Enumerative Combinatorics 3: Catalan numbers

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In the last chapter, as in most of this course, we treated power series as formal objects: even differentiation involves no limiting processes. However, if the coefficients are complex numbers, and the series converge in some neighbourhood of the origin, then analytic methods can be used. These methods can be very powerful. We will see them at work in the derivation of a formula for the *Catalan numbers*, and then give a few examples of combinatorial objects counted by Catalan numbers.

3.1 Analysis

A complex function which is analytic in some neighbourhood of the origin is represented by a convergent power series in a disc about the origin. If an analytic relation between functions holds in a suitable disc, then any connection between the coefficients which can be derived will also be true in the world of formal power series.

The most important formal power series to which this principle can be applied are

(a) The *binomial series* $(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n$, where a is any complex number, and the binomial coefficient is defined as

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}.$$

(b) The *exponential series* $\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}..$

(c) The *logarithmic series* $\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{n}$.

Here is a simple example. The identity

$$(1+x)^a(1+x)^b = (1+x)^{a+b},$$

valid for $|x| < 1$, gives rise to the *Vandermonde convolution*

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}.$$

3.2 Example: Catalan numbers

The Catalan numbers are one of the most important sequences of combinatorial numbers, with a large range of occurrences in apparently different counting problems. I will introduce them with one particular occurrence, and then give a number of different places where they arise. The derivation of the formula for them is on the border between formal and analytic methods, and multivariate versions of this method are useful in areas such as lattice path problems.

Problem Given an algebraic structure with a (non-associative) binary operation $\circ$, in how many different ways can a product of n terms be evaluated by inserting brackets?

For example, the product $a \circ b \circ c \circ d$ has five evaluations:

$$((a \circ b) \circ c) \circ d, (a \circ (b \circ c)) \circ d, (a \circ b) \circ (c \circ d), a \circ ((b \circ c) \circ d), a \circ (b \circ (c \circ d)).$$

Let C_n be the number of evaluations of a product of n terms, for $n \geq 1$, so that $C_1 = C_2 = 1$, $C_3 = 2$, $C_4 = 5$. Let $c(x) = \sum_{n\geq 1} C_n x^n$ be the generating function.

In a bracketing of n terms, the last application of $\circ$ will combine some product of the first m terms with some product of the last $n-m$ terms, for some m with $1 \leq m \leq n-1$. So we have the recurrence relation

$$C_n = \sum_{m=1}^{n-1} C_m C_{n-m} \text{ for } n > 1.$$

Combined with the initial condition $C_1 = 1$, this determines the sequence.

Now consider the product $c(x)^2$. The recurrence relation shows that the terms in x^n in $c(x)^2$ are the same as those in $c(x)$ for $n > 1$; only the terms in x differ, with $c(x)$ containing $1x$ and $c(x)^2$ containing $0x$. So we have

$$c(x) = x + c(x)^2.$$

We can rearrange this as a quadratic equation:

$$c(x)^2 - c(x) + x = 0.$$

The solution of this equation is

$$c(x) = \tfrac{1}{2}\left(1 \pm \sqrt{1-4x}\right).$$

The choice of sign in the square root is determined by the fact that $c(0) = 0$, so we must take the negative sign:

$$c(x) = \tfrac{1}{2}\left(1 - \sqrt{1-4x}\right).$$

From this expression it is possible to extract the coefficient of x^n. According to the Binomial Theorem,

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n,$$

and so

$$C_n = -\tfrac{1}{2}(-4)^n \binom{1/2}{n}.$$

Now

$$\begin{aligned}
\binom{1/2}{n} &= \frac{(1/2)(-1/2)\cdots(-(2n-3)/2)}{n!} \\
&= \frac{1}{2^n}(-1)^{n-1}\frac{1\cdot 3\cdot (2n-3)}{n!} \\
&= \frac{1}{2^n}(-1)^{n-1}\frac{1}{n}\frac{(2n-2)!}{2^{n-1}((n-1)!)^2} \\
&= -2(-\tfrac{1}{4})^n \frac{1}{n}\binom{2n-2}{n-1},
\end{aligned}$$

so finally we obtain

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

The result and its proof call for a few remarks.

First, are these manipulations really valid?

(a) We have used here the Binomial Theorem for exponent 1/2, which is proved analytically by observing that the function $(1+x)^{1/2}$ is analytic in the interior of the unit disc (it has a branchpoint at $x = -1$), and then using the formula for the coefficient of x^n in the Taylor series (differentiate n times, put $x = 0$, divide by $n!$).

(b) It is clear, from back substitution, that the function $c(x) = \frac{1}{2}(1 - \sqrt{1-4x})$ does indeed satisfy the equation $c(x) = x + c(x)^2$; so its coefficients satisfy the recurrence relation and initial condition for the Catalan numbers C_n. Since these data determine the numbers uniquely, our final formula is indeed valid.

Second, this is a case where, even once you know the formula for the Catalan numbers, it is difficult to show directly that they satisfy the recurrence relation. (Spend a few moments trying; you will be convinced of this!)

And third, it is not at all obvious that n divides the binomial coefficient $\binom{2n-2}{n-1}$; but since C_n counts something, it is an integer, and so this divisibility is indeed true.

3.3 Other Catalan objects

Here are a small selection of the many objects counted by Catalan numbers.

The obvious ways of verifying this for a class of objects are either

(a) to verify the Catalan recurrence and initial condition; or

(b) to find a bijection to a known class of Catalan objects.

There are sometimes other less obvious ways, as we will see in the case of Dyck paths.

Where possible I have given an illustration of the five Catalan objects counted by C_4.

Binary trees

A binary tree has a root of degree 2; the other vertices have degree 1 or 3. So every non-root vertex is either a leaf or has two descendants, which we specify as left and right descendants.

The number of binary trees with n leaves is C_n. Figure 1 shows the correspondence with bracketed products: the tree is a "parse tree" for the product.

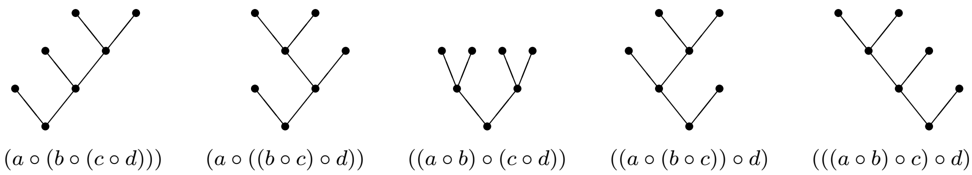

Figure 1: Binary trees and bracketed products

Rooted plane trees

The number of rooted plane trees with n edges is C_{n+1}. Figure 2 shows the rooted plane trees with three edges.

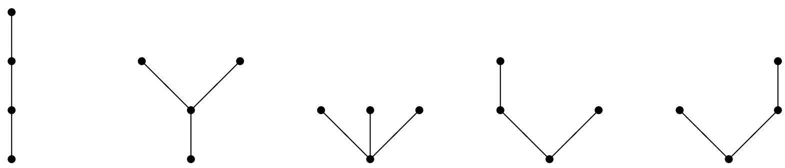

Figure 2: Rooted plane trees

Dissections of polygons

An n-gon can be dissected into triangles by drawing $n-2$ non-crossing diagonals. There are C_{n-1} dissections of an n-gon. Figure 3 shows dissections of a pentagon.

Dyck paths

A *Dyck path* starts at the origin and ends at $(2n, 0)$, moving at each step to the adjacent lattice point in either the north-easterly or south-easterly

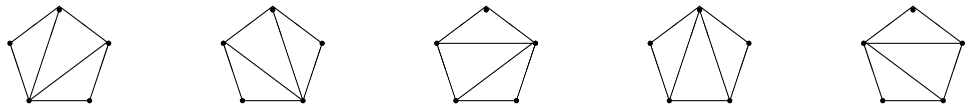

Figure 3: Dissections of a polygon

direction and never going below the X-axis. (An even number of steps is required since each step either increases or decreases the Y-coordinate by 1.)

Figure 4 shows the Dyck paths with $n = 3$.

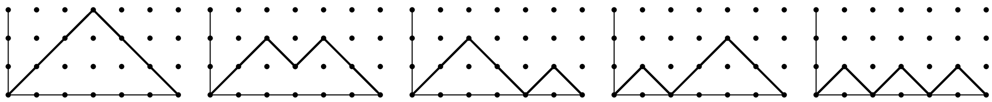

Figure 4: Dyck paths

The number of Dyck paths is C_{n+1}, and of these, C_n never return to the X-axis before the end. I will indicate the proof since it illustrates another technique.

Let D_n be the number of Dyck paths, and E_n the number which never return to the axis. Now a Dyck path begins by moving from $(0, 0)$ to $(1, 1)$ and ends by moving from $(2n - 1, 1)$ to $(2n, 0)$; if it did not return to the axis in between, then removing these "legs" gives a shorter Dyck path. So

$$E_n = D_{n-1}.$$

Suppose that a Dyck path first returns to the axis at $(2k, 0)$. Then it is a composite of a non-returning Dyck path of length $2k$ with an arbitrary Dyck path of length $2(n - k)$; so

$$D_n = \sum_{k=1}^{n} E_k D_{n-k}.$$

Solving these simultaneous recurrences gives the result.

Ballot numbers

An election is held with two candidates A and B, each of whom receives exactly n votes. In how many ways can the votes be counted so that A is never behind in the count?

It is easy to match these ballot numbers with Dyck paths. For $n = 3$, the five counts are AAABBB, AABABB, AABBAB, ABAABB, and ABABAB.

This can be described another way. In a $2 \times n$ array, we place the numbers $1, \ldots, 2n$ in order against the candidates who receive those votes. This gives the representations shown in Figure 5.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}$$

AAABBB AABABB AABBAB ABAABB ABABAB

Figure 5: Tableaux

Note that the numbers increase along each row and down each column.

3.4 Young diagrams and tableaux

The five objects shown are known as *Young tableaux*; they arise in the representation theory of the symmetric group and much related combinatorics.

A *Young diagram* (sometimes called a *Ferrers diagram*) consists of n boxes arranged in left-aligned rows, the number of boxes in each row being a non-decreasing function of the row number. This is simply a graphical representation of a partition of n: for each partition $n = a_1 + a_2 + \cdots$, with $a_1 \geq a_2 \geq \ldots$, we take a_1 boxes in the first row, a_2 in the second, and so on. Now a *Young tableau* is a filling of the boxes with the numbers $1, 2, \ldots, n$ so that each row and each column is in increasing order. You maay like to invent a ballot interpretation for the number of Young tableaux belonging to a given diagram.

This combinatorics is important in describing the representation theory of the symmetric group S_n, the group of all permutations of $\{1, \ldots, n\}$. It is known that the irreducible matrix representations of S_n over the complex numbers are in one-to-one correspondence with the partitions of n (that is, to the Young diagrams); the degree of a representation is equal to the number of Young tableaux belonging to the corresponding diagram. Thus, the five

Young tableaux shown in the preceding section correspond to an irreducible representation of degree 5 of the group S_6.

There is a "hook length formula" for the number of Young tableaux corresponding to a given diagram. The *hook* associated with a cell consists of that cell and all those to its right in the same row or below it in the same column. The *hook length* of a cell is the number of cells in its hook. Now the number of Young tableaux associated with the diagram is equal to $n!$ divided by the product of the hook lengths of all its cells.

Thus for the diagram with two rows of length 3, the formula gives

$$\frac{6!}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1} = 5.$$

3.5 Wedderburn–Etherington numbers

What happens if we count binary trees without the left-right distinction between the two children at each node? In other words, two binary trees will count as "the same" if a sequence of reversals of subtrees above each point converts one to the other.

It can be shown that the recurrence relation for the number W_n of binary trees with this convention (the *Wedderburn–Etherington numbers* is

$$W_n = \begin{cases} \frac{1}{2}\sum_{i=1}^{n-1} W_i W_{n-i} & \text{if } n \text{ is odd,} \\ \frac{1}{2}\left(\sum_{i=1}^{n-1} W_i W_{n-i} + W_{n/2}\right) & \text{if } n \text{ is even,} \end{cases}$$

and that the generating function $w(x)$ satisfies

$$w(x) = x + \tfrac{1}{2}(w(x)^2 + w(x^2)).$$

This is much more difficult to solve. Whereas C_n is roughly 4^n (in the sense that the limit of $C_n^{1/n}$ as $n \to \infty$ is 4), W_n is roughly $2.483\ldots^n$ in the same sense.

Exercises

1 Give a counting proof of the Vandermonde convolution in the case where a and b are natural numbers.

2 Verify some of the formulae for Catalan objects in the notes, either by deriving a recurrence, or by finding bijections between the objects counted.

3 In the analysis of Dyck paths, adopt the convention that $D_0 = 1$ and $E_0 = 0$. Prove that, if $d(x)$ and $e(x)$ are the generating functions, then

$$xd(x) = e(x), \qquad d(x) = 1 + e(x)d(x).$$

Hence derive formulae for D_n and E_n.

4 Use the hook length formula to derive the formula for the Catalan number C_n.

5 Prove the recurrence relation and the equation for the generating function for the Wedderburn–Etherington numbers.