London Taught Course on Spectral Theory

E. B. Davies

Week 5
11 February 2013

This lecture is based on Chapters 4 and 6 of STDO, which contain much more detail than the lectures did. It provides a method of obtaining quantitative bounds on the eigenvalues of a range of differential and other operators.

One assumes that $H = H^*$ in $\mathcal{H}$ and that $He_n = \lambda_n e_n$ for all $n \in \mathbf{N}$ where λ_n is an increasing sequence of eigenvalues that diverges to $+\infty$ as n increases, and e_n is a complete orthonormal sequence in $\mathcal{H}$. The quadratic form of H is defined by

$$Q(f) = \langle Hf, f \rangle$$

for all $f \in \mathrm{Dom}(H)$. The eigenvalue λ_n is then given by

$$\lambda_n = \inf\{\lambda(L) : \dim(L) = n\}$$

where

$$\lambda(L) = \sup\{Q(f) : f \in L, \|f\| = 1\},$$

and L denotes any finite-dimensional subspace of $\mathrm{Dom}(H)$.

The lecture was devoted to proving this and providing applications and generalizations. Theorem 4.5.1, Theorem 4.5.3, Lemma 4.5.4 and Example 4.6.1 correspond quite closely to what was covered. Theorem 6.1.4, Lemma 6.2.1, Theorem 6.2.3 and Theorem 6.3.2 are more general. They prove that one can use the variational Theorem 4.5.1 with finite-dimensional subspaces L of spaces called $W^{1,2}(\Omega)$, $\mathrm{Dom}(H^{1/2})$ and $C_c^\infty(\Omega)$, rather than restricting attention to subspaces of $\mathrm{Dom}(H)$. These extensions are important if one wishes to compare the eigenvalues of operators acting in two different Hilbert spaces.