Enumerative Combinatorics 1: Subsets, Partitions, Permutations

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Enumerative combinatorics is concerned with *counting* discrete structures of various types. There is a great deal of variation both in what we mean by "counting" and in the types of structures we count. Typically each structure has a "size" measured by a non-negative integer n, and "counting" may mean

(a) finding an exact formula for the number $f(n)$ of structures of size n;

(b) finding an approximate or asymptotic formula for $f(n)$;

(c) finding an analytic expression for a generating function for $f(n)$;

(d) finding an efficient algorithm for computing $f(n)$ exactly or approximately;

(e) finding an efficient algorithm for stepping from one of the counted objects to the next (in some natural ordering).

In this course I will mostly be concerned with the first three goals; discussing algorithms would lead too far afield. The exception to this is one particularly important algorithm, a *recurrence relation*, in which the value of $f(n)$ is computed from n and the earlier values $f(0),\ldots,f(n-1)$.

An *asymptotic formula* for $f(n)$ is an analytic function $g(n)$ such that $f(n)/g(n) \to 0$ as $n \to \infty$. There are several types of *generating functions*; the most important for us are the *ordinary generating function* $\displaystyle\sum_{n\geq 0} f(n)x^n$, and the *exponential generating function* $\displaystyle\sum_{n\geq 0} \frac{f(n)x^n}{n!}$.

If you want to learn the state-of-the-art in combinatorial enumeration, I recommend the two volumes of Richard Stanley's *Enumerative Combinatorics*, or the book *Analytic Combinatorics* by Philippe Flajolet and Robert Sedgewick. The On-line Encyclopedia of Integer Sequences is another valuable resource for combinatorial enumeration.

1.1 Subsets

The three most important objects in elementary combinatorics are subsets, partitions and permutations; we briefly discuss the counting functions for these. First, subsets.

The total number of subsets of an n-element set is 2^n. This can be used by noting that this number $f(n)$ satisfies the recurrence relation $f(n) = 2f(n-1)$; this is proved by observing that any subset of $\{1,\ldots,n-1\}$ can be extended to a subset of $\{1,\ldots,n\}$ in two different ways, either including the element n or not.

The *binomial coefficient* $\binom{n}{k}$ is the number of k-element subsets of $\{1,\ldots,n\}$. The formula is

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!\,(n-k)!}.$$

Note that there are k factors in both numerator and denominator. We have $\binom{n}{0} = \binom{n}{n} = 1$. We can extend the definition to all non-negative integers n and k by defining $\binom{n}{k} = 0$ for $k > n$: this fits with the counting interpretation.

The recurrence relation for binomial coefficients is *Pascal's Triangle*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \text{ for } 0 < k < n.$$

For the first term on the right counts subsets containing n, while the second counts subsets not containing n.

Counting subsets by cardinality gives

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

There is a huge literature on "binomial coefficient identities". A few examples are given as exercises.

Anticipating our discussion of formal power series in the next chapter, we now discuss generating functions for binomial coefficients.

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

This is the *Binomial Theorem* for non-negative integer exponents. If we write $(1+x)^n = (1+x)\cdots(1+x)$ and expand the product, then we obtain the term in x^k by choosing x from k of the brackets and 1 from the remaining $n-k$, which can be done in $\binom{n}{k}$ ways; each contributes 1 to the coefficient of x^k, so the theorem holds.

If we multiply this equation by y^n and sum, we obtain the *bivariate generating function* for the binomial coefficients:

$$\begin{aligned}\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}x^k y^n &= \sum_{n\geq 0}(1+x)^n y^n \\ &= \frac{1}{1-(1+x)y} \\ &= \frac{1}{1-y}\cdot\frac{1}{1-xy/(1-y)} \\ &= \sum_{k\geq 0}\frac{y^k}{(1-y)^{k+1}}\,x^k,\end{aligned}$$

so we obtain the other univariate generating function for binomial coefficients:

$$\sum_{n\geq k}\binom{n}{k}y^n = \frac{y^k}{(1-y)^{k+1}}.$$

This formula is actually a rearrangement of the Binomial Theorem for negative integer exponents. The basis of this connection is the following evaluation, for positive integers m and k:

$$\binom{-m}{k} = \frac{-m(-m-1)\cdots(-m-k+1}{k!}$$

$$
\begin{aligned}
&= (-1)^k \frac{(m+k-1)\cdots(m+1)m}{k!} \\
&= (-1)^k \binom{m+k-1}{k}.
\end{aligned}
$$

1.2 Partitions

In this case and the next, we are unable to write down a simple formula for the counting numbers, and have to rely on recurrence relations or other techniques.

The *Bell number* $B(n)$ is the number of partitions of a set of cardinality n. We refine this in the same way we did for subsets. The *Stirling number of the second kind*, $S(n,k)$, is the number of partitions of an n-set into k parts. Thus, $S(0,0)=1$ and $S(0,k)=0$ for $k>0$; and if $n>0$, then $S(n,0)=0$, $S(n,1)=S(n,n)=1$, and $S(n,k)=0$ for $k>n$. Clearly we have

$$\sum_{k=1}^{n} S(n,k) = B(n) \text{ for } n>0.$$

The recurrence relation replacing Pascal's is:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \text{ for } 1 \le k \le n.$$

It turns out that we can turn this into a statement about a generating function, but with a twist. Let

$$(x)_k = x(x-1)\cdots(x-k+1) \text{ (k factors).}$$

Then we have

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k \text{ for } n>0.$$

It is possible to find a traditional generating function for the index n:

$$\sum_{n\ge k} S(n,k)y^n = \frac{y^k}{(1-y)(1-2y)\cdots(1-ky)}.$$

Also, the exponential generating function for the index n is

$$\sum_{n\ge k} \frac{S(n,k)x^n}{n!} = \frac{(\exp(x)-1)^k}{k!}.$$

Summing over k gives the e.g.f. for the Bell numbers:

$$\sum_{n\geq 0} \frac{B(n)x^n}{n!} = \exp(\exp(x) - 1).$$

1.3 Permutations

The number of permutations of an n-set (bijective functions from the set to itself) is the factorial function $n! = n(n-1)\cdots 1$ for $n \geq 0$. The exponential generating function for this sequence is $1/(1-x)$, while the ordinary generating function has no analytic expression (it is divergent for all $x \neq 0$).

Any permutation can be decomposed uniquely into disjoint cycles. So we refine the count by letting $u(n, k)$ be the number of permutations of an n-set which have exactly k cycles (including cycles of length 1). Thus,

$$\sum_{k=1}^{n} u(n, k) = n! \text{ for } n > 0.$$

The numbers $u(n, k)$ are the *unsigned Stirling numbers of the first kind*. The reason for the name is that it is common to use a different count, where a permutation is counted with weight equal to its *sign* (as defined in elementary algebra, for example the theory of determinants). Let $s(n, k)$ be the sum of the signs of the permutations of an n-set which have k cycles. Since the sign of such a permutation is $(-1)^{n-k}$, we have $s(n, k) = (-1)^{n-k}u(n, k)$. The numbers $s(n, k)$ are the *signed Stirling numbers of the first kind*.

We have

$$\sum_{k=1}^{n} s(n, k) = 0 \text{ for } n > 1.$$

This is related to the algebraic fact that, for $n > 1$, the permutations with sign + form a subgroup of the symmetric group of index 2 (that is, containing half of all the permutations), called the *alternating group*.

We will mainly consider signed Stirling numbers below, though it is sometimes convenient to prove a result first for the unsigned numbers.

As usual we take $s(n, 0) = 0$ for $n > 0$ and $s(n, k) = 0$ for $k > n$.

We have $s(n, n) = 1$, $s(n, 1) = (-1)^{n-1}(n-1)!$, and the recurrence relation

$$s(n, k) = s(n-1, k-1) - (n-1)s(n-1, k) \text{ for } 1 \leq k \leq n.$$

From this, we find a generating function:

$$\sum_{k=1}^{n} s(n,k)x^k = (x)_n.$$

Putting $x = 1$ in this equation shows that indeed the sum of the signed Stirling numbers is zero for $n > 1$.

Note that this is the inverse of the relation we found for the Stirling numbers of the second kind. So the matrices formed by the Stirling numbers of the first and second kind are inverses of each other.

Exercises

1. Let A be the matrix of binomial coefficients (with rows and columns indexed by $\mathbb{N}$, and (i,j) entry $\binom{i}{j}$), and B the matrix of "signed binomial coefficients" (as before but with (i,j) entry $(-1)^{i-j}\binom{i}{j}$). Prove that A and B are inverses of each other.

What are the entries of the matrix A^2?

2. Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

3. (a) Prove that the following are equivalent for sequences $(a_0.a_1,\ldots)$ and $(b_0,b_1,\ldots)$, with exponential generating functions $A(x)$ and $B(x)$ respectively:

(ii) $b_0 = a_0$ and $b_n = \displaystyle\sum_{k=1}^{n} S(n,k)a_k$ for $n \geq 1$;

(i) $B(x) = A(\exp(x) - 1)$.

(b) Prove that the following are equivalent for sequences $(a_0.a_1,\ldots)$ and $(b_0,b_1,\ldots)$, with exponential generating functions $A(x)$ and $B(x)$ respectively:

(i) $b_0 = a_0$ and $b_n = \displaystyle\sum_{k=1}^{n} s(n,k)a_k$ for $n \geq 1$;

(ii) $B(x) = A(\log(1 + x))$.

4. Construct a bijection between the set of all k-element subsets of $\{1, 2, \ldots, n\}$ containing no two consecutive elements, and the set of all k-element subsets of $\{1, 2, \ldots, n - k + 1\}$. Hence show that the number of such subsets is $\binom{n-k+1}{k}$.

In the UK National Lottery, six numbers are chosen randomly (without replacement, order unimportant) from the set $\{1, \ldots, 49\}$. What is the probability that the selection contains no two consecutive numbers?