London Taught Course on Spectral Theory

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Week 4
4 February 2013

This first part of this lecture is based on Theorem 2.5.13 of STDO. This may be summarized as stating that every self-adjoint operator is unitarily equivalent to a self-adjoint multiplication operator. This is an operator M acting on the space $L^2(X,\mu)$ according to the formula

$$(Mf)(x) = m(x)f(x),$$

where $m : X \to \mathbf{R}$ is a real-valued function and

$$\mathrm{Dom}(M) = \{f \in L^2 : mf \in L^2\}.$$

The functional calculus is realized by writing

$$(R(z,K))f(x) = (z - m(x))^{-1} f(x)$$

for all $z \notin \mathrm{Spec}(K)$ and all $f \in L^2$. More generally

$$(F(M)f)(x) = F(m(x))f(x)$$

where F is any bounded measurable function on $\mathbf{R}$ and $f \in L^2$.

In particular if X is a countable set and μ is the counting measure on X then $\ell^2(X)$ is the space of square summable functions on X and $m : X \to \mathbf{R}$ is any function and

$$\mathrm{Dom}(M) = \{f : X \to \mathbf{C} : \sum_{x \in X} |f(x)|^2 + |m(x)f(x)|^2 < \infty.\}$$

The second part of the lecture applies the ideas developed during the course to a self-adjoint operator associated with the Hermite polynomials. There are many resources on the web for orthogonal polynomials, but I gave proofs based on the material in this course.