is not locally increasing at any point.*

Since shifted copies of residual sets are residual, as a corollary of the lemma we can see:

Corollary. *For any given $g \in C[0,1]$ and for a typical continuous function $f \in C[0,1]$, $f + g$ is not locally increasing at any point.*

By choosing $g(x) = nx$ (where $n \in \mathbb{N}$ is arbitrary), we can see that $f(x) + nx$ is not locally increasing at any point for a typical continuous function f. In particular, f cannot have a derivative $f'(x) > -n$ at any point. □

Proof of the Lemma

Let A_n denote the set of those continuous functions for which there is an x such that f is locally increasing on $[x - \frac{1}{n}, x + \frac{1}{n}]$. It is enough to show that the sets A_n are closed and they have empty interior. This shows they are nowhere dense.

- A_n is closed for each n: suppose that $f_1, f_2, \dots \in A_n$, and $f_n \to f$ uniformly. We need to show that $f \in A_n$.
  For each f_k there is an x_k so that f_k is locally increasing on $[x_k - \frac{1}{n}, x_k + \frac{1}{n}]$. By choosing a subsequence, we can assume that $x_1, x_2, \dots$ converges, say to a point x. It is easy to verify that f is locally increasing on $[x - \frac{1}{n}, x + \frac{1}{n}]$.
- A_n has empty interior: we need to show that A_n contains no open ball $B(f, r) = \{g \in C[0,1] : \forall x, |f(x) - g(x)| \leq r\}$. Indeed, it is easy to see that for every $f \in C[0,1]$ and for every $r > 0$ we can choose an appropriate saw-tooth function $g \in C[0,1]$ such that $g \in B(f, r)$ but $g \notin A_n$. □

Another Application: Besicovitch Sets

A Besicovitch set is a subset of $\mathbb{R}^n$ which contains a line segment in each direction. Besicovitch sets are also known as Kakeya sets.

In 1917 Besicovitch was working on a problem in Riemann integration, and reduced it to the question of existence of planar sets of measure 0 which contain a line segment in each direction. He then constructed such a set.

Many other ways to construct Besicovitch sets of measure zero have since been discovered. Here we show a proof of T.W. Körner (Studia Math. 158 (2003), no. 1, 65–78.) of the existence of Besicovitch sets in $\mathbb{R}^2$. This proof shows that "a typical Besicovitch set" has measure zero. Of course, in order to understand what we mean by a "typical Besicovitch set" first we need to define an appropriate metric space whose "points" are Besicovitch sets.

Hausdorff Distance

Definition. Let F_1 and F_2 be two non-empty closed subsets of the unit square $[0,1]^2$. The Hausdorff distance $d_H(F_1, F_2)$ is the minimal number r such that the closed r-neighborhood of F_1 contains F_2 and the closed r-neighborhood of F_2 contains F_1.

It is easy to check that d_H is a metric; the resulting metric space is denoted by $\mathcal{F}$.

The "points" of $\mathcal{F}$ are the non-empty closed subsets of $[0,1]^2$.

$\mathcal{F}$ is separable, since finite sets of points with rational coordinates form a countable dense subset of $\mathcal{F}$.

$\mathcal{F}$ is complete

Proof. Let $F_1, F_2, \ldots$ be a Cauchy sequence in $\mathcal{F}$. Let F be the set of limit points of sequences x_k with $x_k \in F_k$. We show that F is the limit of the sequence F_k.

1. F is not too large:
   - Pick $\varepsilon > 0$. Take N so large that $m, n \geq N$ implies $d_H(F_m, F_n) < \varepsilon$.
   - Since F_n is in the ε-neighbourhood of F_N for each $n \geq N$, clearly F is also in the ε-neighbourhood of F_N.
2. F is not too small:
   - Pick N_i strictly increasing so that $m, n \geq N_i$ implies $d_H(F_m, F_n) < \frac{\varepsilon}{2^i}$.
   - For any $x \in F_{N_1}$ there are points $x_k \in F_k$ for each $N_1 < k \leq N_2$ for which $d(x, x_k) < \frac{\varepsilon}{2}$. Similarly, there are points $x_k \in F_k$ for $N_2 < k \leq N_3$ for which $d(x_{N_2}, x_k) < \frac{\varepsilon}{4}$. And so on.
   - This defines a sequence x_k converging less than $\frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots = \varepsilon$ away from x, hence F_{N_1} is in the ε-neighbourhood of F. □

Construction of Besicovitch sets: Main Lemma

Notation. Let $\mathcal{K}$ denote the set of closed subsets of $[0,1]^2$ that can be written as a union of line segments connecting the top and bottom sides of $[0,1]^2$, containing at least one line segment of each direction of angle between 60° and 120°.

One checks easily that $\mathcal{K}$ is a closed subset of $\mathcal{F}$. In particular, $\mathcal{K}$ is complete.

Our aim is to show that a typical element of $\mathcal{K}$ has Lebesgue measure zero. Then, taking the union of three rotated copies of a null set $K \in \mathcal{K}$ we obtain a Besicovitch set of measure zero. It is enough to prove the following lemma:

Main Lemma. Let $I \subset [0,1]$ be any line segment of length ε. Then a typical element of $\mathcal{K}$ intersects each horizontal line segment of the strip $S = \{(x,y) \in [0,1]^2 : y \in I\}$ in a set of linear measure at most 100ε.

Proof of Main Lemma

Let $\mathcal{L}$ denote the set of those $K \in \mathcal{K}$ for which $K \cap S$ can be covered by finitely many triangles whose union meets each horizontal line segment of S in a set of measure less than 100ε. Clearly, $\mathcal{L}$ is an open subset of $\mathcal{K}$.

Claim. *$\mathcal{L}$ is dense.*

Proof. Let $K \in \mathcal{K}$ and $\delta > 0$. Choose finitely many line segments $L_1, L_2, \dots, L_n$ of angles $60^\circ = \theta_1 < \theta_2 < \cdots < \theta_n = 120^\circ$ connecting the top and bottom sides of $[0,1]^2$ so that

1. $d_h(\bigcup_{k=1}^{n-1} L_k, K) < \delta$;
2. $|\theta_{k+1} - \theta_k| < \delta$ for $k = 1, 2, \dots, n-1$.
3. Let P_k denote the intersection of L_k and of the horizontal middle line of S. Then the line segments passing through the points P_k of angle in $[\theta_k, \theta_{k+1}]$ join the top and bottom sides of $[0,1]^2$.

Let L be the union of all line segments as in 3. If δ is small enough, then $d_H(K, L) < \varepsilon$ and $L \in \mathcal{L}$. □

Third Application: Typical Homeomorphisms

A bijection f is called a homeomorphism, if it is continuous and it has a continuous inverse.

Remark. If $f : [0,1] \to [0,1]$ is a homeomorphism then it is either strictly increasing or strictly decreasing. Hence $x, f(x), f(f(x)), \ldots$ is a monotone sequence for each $x \in [0,1]$. However, the homeomorphisms of the square $[0,1]^2$ are more interesting:

Theorem. *There is a homeomorphism $T : [0,1]^2 \to [0,1]^2$ such that $x, T(x), T(T(x)), \ldots$ is dense in $[0,1]^2$ for some x.*

Notation. We denote $x = T^0(x)$, $T(x) = T^1(x)$, $T(T(x)) = T^2(x)$, $T(T(T(x))) = T^3(x)$, etc. Also $T^{-1}(x)$ is the inverse image of x, $T^{-2}(x) = T^{-1}(T^{-1}(x))$, etc.

$T^0(x), T^1(x), T^2(x), \ldots$ is called the orbit of x.
A point x is recurrent, if for any open set $G \ni x$ there is an $n \geq 1$ so that $T^n(x) \in G$. That is, its orbit has a subsequence converging to x.
The point x is periodic, if its orbit is periodic.

Metric on Homeomorphisms

Our aim is to show that for a "typical" homeomorphism there is an x with dense orbit. A metric on the space of all homeomorphisms of $[0,1]^2$:

$$d(S,T) = \max_{x\in[0,1]^2} |T(x) - S(x)| + \max_{x\in[0,1]^2} |T^{-1}(x) - S^{-1}(x)|$$

That is, homeomorphisms $T_1, T_2, \dots$ converge to T if $T_n \to T$ uniformly and $T_n^{-1} \to T^{-1}$ uniformly. With this metric, the space of homeomorphisms is a complete metric space (see homeworks). We make the problem harder and demand in addition that $T : [0,1]^2 \to [0,1]^2$ preserves measure (that is, each set X has the same measure as its image $T(X)$):

Theorem. *For a typical measure preserving homeomorphism $T : [0,1]^2 \to [0,1]^2$, the orbit of a typical point $x \in [0,1]^2$ is dense.*

The set of all measure preserving homeomorphisms is a closed subset of all homeomorphisms, so it is also a complete metric space. We denote it by M.

Poincaré Recurrence Theorem

Lemma (Poincaré Recurrence Theorem)

Let T be a measure preserving homeomorphism. Then all points are recurrent except a set of first category and measure zero.

Proof.

- Let $Q_1, Q_2, \dots$ be an enumeration of all open squares with rational vertices contained in $[0,1]^2$. Need to show that, for each k, $\{x \in Q_k : T^n(x) \notin Q_k \text{ for any } n \geq 1\}$ is of first category and measure zero.
- Fix k and let $R = Q_k \setminus \bigcup_{n=1}^{\infty} T^{-n}(Q_k)$.
- The sets R, $T(R)$, $T^2(R), \dots$ all have the same measure. They are also pairwise disjoint: indeed, if $T^i(R) \cap T^j(R) \neq \emptyset$ for some $i < j$, then $T^{i-j}(Q_k) \cap R \supset T^{i-j}(R) \cap R \neq \emptyset$, contradiction). So they must have zero measure.
- It is also clear that R is (relatively) closed in Q_k, and since it has zero measure, it has empty interior. Closed sets of empty interior are nowhere dense. □

Further Lemmas

Lemma. *For a typical $T \in M$, a typical point $x \in [0,1]^2$ is non-periodic.* *(Proof: Homework.)*

Lemma. *Let $Q_1, Q_2, \ldots$ be an enumeration of squares as before. Then $E_{ij} = \bigcup_{k=1}^{\infty} \{T \in M : Q_i \cap T^{-k}(Q_j) \neq \emptyset\}$ is dense in M.*

Proof. Fix a homeomorphism T for which a typical point x is non-periodic, and a small $\varepsilon > 0$. By Poincaré Recurrence Theorem, a typical $x \in [0,1]^2$ is recurrent. Choose $x_1, x_2, \ldots, x_N$ so that $B(x_1, \varepsilon) \subset Q_i$, $B(x_N, \varepsilon) \subset Q_j$, and $B(x_k, \varepsilon) \cap B(x_{k+1}, \varepsilon) \neq \emptyset$. Choose a recurrent non-periodic point x_k from $B(x_k, \varepsilon)$, and let $n_k \geq 1$ with $T^{n_k}(x_k) \in B(x_k, \varepsilon)$. Furthermore, let x_k be chosen so that it does not belong to the orbit of any of the points $x_1, x_2, \ldots, x_{k-1}$.

It is easy to define a homeomorphism $S \in M$ that moves each point of $[0,1]^2$ by at most 10ε, identity in a neighborhood of all the points $T(x_k), T^2(x_k), \ldots, T^{n_k - 1}(x_k)$ for all k, and it maps $T^{n_k}(x_k)$ to x_{k+1}. Then the homeomorphism $(S \circ T)^{n_1 + n_2 + \cdots + n_N}$ takes x_1 to x_N. Since $d(S \circ T, T) \leq 10\varepsilon$, E_{ij} is dense. □

Proof of the Theorem

Recall that we are proving:

Theorem. *For a typical measure preserving homeomorphism $T: [0,1]^2 \to [0,1]^2$, the orbit of a typical point $x \in [0,1]^2$ is dense.*

Proof.

- By the previous Lemma, $E_{ij} = \bigcup_{k=1}^{\infty} \{T \in M : Q_i \cap T^{-k}(Q_j) \neq \emptyset\}$ is dense in M. It is also clear that E_{ij} is open. Hence $\bigcap_{ij} E_{ij}$ is residual.
- For any $T \in \bigcap_{ij} E_{ij}$ we have $Q_i \cap \left(\bigcup_{k=1}^{\infty} T^{-k}(Q_j)\right) \neq \emptyset$ for all i, j. Hence $G_j = \bigcup_{k=1}^{\infty} T^{-k}(Q_j)$ is a dense open subset of $[0,1]^2$.
- Then $\bigcap_j G_j$ is a residual subset of $[0,1]^2$. For any $x \in \bigcap_j G_j$ and for any j, $T^k(x) \in Q_j$ for some k, hence the orbit of x is dense.

□

Homeworks

1. Prove that $C[0,1]$ is a Polish space.
2. Let $A \subset [0,1]$ be an arbitrary set of first category. Is it true that $f(A)$
   - has measure zero
   - is of first category

   for a typical continuous function f?
3. Show that

   $$d(S,T) = \max_{x \in [0,1]^2} |T(x) - S(x)| + \max_{x \in [0,1]^2} |T^{-1}(x) - S^{-1}(x)|$$

   is a complete metric on the space of all homeomorphisms of $[0,1]^2$.
4. Prove that for a typical measure-preserving homeomorphism of $[0,1]^2$, a typical point $x \in [0,1]^2$ is non-periodic.
5. Is it true that for a typical homeomorphism of $[0,1]^2$ (without assuming measure-preserving) there is an x with dense orbit?

Measurable and Baire Sets

An interval $I \subset \mathbb{R}^d$ is a rectangular parallelepiped with edges parallel to the axes. It is the product of n 1-dimensional intervals. The volume of I is denoted by $|I|$.

The infimum of the sums $\sum |I_k|$, for all sequences $I_1, I_2, \ldots$ of open intervals that cover E, is called the outer (Lebesgue) measure of E. It is denoted by $\lambda^*(E)$.

Facts.

- *If $A \subset B$, then $\lambda^*(A) \leq \lambda^*(B)$.*
- *If $A = \bigcup_{k=1}^{\infty} A_k$, then $\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda^*(A_n)$.*
- *For any interval I, $\lambda^*(I) = \lambda(I)$.*

Definition. A set E is a null set, if $\lambda^*(E) = 0$.

Definition. E is measurable, if for every $\varepsilon > 0$ there is a closed set F and an open set G such that $F \subset E \subset G$ and $\lambda^*(G \setminus F) < \varepsilon$.

Proposition. *Any interval, open set, closed set, null set is measurable.*

Sigma-algebra of Measurable Sets

Theorem. *The class of measurable sets is the σ-algebra generated by open sets together with null sets. The outer measure λ^* is countably additive on measurable sets; it is called Lebesgue measure and it is denoted by λ. It is a complete measure.*

- A class of sets S is called σ-algebra, if it contains the countable intersections, countable unions, and complements of its members:
  $A_1, A_2, \cdots \in S \Longrightarrow \bigcap_{n=k}^{\infty} A_k \in S$, $\bigcup_{k=1}^{\infty} A_k \in S$, $\mathbb{R}^d \setminus A_k \in S$.
- For any class of sets there is a smallest σ-algebra that contains it; namely, the intersection of all such σ-algebras. This is called the σ-algebra generated by the sets.
- A function $f : S \to \mathbb{R}$ is countably additive, if $f(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} f(A_k)$ holds whenever $A_1, A_2, \ldots$ are disjoint members of S.
- When every subset of a null set belongs to S, it is said to be complete.

Regularity Properties

Since the σ-algebra of measurable sets includes all intervals, it follows that it includes all open sets, all closed sets, all F_σ sets (countable unions of closed sets), all G_δ sets (countable intersections of open sets), etc. Moreover:

Theorem. *A set E is measurable if and only if it can be written as an F_σ set plus a null set (or as a G_δ set minus a null set).*

Proof. If E is measurable, for each n there is a closed set $F_n \subset E$ and an open set $G_n \supset E$ such that $\lambda(G_n \setminus F_n) < 1/n$. Then $\bigcup_{n=1}^\infty F_n$ is F_σ, and $E \setminus \bigcup_{n=1}^\infty F_n$ is a null set since it is contained in $G_n \setminus F_n$ for each n. Similarly, $\bigcap_{n=1}^\infty G_n$ is G_δ, and $(\bigcap_{n=1}^\infty G_n) \setminus E$ is a null set. □

Theorem.

- *For any set E, $\lambda^*(E) = \inf\{\lambda(G) : E \subset G, G$ is open$\}$.*
- *If E is measurable, then $\lambda^*(E) = \sup\{\lambda(F) : F \subset E, F$ is bounded and closed$\}$. (Conversely, if this equation holds and $\lambda^*(E) < \infty$, then E is measurable.)*

Lebesgue Density Theorem

Definition. A measurable set E has density d at a point x, if $\lim_{r\to 0} \frac{\lambda(E\cap B(x,r))}{\lambda(B(x,r))}$ exists and is equal to d.

Notation. Let $\phi(E)$ denote the set of those points $x \in \mathbb{R}^d$ at which E has density 1. (Then E has density 0 at each point of $\phi(\mathbb{R}^d \setminus E)$.)

Theorem (Lebesgue Density Theorem)

For any measurable set E, $\lambda(E \triangle \phi(E)) = 0$.

This implies that E has density 1 at almost all points of E and density zero at almost all points of $\mathbb{R}^d \setminus E$. For instance, it is impossible that a measurable set has density $1/2$ everywhere.

Lemma (Vitali Covering Theorem)

Let $\mathcal{B}$ be a collection of balls centered at the points of a set E, so that $\mathcal{B}$ contains arbitrary small balls at each point $x \in E$. Then $\mathcal{B}$ has a subcollection $\mathcal{B}'$ that consists of disjoint balls of $\mathcal{B}$ and covers all points of E except for a null set.

Proof of Lebesgue Density Theorem

- It is enough to show that $E \setminus \phi(E)$ is a null set. We may also assume that E is bounded. Fix $\varepsilon > 0$. Let A denote the set of those $x \in E$ for which there is an arbitrary small ball $B(x, r)$ with $\frac{\lambda(E \cap B(x,r))}{\lambda(B(x,r))} < 1 - \varepsilon$. It is enough to show that A is a null set.
- Let $G \supset A$ be an open set with $\lambda(G) < \frac{\lambda(A)}{1-\varepsilon}$, and let $\mathcal{B}$ denote the collection of all balls $B(x, r) \subset G$ as above. We apply Vitali Covering Theorem to choose a subcollection $\mathcal{B}'$ that covers A except for a null set.
- The total measure of the balls of $\mathcal{B}'$ is at most $\lambda(G) < \frac{\lambda(A)}{1-\varepsilon}$ and at least

$$\sum_{B \in \mathcal{B}'} \frac{\lambda(E \cap B)}{1 - \varepsilon} = \frac{\lambda(E \cap (\bigcup_{B \in \mathcal{B}'} B))}{1 - \varepsilon} \geq \frac{\lambda(A)}{1 - \varepsilon},$$

a contradiction. □

Proof of Vitali Covering Theorem

We can assume that E is bounded. We construct inductively a disjoint sequence of members of $\mathcal{B}$.

- Let ρ_0 be the supremum of the radii of the balls in $\mathcal{B}$. We can assume that $\rho_0 < \infty$. Choose B_1 of radius at least $\rho_0/2$.
- Having chosen $B_1, \dots, B_n$, let $\mathcal{B}^n$ be the set of members of $\mathcal{B}$ that are disjoint from $B_1, \dots, B_n$. Let ρ_n be the supremum of the radii of the balls in $\mathcal{B}^n$ and choose $B_{n+1} \in \mathcal{B}^n$ of radius larger than $\rho_n/2$.

Claim. *$E = A \setminus \bigcup_{n=1}^{\infty} B_n$ is Lebesgue null.*

Proof.

- The balls $3B_1, 3B_2, \dots$ cover A, so $\lambda(A) \leq 3^d \sum_{n=1}^{\infty} \lambda(B_n)$. Choose N_1 so that $\sum_{n=1}^{N_1} \lambda(B_n) \geq \frac{1}{4^d} \lambda(A)$.
- Similarly, $3B_{N_1+1}, 3B_{N_1+2}, \dots$ cover $A \setminus \bigcup_{n=1}^{N_1} B_n$, therefore there is an N_2 so that $\sum_{n=N_1+1}^{N_2} \lambda(B_n) \geq \frac{1}{4^d} \lambda(A \setminus \bigcup_{n=1}^{N_1} B_n)$.
- Etc. By induction, $\lambda(A \setminus \bigcup_{n=1}^{N_k} B_n) \leq (1 - \frac{1}{4^d})^{k+1} \lambda(A) \to 0$. □

Property of Baire

Recall that a set A is measurable if it belongs to the σ-algebra generated by open sets together with null sets.

Definition. A set A has Baire property, if it can be represented in the form $A = G \triangle P$, where G is open and P is nowhere dense.

Theorem. *The class of sets having the Baire property is a σ-algebra. It is the σ-algebra generated by open sets together with sets of first category.*

Proof.

1. Complement: If $A = G \triangle P$ then $\mathbb{R}^d \setminus A = (\mathbb{R}^d \setminus G) \triangle P$, and if G is open then $\mathbb{R}^d \setminus G$ is closed. Any closed set is the union of its interior and a nowhere dense set.
2. Countable union: If $A_n = (G_n \setminus P_n) \cup Q_n$ where G_n is open and P_n, Q_n are of first category, then $G = \bigcup_{n=1}^{\infty} G_n$ is open, $P = \bigcup_{n=1}^{\infty} P_n$, $Q = \bigcup_{n=1}^{\infty} Q_n$ are of first category, and $G \setminus P \subset \bigcup_{n=1}^{\infty} A_n \subset G \cup Q$.
3. Countable intersection: $\bigcap_{n=1}^{\infty} A_n = \mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} (\mathbb{R}^d \setminus A_n)$. □

Measure and Category: Similarities and Differences

Theorem. *A set has Baire property if and only if it can be written as a G_δ set plus a first category set (or an F_σ set minus a first category set).*

Proof. Write $A = (G \setminus P) \cup Q$, where G is open, P, Q are of first category. Since the closure of a nowhere dense set is nowhere dense, P can be covered by an F_σ set F of first category. Then $A = (G \setminus F) \cup ((G \cap F) \setminus P) \cup Q$, where $G \setminus F$ is G_δ, $((G \cap F) \setminus P) \cup Q \subset F \cup Q$ is of first category. □

Theorem. *For any set A having the Baire property there is a unique regular open set G (i.e. $int(cl(G)) = G$) and a set P of first category such that $A = G \triangle P$.*

Proof. Write $A = H \triangle P$ where H is open and P is of first category. Then $G = \text{int}(\text{cl}(\text{int}(\text{cl}(H)))) \supset H$ is regular open, and $G \setminus H$ is nowhere dense (see homeworks). □

Steinhaus Theorem

Theorem (Steinhaus) *If A has positive measure, then its difference set $A - A = \{a - b : a, b \in A\}$ contains an open neighborhood of the origin.*

Theorem. *If A is of second category, then $A - A$ contains an open neighborhood of the origin.*

Proof.

- If A is of second category, write $A = G \triangle P$, G is non-empty open, P is of first category. Take a ball $B \subset G$. If $x \in \mathbb{R}^d$ has length small enough, then $B \cap (B + x)$ is a non-empty open set, so there is a $y \in (B \cap (B + x)) \setminus (P \cup (P + x))$. Then $x = y - (y - x)$, $y \in A$, $y - x \in A$.
- If A is of positive measure, there is a ball B centered at a density point of A such that $P = B \setminus A$ has measure less than $\lambda(B)/2$. If x is small enough then $(B \cap (B + x)) \setminus (P \cup (P + x))$ has positive measure, so it has a point y and $x = y - (y - x)$, $y \in A$, $y - x \in A$. □

Lusin Theorem

Definition.

- A function f is called measurable, if $f^{-1}(G)$ is measurable for every open set G.
- A function f is Baire, if $f^{-1}(G)$ has the Baire property for every open set G.

Theorem (Lusin) *A function f is measurable if and only if for each $\varepsilon > 0$ there is a set E with $\lambda(E) < \varepsilon$ such that the restriction of f to $\mathbb{R}^d \setminus E$ is continuous.*

Theorem. *A function f is has the Baire property if and only there is a set P of first category such that the restriction of f to $\mathbb{R}^d \setminus P$ is continuous.*

Remark. A measurable function need not to be continuous on the complement of a null set.

Proof of Luzin Theorem

1. Category:
   - Let $I_1, I_2 \ldots$ be an enumeration of rational open intervals. If f is Baire, write $f^{-1}(I_n) = G_n \triangle P_n$, $P = \bigcup P_n$. The restriction g of f to $\mathbb{R}^d \setminus P$ is continuous, since $g^{-1}(I_n) = G_n \setminus P$, and $G_n \setminus P$ is relatively open in $\mathbb{R}^d \setminus P$.
   - Conversely, if the restriction g of f to the complement of P is continuous, then for any open set G, $g^{-1}(G) = H \setminus P$, H is open, P is of first category, $H \setminus P \subset f^{-1}(G) \subset H \cup P$.
2. Measure:
   - If f is measurable, for each rational I_i there is a closed F_i and open G_i so that $F_i \subset f^{-1}(I_i) \subset G_i$, $\lambda(G_i \setminus F_i) < \varepsilon/2^i$. Let $E = \bigcup(G_i \setminus F_i)$. Then $\lambda(E) < \varepsilon$, and if g denotes the restriction of f to $\mathbb{R}^d \setminus E$, $g^{-1}(I_i) = f^{-1}(I_i) \setminus E = G_i \setminus E$.
   - Conversely, suppose that there are sets $E_1, E_2, \ldots,$ $\lambda(E_i) < 1/i$, the restriction f_i of f to $\mathbb{R}^d \setminus E_i$ is continuous. Then for any open set G, $f_i^{-1}(G) = H_i \setminus E_i$, H_i is open. Putting $E = \bigcap E_i$ we have $f^{-1}(G) \setminus E = \bigcup(f^{-1}(G) \setminus E_i) = \bigcup f_i^{-1}(G) = \bigcup(H_i \setminus E_i)$. Since all the sets $H_i \setminus E_i$ are measurable and E is null, $f^{-1}(G)$ is measurable.

Egoroff Theorem

Theorem (Egoroff) *If $f_1, f_2, \ldots$ are measurable functions and $f_n(x) \to f(x)$ at each point x of a set E of finite measure, then for each $\varepsilon > 0$ there is a set F with $\lambda(F) < \varepsilon$ so that $f_1, f_2, \ldots$ converges uniformly on $E \setminus F$.*

Proof. Let $E_{n,k} = \{x \in E : |f_i(x) - f(x)| \geq 1/k \text{ for some } i \geq n\}$. Then, for each fixed k, the sets $E_{1,k}, E_{2,k}, \ldots$ are decreasing and they have empty intersection. So if n_k is large enough then $\lambda(E_{n_k,k}) < \varepsilon/2^k$. We can take $F = \bigcup_k E_{n_k,k}$. □

Remark. The category analogue of Egoroff Theorem fails. There are functions $f_1, f_2, \cdots : \mathbb{R} \to \mathbb{R}$ so that $f_n(x) \to 0$ for each $x \in \mathbb{R}$, but any set on which f_n converges uniformly is nowhere dense.

Fubini Theorem

In what follows, we fix d_1, d_2, and understand 'measurable', 'null set', 'has Baire property', etc in the appropriate spaces.

Notation. For any $A \subset \mathbb{R}^{d_1}$ and $B \subset \mathbb{R}^{d_2}$, $A \times B = \{(x, y) : x \in A, y \in B\} \subset \mathbb{R}^{d_1+d_2}$. For any $E \subset \mathbb{R}^{d_1+d_2}$, the set $E_x = \{y : (x, y) \in E\} \subset \mathbb{R}^{d_2}$ is a vertical section of E. Horizontal sections are defined analogously.

Fubini Theorem

1. If E is measurable, then E_x is measurable for all x except a set of measure zero.
2. If E is a null set, then E_x is a null set for all x except a set of measure zero.
3. If E is measurable and it has positive measure, then E_x has positive measure for positively many x.

Moreover, $\lambda(E) = \int \lambda(E_x)\, dx$.

Proof. See any standard textbook on measure theory.

Kuratowski-Ulam Theorem

The category analogue of Fubini Theorem is the following:

Theorem. (Kuratowski-Ulam)

1. *If E has Baire property, then E_x has Baire property for all x except a set of first category.*
2. *If E is of first category, then E_x is of first category for all x except a set of first category.*
3. *If E is a Baire set of second category, then E_x is a Baire set of second category for a set of x's of second category.*

Lemma. *If E is nowhere dense, then E_x is nowhere dense for all x except a set of first category.*

Proof of Kuratowski-Ulam Theorem

Proof of Kuratowski-Ulam Theorem. It is clear that the lemma implies part 2. To show 1, let $E = G \triangle P$, G is open, P is of first category. Every section of an open set is open, hence E_x has Baire property whenever P_x is of first category. By 2, this is the case for all x except a set of first category. In 3, G is non-empty and using 2 again we can see that E_x is of second category for every x for which E_x meets G. □

Proof of Lemma. Since the closure of a nowhere dense set is nowhere dense, we can assume without loss of generality that E is closed. Let G be its complement, then G is open and dense.
For any rational open interval $I_n \in \mathbb{R}^{d_2}$, let G_n be the projection of the part of G that lies in the horizontal strip determined by I_n:

$$G_n = \{x : (x, y) \in G \text{ for some } y \in I_n\}.$$

Each G_n is dense and open, so the complement of $\bigcap G_n$ is of first category. If $x \in \bigcap G_n$ then G_x is open and dense, hence E_x is nowhere dense. □

Homeworks

1. Show that for any open set G, $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(G))))$ is regular open, and $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(G)))) \setminus G$ is nowhere dense.
2. Find functions $f_1, f_2, \cdots : \mathbb{R} \to \mathbb{R}$ so that $f_n(x) \to 0$ for each $x \in \mathbb{R}$, but any set on which f_n converges uniformly is nowhere dense.
3. Learn the proof of Fubini theorem.
4. For any two topological spaces X and Y, the product topology on $X \times Y$ is generated by the sets $G \times H$, where $G \subset X$, $H \subset Y$ are open. In other words, $A \subset X \times Y$ is open if and only if for any $(x, y) \in A$ there are neighbourhoods of x in X and y in Y whose product is in A.
    - Find Baire spaces X and Y for which Kuratowski-Ulam Theorem fails.
    - Show that Kuratowski-Ulam Theorem holds if X and Y are Polish spaces.

Cardinality Revisited

Definition. Suppose that P is a set and that $\leq$ is a relation on P. Then $\leq$ is a partial order if it is reflexive, antisymmetric, and transitive, i.e., for all $a, b, c \in P$ we have

- $a \leq a$ (reflexivity)
- if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)
- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)

A partial order is a (total) order if for all $a, b \in P$ we have

- $a \leq b$ or $b \leq a$ (totality)

If P is a partially ordered set, and if S is a (totally) ordered subset, S is called a chain.

In a partially ordered set, the concepts of a is the greatest element ($a \geq x$ for all $x \in P$) and a is maximal ($x \geq a$ implies $x = a$) are not the same.

The least element, minimal elements are defined analogously. We can also define lower bound, upper bound, greatest lower bound (infimum), least upper bound (supremum) of any subset of P.

Zorn Lemma

If it exists, the greatest element of P is unique. If there is no greatest element, there can be many maximal elements. Also, in infinite sets, maximal elements may not exists. An important tool to ensure the existence of maximal elements under certain conditions is Zorn Lemma:

Zorn Lemma. Every partially ordered set, in which every chain has an upper bound, contains at least one maximal element.

This is not a Proof. We are going to define elements $a_0 < a_1 < a_2 < \ldots$ in P. This sequence is really long: the indices are not just the natural numbers, but there are also 'infinite indeces', called ordinals.

We pick $a_0 \in P$ arbitrary. If some of the a's have been already defined and they form a totally ordered subset T, then, by the assumption of the Lemma, this T has an upper bound. If there is no maximal element, we can choose the next a to be larger than the upper bound of T. The infinite chain of a's we obtain has no maximal element, contradiction. □

Well-ordering

Definition. A well-ordering on a set P is a total order on P with the property that every non-empty subset of P has a least element.

Theorem. *In a well-ordered set every element, except a possible greatest element, has a unique successor. Every subset which has an upper bound has a least upper bound. There may be elements (besides the least element) which have no predecessor.*

Examples.

- The standard ordering of $\mathbb{N}$ is a well-ordering, but standard ordering of $\mathbb{Z}$ is not.
- The following ordering on $\mathbb{Z}$ is a well-ordering: $x \leq y$ if and only if one of the following conditions hold:
  - $x = 0$
  - x is positive and y is negative
  - x and y are both positive, and x is smaller than y
  - x and y are both negative, and y is smaller than x

  That is, $0 < 1 < 2 < \cdots < -1 < -2 < \ldots$

  -1 has no predecessor.

An Introduction to Ordinals

Ordinals describe the position of an element in a sequence. They may be used to label the elements of any given well-ordered set, the smallest element being labeled 0, the one after that 1, the next one 2, and so on. After all natural numbers comes the first infinite ordinal, ω, and after that come $\omega + 1$, $\omega + 2$, $\omega + 3, \ldots$, after all these come $\omega + \omega$,... (Exactly what addition means we will not define: we just consider them as names.)

Ordinals measure the "length" of the whole set by the least ordinal which is not a label for an element of the set. For instance, for $\{0, 1, 2, \ldots, 11\}$ the first label not used is 12, for $\mathbb{N}$ it is ω, for $\mathbb{Z}$ with the well-ordering defined above it is $\omega + \omega$.

Now we don't want to distinguish between two well-ordered sets if they have the same ordering, i.e. if there is an order-preserving bijection between them. An ordinal is defined as an equivalence class of well-ordered set.

Ordering the Ordinals

Definition. Let P and Q be well-ordered sets with ordinal numbers α and β. We say that $\alpha < \beta$, if A is order isomorphic to an initial segment of B, that is, there is a $b \in B$ and an order-preserving bijection between A and $\{x \in B : x < b\}$.
Similarly as in the "proof" of Zorn Lemma, it "can be shown" that this is a total order: any two ordinals are comparable. Moreover:

Fact. *The order defined above is a well-ordering of the ordinals.*

Definition. If α is an ordinal, its successor is denoted by $\alpha + 1$. Those ordinals that have a predecessor, i.e. they can be written in the form $\alpha + 1$, are called successor ordinals. An ordinal which is neither zero nor a successor ordinal is called a limit ordinal.

Transfinite Induction

Transfinite induction is an extension of mathematical induction to well-ordered sets. Suppose that $P(\alpha)$ is true whenever $P(\beta)$ is true for all $\beta < \alpha$. Then transfinite induction tells us that P is true for all ordinals.

Usually the proof is broken down into three cases:

- Zero case: Prove that P(0) is true.
- Successor case: Prove that for any successor ordinal $\alpha + 1$, $P(\alpha + 1)$ follows from $P(\alpha)$ (or, if necessary, follows from "$P(\beta)$ for all $\beta \leq \alpha$").
- Limit case: Prove that for any limit ordinal α, $P(\alpha)$ follows from "$P(\beta)$ for all $\beta < \alpha$."

Transfinite induction can be used not only to prove things, but also to define them: in order to define a_α for ordinals α, one can assume that it is already defined for all smaller β. We use transfinite induction and properties of a_β's to show that a_α can be defined. This method is called transfinite recursion.

Cardinality and the Continuum Hypothesis

Each ordinal has an associated cardinality, the cardinality of the well-ordered set representing the ordinal. The smallest ordinal having a given cardinality is called the initial ordinal of that cardinality.

The α^{th} infinite initial ordinal is denoted by ω_α. So $\omega_0 = \omega$ is the ordinal corresponding to the natural order on $\mathbb{N}$, ω_1 is the smallest uncountable ordinal, ω_2 is the smallest ordinal whose cardinality is greater than the cardinality of ω_1, and so on. The smallest ordinal that is larger than ω_n for each n is ω_ω, then comes $\omega_{\omega+1}$, etc.

"Any set can be well-ordered", so its cardinality can be written in the form ω_α for some ordinal α. Let c denote the cardinality of a set of continuum many points. How can we find α so that $c = \omega_\alpha$?

Continuum Hypothesis (CH) The continuum hypothesis says that there is no set whose size is strictly between that of the integers and that of the real numbers. That is, $c = \omega_1$.

Cohen proved that the continuum hypothesis is neither provable nor disprovable.

Borel Sets

Recall that sets belonging to the σ-algebra generated by open and null sets are called measurable sets, and the elements of the σ-algebra generated by open and first category sets are the sets with Baire property. We now consider the σ-algebra generated by open sets only. The sets belonging to this σ-algebra are called Borel sets. More generally:

Definition. A topological space is a set X together with a collection T of subsets of X satisfying the following:

- $\emptyset \in T$, $X \in T$
- The union of any collection of sets in T is also in T.
- The intersection of any finite collection of sets in T is also in T.

The elements of X are called points, the sets in T are the open sets, and their complements in X are the closed sets. Interior and closure are defined in the usual way.
Sets belonging to the σ-algebra generated by open sets are called Borel sets.

Borel Hierarchy

Borel sets are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections.
More precisely, let X be a topological space, and let P be any collection of subsets of X. We will use the notation:

- G = all open sets of X
- F = all closed sets of X
- P_σ = all countable unions of elements of P
- P_δ = all countable intersections of elements of P

So

- G_δ = countable intersections of open sets
- F_σ = countable unions of closed sets
- $G_{\delta\sigma}$ = countable unions of G_δ sets
- $F_{\sigma\delta}$ = countable intersections of F_σ sets
- $G_{\delta\sigma\delta}$ = countable intersections of $G_{\delta\sigma}$ sets
- $F_{\sigma\delta\sigma}$ = countable unions of $F_{\sigma\delta}$ sets
- etc

We may also need to define classes obtained in countable many, but more than ω steps.

Borel Hierarchy Continued

Denote

$P^0 = F$	$Q^0 = G$
$P^1 = G_\delta$	$Q^1 = F_\sigma$
$P^2 = F_{\sigma\delta}$	$Q^2 = G_{\delta\sigma}$
$\vdots$	$\vdots$
$P^\alpha = \left(\bigcup_{\beta<\alpha} Q^\beta\right)_\delta$	$Q^\alpha = \left(\bigcup_{\beta<\alpha} P^\beta\right)_\sigma$

for all $\alpha < \omega_1$. For each α, if $A \in P^\alpha$ then $X \setminus A \in Q^\alpha$ and vica versa. Also, $P^\beta \subset Q^\alpha$ and $Q^\beta \subset P^\alpha$ if $\beta < \alpha$. Any set belonging to any of these classes is a Borel set. The converse is also true:

Claim. $\mathcal{B} = \bigcup_{\alpha<\omega_1} P^\alpha = \bigcup_{\alpha<\omega_1} Q^\alpha =$ *Borel sets.*

Proof. We need to show that $\mathcal{B}$ is a σ-algebra. It is enough to show that if $A_1, A_2, \dots \in \mathcal{B}$ then $\bigcup A_n \in \mathcal{B}$, $\bigcap A_n \in \mathcal{B}$. Let $\alpha_1, \alpha_2, \dots$ such that $A_n \in P^{\alpha_n}$, and choose α that is larger than each α_n (see homeworks). Then $\bigcup A_n \in Q^\alpha$, $\bigcap A_n \in P^{\alpha+1}$. $\square$

Product of Topological Spaces

Question. We have seen that any Borel set can be obtained in ω_1 steps. But do we really need ω_1 steps? We will see that the answer is yes. In order to prove this we need some preparations.

Definition. If X and Y are topological spaces, then $X \times Y$ is defined to be the topological space whose points are pairs $\{(x, y) : x \in X, y \in Y\}$ and a set $A \subset X \times Y$ is open if and only if it is the union of sets of form $G \times H$, where $G \subset X$, $H \subset Y$ are open.

Definition. More generally, if X_i $(i \in I)$ are topological spaces, then the points of the product $\prod_{i \in I} X_i$ are the points of the Cartesian product of the sets X_i, and a set A is open if and only if it can be written as a union of cylinder sets. A cylinder set is a product of subsets $A_i \subset X_i$, finitely many of which are arbitrary open sets and all the others are the whole space X_i.

Fact. *A sequence of points $x_n \in \prod_{i \in I} X_i$ converges to $x \in \prod_{i \in I} X_i$ if and only if each coordinate of the sequence x_n converges to the appropriate coordinate of x.*

Basic Examples

- Product of Euclidean spaces $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_n} = \mathbb{R}^{d_1+d_2+\cdots+d_n}$.
- For any topological space X, we denote by $X^{\mathbb{N}}$ the countable product $X \times X \times X \times \ldots$.

Theorem. *Consider $\{0,1\}$ with the discrete topology (i.e. all four subsets are open). Then $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set.*

Proof. For any infinite sequence $\mathbf{i} = \{i_1, i_2, \ldots\} \in \{0,1\}^{\mathbb{N}}$, let $f(\mathbf{i}) = \sum \frac{2i_k}{3^k}$. This is a bijection between $\{0,1\}^{\mathbb{N}}$ and the Cantor set, and it maps the cylinder sets of $\{0,1\}^{\mathbb{N}}$ to the points of the intervals of length $1/3^k$, $k \in \mathbb{N}$ of the construction of the Cantor set. Therefore it is easy to check that the image and preimage of each open set is open, f is a homeomorphism. □

Theorem. *Consider $\mathbb{N}$ with the discrete topology (i.e. all subsets are open). Then $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.*

Proof that $\mathbb{N}^{\mathbb{N}}$ is Homeomorphic to the Irrationals

- Let $q_1, q_2 \dots$ be an enumeration of $\mathbb{Q}$. We define inductively for each finite sequence of natural numbers $\mathbf{n} = \{n_1, n_2, \dots, n_k\}$ an open interval $I_{\mathbf{n}}$ as follows.
  - Let $I_1, I_2, \dots$ be an enumeration of the components of $\mathbb{R} \setminus (\mathbb{Z} \cup \{q_1\})$.
  - If $I_{\mathbf{n}} = (a, b)$ has already been defined, choose $x_i \in (a, b) \cap \mathbb{Q}$ for each $i \in \mathbb{Z}$ such that $\lim_{i \to -\infty} x_i = a$, $\lim_{i \to +\infty} x_i = b$, $0 < x_{i+1} - x_i < 1/(k+1)$.
  - If $q_{k+1} \in I_{\mathbf{n}}$ then we also require $x_0 = q_{k+1}$.
  - Let $I_{\mathbf{n}1}$, $I_{\mathbf{n}2}$, ... be an enumeration of the components of $(a, b) \setminus \{x_i : i \in \mathbb{Z}\}$.
- Then $\mathrm{cl}(I_{n_1 n_2 \dots n_k}) \subset I_{n_1 n_2 \dots n_k n_{k+1}}$, and since the length of $I_{n_1 n_2 \dots n_k}$ is at most $1/k$, $I_{n_1} \cap I_{n_1 n_2} \cap I_{n_1 n_2 n_3} \cap \dots$ is a point for any $(n_1, n_2, \dots) \in \mathbb{N}^{\mathbb{N}}$. We denote it by $f(n_1, n_2, \dots)$. This defines a bijection between $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R} \setminus \mathbb{Q}$.
- The image of the cylinder sets are those points of $\mathbb{R} \setminus \mathbb{Q}$ that are in an interval $I_{\mathbf{n}}$, so the image and preimage of each open set is open, f is a homeomorphism.

Universal Sets

Definition. If $A \subset X \times Y$, then $A_x = \{y \in Y : (x, y) \in A\}$, $A^y = \{x \in X : (x, y) \in A\}$ are called the sections of A.

Definition. The set A is called a universal P^α set in $X \times Y$, if

- A is P^α (according to the product topology), and
- For any P^α set $B \subset X$ there is a y so that $A^y = B$.

Universal Q^α sets are defined analogously.

Example. Let X be a Polish space, and let $x_1, x_2, \dots$ be a countable dense set in X, and let $B_1, B_2, \dots$ be an enumeration of all balls with centre x_k and rational radius. Then

$$\Big\{(x, (n_1, n_2, \dots)) \in X \times \mathbb{N}^{\mathbb{N}} : x \in \bigcup_{k=1}^{\infty} B_{n_k}\Big\}$$

is a universal open set in $X \times \mathbb{N}^{\mathbb{N}}$. Its complement is universal closed.

Borel Hierarchy Continued

Theorem. *If X, Y are Polish spaces and Y is uncountable, then for each α there are universal P^α and Q^α sets in $X \times Y$.*

Corollary. *If X is an uncountable Polish space and $\alpha < \omega_1$, then the sets $P^\alpha \setminus Q^\alpha$, $Q^\alpha \setminus P^\alpha$, $P^\alpha \setminus \bigcup_{\beta<\alpha}(P^\beta \cup Q^\beta)$, $Q^\alpha \setminus \bigcup_{\beta<\alpha}(P^\beta \cup Q^\beta)$ are non-empty.*

Proof. Let A be a universal P^α set in $X \times X$. Let B be its intersection with the diagonal $B = \{x \in X : (x, x) \in A\}$. Then B is P^α. But $A \notin Q^\alpha$. Indeed, suppose that $A \in Q^\alpha$. Then $X \setminus A \in P^\alpha$, so there is a y so that $X \setminus A = B^y$. Both $y \in A$ and $y \notin A$ lead to a contradiction. So $A \in P^\alpha \setminus Q^\alpha$, and then $X \setminus A \in Q^\alpha \setminus P^\alpha$. This also shows that $P^\alpha \setminus \bigcup_{\beta<\alpha}(P^\beta \cup Q^\beta)$ and $Q^\alpha \setminus \bigcup_{\beta<\alpha}(P^\beta \cup Q^\beta)$ are non-empty, since $P^\alpha = \bigcup_{\beta<\alpha}(P^\beta \cup Q^\beta)$ would imply $Q^\alpha = \bigcup_{\beta<\alpha}(P^\beta \cup Q^\beta)$. □

Universal P^α in $X \times \mathbb{N}^\mathbb{N}$

- We prove by transfinite induction. We have already seen that there are universal open and closed sets, so the statement is true for $\alpha = 0$. Let $B_1, B_2, \ldots$ be balls as in that proof.
- There are countably many ordinals less than α. Let $\beta_1, \beta_2, \ldots$ be an enumeration of all ordinals less than α that contains each $\beta < \alpha$ infinitely many times, and let A_n be a universal P^{β_n} set in $X \times \mathbb{N}^\mathbb{N}$.
- Choose continuous functions $\phi_n : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ so that for any sequence $v_1, v_2, \dots \in \mathbb{N}^\mathbb{N}$ there is a $v \in \mathbb{N}^\mathbb{N}$ so that $\phi_n(v) = v_n$ for each n (see homeworks). Let $B_n = \{(x, v) \in X \times \mathbb{N}^\mathbb{N} : (x, \phi_n(v)) \in A_n\}$. Then B_n is P^{β_n}, since B_n is the preimage of A_n under the continuous mapping $(x, v) \to (x, \phi_n(v))$. Therefore $B = \bigcup_{n=1}^{\infty} B_n$ is Q^α. We show that B is universal Q^α (and then its complement is universal P^α).

Universal P^α in $X \times \mathbb{N}^\mathbb{N}$ continued

- Let C be an arbitrary Q^α set in X. Then C can be written as $\bigcup C_n$, where C_n is P^{β_n} in X (we use here that $\beta_1, \beta_2, \dots$ contains every $\beta < \alpha$ infinitely many times). Since A_n is universal P^{β_n}, there is $v_n \in \mathbb{N}^\mathbb{N}$ so that $C_n = A_n^{v_n}$.
- Choose v for which $\phi_n(v) = v_n$. Then $B^v = \bigcup_n B_n^v = \bigcup_n \{x : (x, \phi_n(v)) \in A_n\} = \bigcup_n \{x : (x, v_n) \in A_n\} = \bigcup_n A_n^{v_n} = \bigcup_n C_n = C$. □

Lemma. *Every uncountable Polish space has a subset homeomorphic to $\mathbb{N}^\mathbb{N}$.*

Proof. See homeworks.

Proof of the existence of universal P^α, Q^α sets in $X \times Y$:
Let $Z \subset X$ be homeomorphic to $\mathbb{N}^\mathbb{N}$, and let A be a universal P^α set in $X \times Z$. Let B be a P^α set in $X \times Y$ for which $A = (X \times Y) \cap B$. Then B is a universal P^α set in $X \times Y$. □

Another Application of Universal Sets

Universal sets can often be used to construct sets with unexpected properties. An example is the following:

Theorem. *There is a set $A \subset \mathbb{R}^2$ such that:*

- *A contains exactly one point on each horizontal line.*
- *Every open cover $G \supset A$ contains a horizontal line.*

Proof.

- Let H be a universal closed set in $\mathbb{R}^2 \times \mathbb{R}$. For each y for which there is an x with $(x, y, y) \in H$, choose such an $x = x_y$. Otherwise choose x_y arbitrarily. Let $A = \{(x, y) : x = x_y\}$.
- Let $G \supset A$ be open. Our aim is to show that G contains a horizontal line, i.e. $F = \mathbb{R}^2 \setminus G$ does not meet every horizontal line. Since F is closed, there is a z such that $H^z = F$.
- F cannot meet the horizontal line $\{(x, y) : y = z\}$, since otherwise $(x, z) \in F$, $(x, z, z) \in H$, $(x_z, z, z) \in H$, $(x_z, z) \in A$, $(x_z, z) \notin F$, $(x_z, z, z) \notin H$, contradiction. □

Remarks

In the last construction, the points x_y can be chosen in such a way that A is Borel (see homeworks). Arguments using universal sets usually lead to constructions of Borel sets. Constructing non-Borel sets are often much easier, and analysts regard them to be cheating.

Question. It is an open problem whether there is a Borel set A that meets every 'almost horizontal' curve in a set of linear measure zero, but every open set G covering A meets an 'almost horizontal' curve in a large set: the measure of its projection to the x axis is at least 1.

Let $\varepsilon > 0$ be fixed. We say that a curve is almost horizontal, if its chords have angle at most ε, or, equivalently, they are graphs of a Lipschitz function with Lipschitz constant at most ε. It is an easy exercise to show that A cannot be F_σ. It is already an open problem whether A can be G_δ.
The assumption that A is Borel is important. We will see later that there are non-Borel counterexamples.

Homeworks

1. Show that for any sequence of countable ordinals $\alpha_1, \alpha_2, \ldots$ there is a countable ordinal α that is larger than each α_n.
2. Let $X_1, X_2, \ldots$ be topological spaces of finitely many points, each of them is equipped with the discrete topology (all subsets are open). Show that $\prod X_i$ is homeomorphic to the Cantor set.
3. Prove that every uncountable Polish space has a subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Hint:
   - Show that every uncountable Polish space has a subset homeomorphic to the Cantor set.
   - Show that the Cantor set has a subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$.
4. Find continuous functions $\phi_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ so that for any sequence $v_1, v_2, \cdots \in \mathbb{N}^{\mathbb{N}}$ there is a $v \in \mathbb{N}^{\mathbb{N}}$ so that $\phi_n(v) = v_n$ for each n.
5. Show that there is a Borel set $A \subset \mathbb{R}^2$ that contains exactly one point on each horizontal line, and such that every open cover $G \supset A$ contains a horizontal line.

Cheating Constructions

Transfinite induction is often used in analysis to construct pathological examples of sets and functions, (or, assuming continuum hypothesis, to show that certain statements cannot be proved/disproved). Some illustrative examples are the following:

Example 1: Previous construction revisited. Let y_α $(\alpha < c)$ be a well-ordering of $\mathbb{R}$ and let G_α $(\alpha < c)$ be an enumeration of those open subsets of $\mathbb{R}^2$ that do not contain any horizontal line. For each α, choose a point $(x_\alpha, y_\alpha) \notin G_\alpha$. The set $A = \{(x_\alpha, y_\alpha) : \alpha < c\}$ contains one point on each horizontal line, and $A \not\subset G_\alpha$ for any α.

Example 2: Positive sets without collinear points. There is a planar set of full (outer) measure that has no three collinear points. Indeed, let F_α $(\alpha < c)$ be an enumeration of closed sets of positive measure. For each α choose a point $x_\alpha \in F_\alpha$ that is not on the line connecting x_β, x_γ for any $\beta, \gamma < \alpha$ (cf. Homework 1). Then the complement of $A = \{x_\alpha : \alpha < c\}$ does not contain any closed set of positive measure.

Bernstein Sets

Theorem (Bernstein) *There is a set $A \subset \mathbb{R}$ such that both A and its complement intersect each closed subset of $\mathbb{R}$ of continuum many points. Such sets are called Bernstein sets.*

Proof.

- There are continuum many uncountable closed subsets of $\mathbb{R}$. Let F_α $(\alpha < c)$ be a well ordering of such sets. They all have continuum many points, since they contain a copy of $\mathbb{N}^\mathbb{N}$.
- Choose two points $x_0, y_0 \in F_0$ arbitrarily. In the α^{th} step choose $x_\alpha, y_\alpha \in F_\alpha$ that are different from all the points x_β, y_β chosen before. This can be done, since F_α has continuum many points (i.e. more than the what we have chosen before).
- Let $A = \{x_\alpha : \alpha < c\}$. Since $x_\alpha \in A$, it meets all the sets F_α. Since $y_\alpha \notin A$, its complement meets each F_α. □

Properties of Bernstein Sets

Theorem. *Bernstein sets are non-measurable.*

Proof. If A is measurable, then either A or its complement has positive measure, and so contains a closed set F of positive measure. Every set of positive measure is uncountable. □

Theorem. *Bernstein sets do not have Baire property.*

Proof. Suppose that A has Baire property. Either A or its complement is of second category. If A is of second category, then $A = G \triangle P$, where G is a non-empty open and P is a first category set.

Write $P = \bigcup P_n$ where P_n is nowhere dense, and choose two disjoint closed intervals $I_0, I_1 \subset G$ disjoint from P_1. Inductively, if the intervals $I_{\mathbf{i}}$ have been defined for each sequence $\mathbf{i}$ of 0's and 1's of length k, choose two disjoint closed intervals $I_{\mathbf{i}0}, I_{\mathbf{i}1} \subset I_{\mathbf{i}}$ disjoint from P_{k+1}. Then $C = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{i} \in \{0,1\}^k} I_{\mathbf{i}}$ is a closed subset of $G \setminus P \subset A$ and has continuum many points. □

A Construction Using (CH): Sierpinski Theorem

Theorem (Sierpinski)

- *Assuming (CH), there is a set $A \subset \mathbb{R}^2$ that has countably many points on each vertical line and misses only countably many points on each horizontal line.*
- *The converse is also true. If there is a set $A \subset \mathbb{R}^2$ that has countably many points on each vertical line and misses only countably many points on each horizontal line, then (CH) holds.*

Corollary. *The existence of a set A described above is neither provable nor disprovable.*

Proof of (CH)$\Longrightarrow$ existence of A.
Let x_α $(\alpha < c)$ be a well-ordering of $\mathbb{R}$, and let $A = \{(x_\alpha, x_\beta) : \beta < \alpha\}$. Because of (CH), for each α there are only countably many β with $\beta < \alpha$. □

Proof of Existence of $A \implies$ (CH)

Let x_α $(\alpha < c)$ be a well-ordering of $\mathbb{R}$. We define another well ordering as follows:

1. For any given $y \in \mathbb{R}$, let $\alpha = \alpha(y)$ be the least index for which $(x_\alpha, y) \in A$.
2. For any given α, there are only countably many points on the vertical line $x = x_\alpha$. We order these points into a sequence.
3. For any $u, v \in \mathbb{R}$ we define $u < v$ if
   - either $\alpha(u) < \alpha(v)$, or
   - $\alpha(u) = \alpha(v) = \alpha$ and $(x_\alpha, u) < (x_\alpha, v)$ according to the ordering defined in Step 2.

This is a well ordering. Since each horizontal line misses only countably many points, $\alpha(y) < \omega_1$ for each α, and our ordered sequence has length ω_1.

Sierpinski-Erdős Duality Theorem

Theorem (Sierpinski) *Assuming (CH), there exists a one-to-one mapping* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(E)$ *has measure zero if and only if* E *is of first category.*

Theorem (Erdős) *Assuming (CH), there exists a one-to-one mapping* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f = f^{-1}$ *and* $f(E)$ *has measure zero if and only if* E *is of first category.*

Corollary (Duality Principle) *Let* P *be any proposition involving solely the notions of nullset, first category, and notions of set theory (cardinality, disjointness, etc) that are invariant under one-to-one transformations. Let* P^* *be the proposition obtained by replacing "nullset" by "first category set". Then each of the propositions* P *and* P^* *implies the other, assuming (CH).*

Proof of Sierpinski-Erdős Duality Theorem

- Decompose $\mathbb{R}$ into the disjoint union of two sets A and B, where A is of first category and B is a null set.
- We choose X_α $(\alpha < c)$ Lebesgue null subsets of A, and Y_α $(\alpha < c)$ first category subsets of B, such that
  - $X_\beta \subset X_\alpha$, $Y_\beta \subset Y_\alpha$ for any $\beta < \alpha$
  - $X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta$, $Y_\alpha \setminus \bigcup_{\beta<\alpha} Y_\beta$ are uncountable
  - each G_δ Lebesgue null subset of A is contained in X_α for large enough α, and each F_σ first category subset of B is contained in Y_α for large enough α
- Let f be a bijection that maps X_0 onto Y_0 and maps $X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta$ onto $Y_\alpha \setminus \bigcup_{\beta<\alpha} Y_\beta$ for each α. This defines $f : A \to B$. We define f on B to be the inverse of $f : A \to B$.
- Let C be an arbitrary Lebesgue null set. Then $f(C) = f(A \cap C) \cup f(B \cap C) \subset f(A \cap C) \cup A$. Since every Lebesgue null set is contained in a G_δ Lebesgue null set, there is an α such that $A \cap C \subset X_\alpha$. Hence $f(C) \subset f(X_\alpha) \cup A = Y_\alpha \cup A$ is of first category. Similarly, if D is of first category then $f(D)$ is Lebesgue null.

Homeworks

1. Show that less than continuum many lines cannot cover any closed planar set of positive measure. (Hint: show that any closed set of positive measure contains a copy of the Cantor set without 3 collinear points)
2. Use transfinite induction and (CH) to construct a (non-Borel) set $A \subset \mathbb{R}^2$ that meets every 'almost horizontal' curve in a set of linear measure zero, but every open set G covering A meets an 'almost horizontal' curve in a set of full linear measure. (Hint: well-order all curves and all open sets).
3. Find interesting applications of the Duality Principle in Chapter 20 of the book 'Oxtoby: Measure and Category'.

Determinacy of Borel Games

We consider two players, Player (I) and Player (II), with Player (I) going first. They play "forever", that is, their plays are indexed by the natural numbers. When they are finished, a predetermined condition decides which player won.

Consider a subset $A \subset \mathbb{N}^{\mathbb{N}}$. In the game G_A, (I) plays a natural number n_0, then (II) plays n_1, then (I) plays n_2, and so on. Then (I) wins the game if and only if $(n_0, n_1, n_2, \dots) \in A$, otherwise (II) wins.

A strategy for a player is a way of playing in which his moves are entirely determined by the foregoing moves: a strategy for Player (I) is a function that accepts as an argument any finite sequence of natural numbers of even length, and returns a natural number. If σ is such a strategy and $(n_0, \dots, n_{2k-1})$ is a sequence of natural numbers, then $n_{2k} = \sigma(n_0, \dots, n_{2k-1})$ is the next move (I) will make, if he is following the strategy σ. Strategies for (II) are just the same, substituting "odd" for "even".

Determinacy

A strategy is winning, if the player following it must necessarily win, no matter what his opponent plays. If σ is a strategy for (I), then σ is a winning strategy for (I) in the game G_A if, for any sequence of natural numbers $(n_1, n_3, n_5 \dots)$, the sequence of plays produced by σ

$$n_0 = \sigma(\emptyset),\ n_1,\ n_2 = \sigma(n_0, n_1),\ n_3,\ n_4 = \sigma(n_0, n_1, n_2, n_3),\ \dots$$

is an element of A.

A game is determined if there is a winning strategy for one of the players.

Note that there cannot be a winning strategy for both players for the same game, for if there were, the two strategies could be played against each other. The resulting outcome would then, by hypothesis, be a win for both players, which is impossible. But it may happen that none of the players have a winning strategy, i.e. the game is not determined.

Rules of the Game

The "rules" of the game are also encoded in the set A: if one of the players chooses a natural number that is not allowed by the rules of the particular game, he loses, i.e. no matter how the players continue playing, the sequence obtained will belong to the complement of A.

However, sometimes it is more convenient to separate the "rules" from A. Let T be a set of finite sequences, such that every initial segment (including $\emptyset$) of an element of T is also in T, and such that every element of T is a proper initial segment of an element in T. Such a set is called a tree. Let $\mathcal{F}(T)$ denote the collection of all infinite sequences $(n_1, n_2, \dots)$ all of whose finite initial segments belong to T. For each $A \subset \mathcal{F}(T)$ we define the game $G(A, T)$: Player (I) picks $(n_0) \in T$, (II) picks n_1 with $(n_0, n_1) \in T$, (I) picks n_2 with $(n_0, n_1, n_2) \in T$, etc. A strategy for (I) is a function σ whose domain is the set of all elements of T of even length such that always $(n_0, \dots, n_{2k-1}, \sigma(n_0, \dots, n_{2k-1})) \in T$. A strategy for (II) is similarly defined. The game $G(A, T)$ is determined if one of the players have a winning strategy.

Finite Games are Determined

Familiar games, such as chess or tic-tac-toe, are always finished in a finite number of moves. If such a game is modified so that a particular player wins under any condition where the game would have been called a draw, then it is always determined.

The proof that such games are determined is rather simple: Player (I) simply plays not to lose; that is, he plays to make sure that Player (II) does not have a winning strategy after (I)'s move. If Player (I) cannot do this, then it means Player (II) had a winning strategy from the beginning. On the other hand, if Player (I) can play in this way, then he must win, because the game will be over after some finite number of moves, and he can't have lost at that point.

This proof does not actually require that the game always be over in a finite number of moves, only that it be over in a finite number of moves whenever (II) wins. This condition, topologically, is that the set A is closed (see the definition next slide).

Topology of $\mathcal{F}(T)$

We give $\mathcal{F}(T)$ the topology inherited from the product topology of $\mathbb{N}^{\mathbb{N}}$: a subset of $\mathcal{F}(T)$ is open if it is a union of cylinder sets, i.e. sets of form $\{x : p \text{ is an initial segment of } x\}$ with $p \in T$.

More generally:
Let T be an arbitrary tree, i.e. a set of sequences of finite length (not necessary of natural numbers!) s.t.

- every initial segment of an element of T is also in T
- every element of T is a proper initial segment of an element in T.

Let $\mathcal{F}(T)$ denote the collection of all infinite sequences all of whose finite initial segments belong to T. The cylinder sets of $\mathcal{F}(T)$ are sets of form $\{x : p \text{ is an initial segment of } x\}$ with $p \in T$. A subset of $\mathcal{F}(T)$ is open if it is a union of cylinder sets. A subset $A \subset \mathcal{F}(T)$ is Borel if it is in the σ-algebra generated by the open subsets of $\mathcal{F}(T)$.

We say that the game $G(A, T)$ is closed if A is closed, $G(A, T)$ is open if A is open, etc.

Borel Games are Determined

Recall that if the Banach-Mazur game for a set A is determined, then either (II) has a winning strategy, in which case A is of first category, or (I) has a winning strategy, in which case the complement of A is of first category inside some interval I.
As a corollary we can see that if A is a Bernstein set, i.e. both A and its complement meet each closed subset of $\mathbb{R}$ in continuum many points, then the Banach-Mazur game is not determined.
Bernstein sets are never Borel sets. Our main theorem is:

Theorem (Martin) *All Borel games are determined.*

Remark. If T consists of finite sequences of natural numbers, i.e. $\mathcal{F}(T) \subset \mathbb{N}^{\mathbb{N}}$, then the classes "Borel games $G(A, T)$" and "games G_A where A is Borel" coincide. Indeed, for every such T, $\mathcal{F}(T)$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$. Therefore $A \subset \mathcal{F}(T) \subset \mathbb{N}^{\mathbb{N}}$ is a Borel subset of $\mathbb{N}^{\mathbb{N}}$ if and only if it is a Borel subset of $\mathcal{F}(T)$.

Proof

The idea of the proof of Martin's Theorem is to associate to the game $G(A, T)$ an auxiliary game $G(\tilde{A}, \tilde{T})$, which is known to be determined, in such a way that a winning strategy for any of the players in $G(\tilde{A}, \tilde{T})$ gives a winning strategy for the corresponding player in $G(A, T)$. In the game $G(\tilde{A}, \tilde{T})$ the players play essentially a run of the game $G(A, T)$, but furthermore they choose in each turn some additional objects, whose role is to ensure that the payoff set becomes simpler.

First we will show that closed and open games are determined. Then we show how to find an auxiliary game for a closed or open game. Then, using transfinite induction, we will show how to find an auxiliary game if $A = \bigcup A_i \in Q^\alpha$, provided that we have already defined the auxiliary game for each A_i, where $A_i \in P^{\beta_i}$, $\beta_i < \alpha$, and we will show how to find an auxiliary game for the complement of A (which is in P^α).

Step 1: Closed Games

Theorem (Gale-Stewart) *All closed games are determined.*

Note that by symmetry, all open games are determined as well.

Proof. Suppose that (II) does not have a winning strategy. Then let (I) play according to the "play not to lose" strategy: if (II) does not have a winning strategy after some steps, (I) can always move in such a way that (II) will not have a winning strategy after (I)'s move.
Suppose that this is not a winning strategy, (I) loses. Since the complement of A is open, the sequence they obtain belongs to a cylinder set that is disjoint from A. This is a contradiction, since cylinder sets are determined by some finite number of coordinates, so (I) already lost the game after finitely many steps. □

Notation. If (I) has a winning strategy, then those positions from which he can win is a subtree of T. We denote this subtree by T_A. We call it the winning subtree.

Step 2: Auxiliary Games of Closed Games $G(A, T)$

Fix an even natural number k. Define $\tilde{T} = \tilde{T}(T, A, k)$ as follows:

- Sequences of length at most k in $\tilde{T}$ are the same as in T. If $(a_0, a_1, \ldots, a_{k-1})$ have been already chosen, then in the kth step, (I) chooses a subtree $T_I \subset T$ and an a_k, such that:
  - T_I is a (I)-imposed subtree, i.e. if $(b_0, \ldots, b_j) \in T_I$, j is even, and $(b_0, \ldots, b_j, b_{j+1}) \in T$, then $(b_0, \ldots, b_j, b_{j+1}) \in T_I$
  - $(a_0, a_1, \ldots, a_{k-1}, a_k) \in T_I$.
- In the next step, (II) chooses a subtree $T_{II} \subset T_I$ and an a_{k+1}. For choosing T_{II} he has two options:
  - winning option: T_{II} can be the set of all initial segments and continuations of a sequence $p \in T_I$, such that $(a_0, \ldots, a_k)$ is an initial segment of p, and the cylinder set determined by p is in the complement of A.
  - losing option: if (II) has a strategy to ensure that the final sequence will be in A, he can choose its "winning subtree" $T_{II} = (T_I)_A$. Then of course $\mathcal{F}(T_{II}) \subset A$.

  Player (II) chooses a_{k+1} so that $(a_0, a_1, \ldots, a_k, a_{k+1}) \in T_{II}$, and they continue choosing $a_{k+2}, a_{k+3}, \ldots$ so that $(a_0, a_1, \ldots, a_k, a_{k+1}, \ldots, a_j) \in T_{II}$ for all j.

The set $\tilde{A}$

After countably many steps, the players define a sequence

$$(a_0, a_1, \dots, a_{k-1}, (T_I, a_k), (T_{II}, a_{k+1}), a_{k+2}, a_{k+3}, \dots)$$

whose initial segments are in $\tilde{T}$, i.e. the sequence is in $\mathcal{F}(\tilde{T})$. There is a natural projection $\pi : \mathcal{F}(\tilde{T}) \to \mathcal{F}(T)$ that maps the sequence above to $(a_0, a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots) \in \mathcal{F}(T)$. For any $B \subset \mathcal{F}(T)$ we define $\tilde{B} = \pi^{-1}(B)$, i.e. the sequence above belongs to $\tilde{B}$ if and only if $(a_0, a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots) \in B$.

A set is called clopen, if it is both closed and open.

Claim. *$\tilde{A} \subset \mathcal{F}(\tilde{T})$ is clopen.*

Proof. If in the $(k+1)$th step (II) chooses the winning option then he wins $G(\tilde{A}, \tilde{T})$ and if he chooses the losing option then he loses $G(\tilde{A}, \tilde{T})$. So both $\tilde{A}$ and its complement can be written as a union of cylinder sets that are determined by sequences obtained in the $(k+1)th$ step. □

Transfer of Strategies of Player (I)

Suppose that σ is a strategy of Player (I) in $\tilde{T}$ (not necessarily a winning strategy). We define a strategy σ_0 in T, as follows:

- (I) starts following the same strategy σ as in $\tilde{T}$. In the kth step he chooses a_k if σ tells him to choose (T_I, a_k). Then (II) chooses an a_{k+1}, which is automatically in T_I since T_I is (I)-imposed.
- Of course, in this game Player (II) does not choose any T_{II}. Nevertheless, Player (I) "assumes" that Player (II) did choose T_{II}, moreover, he assumes that Player (II) chose the losing option $T_{II} = (T_I)_A$. He keeps assuming this and proceeds according to his strategy σ until (if ever) he finds out that he was wrong, i.e. $(T_I)_A$ does not exist, or they arrive at a finite sequence p that does not belong to $(T_I)_A$.

Transfer of Strategies of Player (I) Continued

- When this happens, Player (I) plays a winning strategy for $G(\mathcal{F}(T_I) \setminus A, T_I)$, reaching (since A is closed) a sequence p that determines a cylinder set that belongs to the complement of A. Now (I) assumes that (II) took the winning option in the $(k+1)$th step with this p, and proceeds with σ accordingly.

Claim. *For any sequence $(a_0, a_1, \ldots, a_k, a_{k+1}, \ldots) \in \mathcal{F}(T)$ that is consistent with the strategy σ_0, there are T_I, T_{II} so that $(a_0, a_1, \ldots, (T_I, a_k), (T_{II}, a_{k+1}), \ldots) \in \mathcal{F}(\tilde{T})$ is consistent with σ.*

Corollary. *For any $B \subset \mathcal{F}(T)$, if σ is a winning strategy in $G(\tilde{B}, \tilde{T})$, then σ_0 is a winning strategy in $G(B, T)$.*

Transfer of Strategies of Player (II)

Suppose now that σ is a strategy of Player (II) in $\tilde{T}$. We define a strategy σ_0 in T, as follows:

- Player (II) starts following the same strategy σ as in $\tilde{T}$. In the kth step (I) chooses a_k, but he does not choose any T_I. Player (II) considers all possible choices of T_I for which his strategy σ would have told him to reply with the winning option. For each such T_I there is a finite sequence p so that (II) would choose T_{II} to be the initial segments and continuations of p. Let P be the collection of all these finite sequences p, and let $B \subset \mathcal{F}(T)$ be the set of all infinite sequences that do not have any initial segment belonging to P. Then B is closed.
- Now (II) assumes that (I) chose $T_I = T_B$, i.e. the winning subtree for B. He follows his strategy σ accordingly, until (if ever) he finds out that his assumption was wrong, i.e. either T_B does not exists, or $(a_0, a_1, \ldots, a_k) \notin T_B$, or they arrive at a finite sequence that is not in T_{II}.

Transfer of Strategies of Player (II) Continued

- Note that if T_B exists and $(a_0, a_1, \ldots, a_k) \in T_B$ then, because of the definition of B, (II) has to choose the losing option, i.e. $T_{II} = (T_B)_A$. This is a (II)-imposed subtree of T_B. When (II) finds out that he was wrong, he has a strategy to ensure that the final sequence will not be in B. Then he follows this strategy, until a sequence $p \in P$ is reached. Then he assumes that in the kth step (I) chose a T_I so that his strategy σ called for a winning T_{II} defined by this p, and then follows σ accordingly.

Then, similarly as for strategies of (I):

Claim. *For any sequence $(a_0, a_1, \ldots, a_k, a_{k+1}, \ldots) \in \mathcal{F}(T)$ that is consistent with the strategy σ_0, there are T_I, T_{II} so that $(a_0, a_1, \ldots, (T_I, a_k), (T_{II}, a_{k+1}), \ldots) \in \mathcal{F}(\tilde{T})$ is consistent with σ.*

Corollary. *For any $B \subset \mathcal{F}(T)$, if σ is a winning strategy in $G(\tilde{B}, \tilde{T})$, then σ_0 is a winning strategy in $G(B, T)$.*

Step 3: Induction

Let T be an arbitrary tree, let A be a Borel subset of $\mathcal{F}(T)$, and let k be an even integer. We will construct a tree $\tilde{T} = \tilde{T}(T, A, k)$ and a projection $\pi : \tilde{T} \to T$, so that

- For any finite sequence $x \in \tilde{T}$, $\pi(x)$ is a finite sequence in T of the same length. The sequences of length at most k are the same in T and $\tilde{T}$, and π on these sequences is the identity.
- If x is an initial segment of y then $\pi(x)$ is an initial segment of $\pi(y)$.

Then π can be extended to the infinite sequences $\mathcal{F}(\tilde{T}) \to \mathcal{F}(T)$. For any $B \subset \mathcal{F}(T)$, we denote $\tilde{B} = \pi^{-1}(B)$. We will define $\tilde{T}$ s.t.

- $\tilde{A}$ is clopen in $\mathcal{F}(\tilde{T})$.

Furthermore, we will find for each strategy σ (of either player) in $\tilde{T}$ a strategy σ_0 of the same player in T, so that

- σ_0 restricted to sequences of length at most n depends only on σ restricted to sequences of length at most n.
- If $x \in \mathcal{F}(T)$ is a play consistent with σ_0, then there is a $y \in \mathcal{F}(\tilde{T})$ so that $\pi(y) = x$, and y is consistent with σ.

Induction Continued

As before, we can see that for any $B \subset \mathcal{F}(T)$, if σ is a winning strategy in $G(\tilde{B}, \tilde{T})$ then σ_0 is a winning strategy in $G(B, T)$. In particular, if $G(\tilde{B}, \tilde{T})$ is determined then $G(B, T)$ is determined.

Since $\tilde{A} \subset \mathcal{F}(\tilde{T})$ is closed, it follows from Step 1 that $G(\tilde{A}, \tilde{T})$ is determined. So if we indeed can find for any T and for any Borel set $A \subset \mathcal{F}(T)$ (and for some k) the tree $\tilde{T}$ and the mapping $\sigma \to \sigma_0$ as described on the previous slide, then we proved that $G(A, T)$ is determined.

- We have already seen how to find $\tilde{T}$ and $\sigma \to \sigma_0$ if A is closed.
- If we can find $\tilde{T}$ and $\sigma \to \sigma_0$ for some Borel set A, then we can choose the same $\tilde{T}$ and $\sigma \to \sigma_0$ for the complement of A; indeed, the only assumption on A was that $\tilde{A} \subset \mathcal{F}(\tilde{T})$ must be clopen. If $\tilde{A}$ is clopen then so is $\tilde{A}^c$.
- Therefore it is enough to construct $\tilde{T}$ and $\sigma \to \sigma_0$ for $A \in Q^\alpha$, and we can assume that we already know how to construct these objects for any set $B \in \bigcup_{\beta<\alpha}(P^\beta \cup Q^\beta)$.

Construction of a tree $\hat{T}$

Let $A = \bigcup A_i$, where $A_i \in P^{\beta_i}$, $\beta_i < \alpha$.

- Let $T_1 = \tilde{T}(T, A_1, k)$, and find for each strategy σ_1 in T_1 a strategy σ_0 in T satisfying the induction hypothesis. Let π_1 denote the projection $T_1 \to T$.
- Since π_1 is continuous, $\pi_1^{-1}(A_2)$ is a P^{β_2} subset of $\mathcal{F}(T_1)$. Let $T_2 = \tilde{T}(T_1, \pi_1^{-1}(A_2), k+2)$, and find for each strategy σ_2 in T_2 a strategy σ_1 in T_1 satisfying the induction hypothesis. Let π_2 denote the projection $T_2 \to T_1$.
- Etc. Let $T_n = \tilde{T}(T_{n-1}, \pi_{n-1}^{-1} \circ \pi_{n-2}^{-1} \circ \cdots \circ \pi_1^{-1}(A_n), k + 2n - 2)$, and find for each strategy σ_n in T_n a strategy σ_{n-1} in T_{n-1} satisfying the induction hypothesis. Let π_n denote the projection $T_n \to T_{n-1}$.

Recall that sequences of length at most k are the same in T and T_1, sequences of length at most $k+2$ are the same in T_1 and T_2, etc. Let $\hat{T}$ be the set of all sequences of length at most $k+2n$ in T_n, for all n.

Construction of $\tilde{T}$

It is clear that $\hat{T}$ is a tree, and there is a natural projection $\hat{\pi} : \hat{T} \to T$, defined by $\hat{\pi}(x) = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n(x)$ if $x \in \hat{T}$ has length at most $k + 2n$.

Denote $\hat{B} = \hat{\pi}^{-1}(B) \subset \mathcal{F}(\hat{T})$ for any $B \subset \mathcal{F}(T)$.

We know that $\pi_i^{-1} \circ \pi_{i-1}^{-1} \circ \cdots \circ \pi_1^{-1}(A_i)$ is clopen in $\mathcal{F}(T_i)$. Since the projection $\hat{T} \to T_i$ is continuous, $\hat{A}_i$ is clopen in $\mathcal{F}(\hat{T})$. Hence $\hat{A} = \bigcup \hat{A}_i$ is open in $\hat{T}$. It is not necessarily true that $\hat{A}$ is closed in $\hat{T}$. But since it is open, we can apply the induction hypothesis once more and choose $\tilde{T} = \tilde{T}(\hat{T}, \hat{A}, k)$. Then $\tilde{A} \subset \mathcal{F}(\tilde{T})$ is clopen, and all requirements are satisfied.

We still have to show that for any strategy σ in $\hat{T}$ there is a strategy τ in T of the same player so that

- τ restricted to sequences of length at most n depends only on σ restricted to sequences of length at most n.
- If $x \in \mathcal{F}(T)$ is a play consistent with τ, then there is a $y \in \mathcal{F}(\hat{T})$ so that $\hat{\pi}(y) = x$, and y is consistent with σ.

Construction of the Strategy τ

Let σ be a strategy of Player (I) in $\hat{T}$. We denote $T_0 = T$, and define a strategy τ_n on T_n for each $n \geq 0$.

- Sequences of length at most $k + 2n$ are the same in T_n and $\hat{T}$. On these sequences we define $\tau_n = \sigma$.
- Sequences $(a_0, a_1, \ldots, a_j)$, $j = k + 2m - 1$, $m > n$ are the same in T_m as in $\hat{T}$. Therefore we can choose a strategy in T_m, denoted by $\sigma_m^{(m)}$, that coincides with σ on sequences of this length. Then the construction of $\hat{T}$ gives strategies $\sigma_{m-1}^{(m)}$ in T_{m-1}, $\sigma_{m-2}^{(m)}$ in T_{m-2}, etc, $\sigma_n^{(m)}$ in T_n. We define $\tau_n = \sigma_n^{(m)}$ on sequences of length j, $j = k + 2m - 1$. Since $\sigma_{m-1}^{(m)}$ restricted to sequences of length at most ℓ depends only on $\sigma_m^{(m)}$ restricted to sequences of length at most ℓ, it does not matter which $\sigma_m^{(m)}$ we start with, we always obtain the same $\sigma_{m-1}^{(m)}, \sigma_{m-2}^{(m)}, \ldots, \sigma_n^{(m)}$ on sequences of this length.

If σ is a strategy of Player (II), τ_n is defined analogously. Finally, we define $\tau = \tau_0$.

Goodbye

The proof is finished if we can show that indeed if $x \in \mathcal{F}(T)$ is a play consistent with τ, then there is a $y \in \mathcal{F}(\hat{T})$ so that $\hat{\pi}(y) = x$, and y is consistent with σ.

It follows from the definition of τ_n that, for each n and for each play $x_{n-1} \in \mathcal{F}(T_{n-1})$ that is consistent with τ_{n-1}, there is a play $x_n \in \mathcal{F}(T_n)$ that is consistent with τ_n so that $\pi_n(x_n) = x_{n-1}$. So if $x \subset \mathcal{F}(T)$ is consistent with τ, then there is a play $x_1 \in \mathcal{F}(T_1)$ that is consistent with τ_1 and for which $\pi_1(x_1) = x_0$. Similarly, there is a play $x_2 \in \mathcal{F}(T_2)$ that is consistent with τ_2 and for which $\pi_2(x_2) = x_1$. Etc.

Since the initial segments of x_n and x_{n+1} of length $k + 2n$ are the same, the sequence $x, x_1, x_2, \ldots$ converges to a sequence y in $\mathcal{F}(\hat{T})$. Also, since the strategy σ agrees with the strategy τ_n on positions of fixed length, for all large n, y is consistent with σ. □