Enumerative Combinatorics 2: Formal power series

Peter J. Cameron

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Probably you recognised in the last chapter a few things from analysis, such as the exponential and geometric series; you may know from complex analysis that convergent power series have all the nice properties one could wish. But there are reasons for considering non-convergent power series, as the following example shows.

Recall the generating function for the factorials:

$$F(x) = \sum_{n\geq 0} n!x^n,$$

which converges nowhere. Now consider the following problem. A permutation of $\{1,\ldots,n\}$ is said to be *connected* if there is no number m with $1 \leq m \leq n-1$ such that the permutation maps $\{1,\ldots,m\}$ to itself. Let C_n be the number of connected permutations of $\{1,\ldots,n\}$. Any permutation is composed of a connected permutation on an initial interval and an arbitrary permutation of the remainder; so

$$n! = \sum_{m=1}^{n} C_m(n-m)!.$$

Hence, if

$$G(x) = 1 - \sum_{n\geq 1} C_n x^n,$$

we have $F(x)G(x) = 1$, and so $G(x) = 1/F(x)$.

Fortunately we can do everything that we require for combinatorics (except some asymptotic analysis) without assuming any convergence properties.

2.1 Formal power series

Let R be a commutative ring with identity. A *formal power series* over R is, formally, an infinite sequence $(r_0, r_1, r_2, \ldots)$ of elements of R; but we always represent it in the suggestive form

$$r_0 + r_1 x + r_2 x^2 + \cdots = \sum_{n \geq 0} r_n x^n.$$

We denote the set of all formal power series by $R[[x]]$.

The set $R[[x]]$ has a lot of structure: there are many things we can do with formal power series. All we require of any operations is that they only require adding or multiplying finitely many elements of R. No analytic properties such as convergence of infinite sums or products are required to hold in R.

(a) *Addition:* We add two formal power series term-by-term.

(b) *Multiplication:* The rule for multiplication of formal power series, like that of matrices, looks unnatural but is really the obvious thing: we multiply powers of x by adding the exponents, and then just gather up the terms contributing to a fixed power. Thus

$$\left(\sum a_n x^n\right) \cdot \left(\sum b_n x^n\right) = \sum c_n x^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Note that to produce a term of the product, only finitely many additions and multiplications are required.

(c) *Infinite sums and products:* These are not always defined. Suppose, for example, that $A^{(i)}(x)$ are formal power series for $i = 0, 1, 2, \ldots$; assume that the first non-zero coefficient in $A^{(i)}(x)$ is the coefficient of x^{n_i}, where $n_i \to \infty$ as $i \to \infty$. Then, to work out the coefficient of x^n in the infinite sum, we only need the finitely many series $A^{(i)}(x)$ for which $n_i \leq n$. Similarly, the product of infinitely many series $B^{(i)}$ is defined provided that $B^{(i)}(x) = 1 + A^{(i)}(x)$, where $A^{(i)}$ satisfy the condition just described.

(d) *Substitution:* Let $B(x)$ be a formal power series with constant term zero. Then, for any formal power series $A(x)$, the series $A(B(x))$ obtained by substituting $B(x)$ for x in $A(x)$ is defined. For, if $A(x) = \sum a_n x^n$, then $A(B(x)) = \sum a_n B(x)^n$, and $B(x)^n$ has no terms in x^k for $k < n$.

(e) *Differentiation:* of formal power series is always defined; no limiting process is required. The derivative of $\sum a_n x^n$ is $\sum n a_n x^{n-1}$, or alternatively, $\sum (n+1) a_{n+1} x^n$.

(f) *Negative powers:* We can extend the notion of formal power series to *formal Laurent series*, which are allowed to have finitely many negative terms:

$$\sum_{n \geq -m} a_n x^n.$$

Infinitely many negative terms would not work since multiplication would then require infintely many arithmetic operations in R.

(g) *Multivariate formal power series:* We do not have to start again from scratch to define series in several variables. For $R[[x]]$ is a commutative ring with identity, and so $R[[x, y]]$ can be defined as the set of formal power series in y over $R[[x]]$.

As hinted above, $R[[x]]$ is indeed a commutative ring with identity: verifying the axioms is straightforward but tedious, and I will just assume this. With the operation of differentiation it is a *differential ring.*

Recall that a *unit* in a ring is an element with a multiplicative inverse. The units in $R[[x]]$ are easy to describe:

Proposition 2.1 *The formal power series $\sum r_n x^n$ is a unit in $R[[x]]$ if and only if r_0 is a unit in R.*

Proof If $\left(\sum r_n x^n\right)\left(\sum s_n x^n\right) = 1$, then looking at the constant term we see that $r_0 s_0 = 1$, so r_0 is a unit.

Conversely, suppose that $r_0 s_0 = 1$. Considering the coefficient of x^n in the above equation with $n > 0$, we see that

$$\sum_{k=0}^{n} r_k s_{n-k} = 0,$$

so we can find the coefficients s_n recursively:

$$s_n = -s_0 \left(\sum_{k=1}^{n} r_k s_{n-k} \right).$$

This argument shows the very close connection between finding inverses in $R[[x]]$ and solving linear recurrence relations.

2.2 Example: partitions

We are considering partitions of a number n, rather than of a set, here. A *partition* of n is an expression for n as a sum of positive integers arranged in non-increasing order; so the five partitions of 4 are

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1.$$

Let $p(n)$ be the number of partitions of n.

Theorem 2.2 (Euler's Pentagonal Numbers Theorem)

$$p(n) = \sum_{k\geq 1} (-1)^{k-1} \left(p(n - k(3k-1)/2) + p(n - k(3k+1)/2) \right),$$

where the sum contains all terms where the argument $n - k(3k \pm 1)/2$ *is non-negative.*

This is a very efficient recurrence relation for $p(n)$, allowing it to be computed with only about $\sqrt{n}$ arithmetic operations if smaller values are known. For example, if we know

$$p(0) = 1, \quad p91) = 1, \quad p(2) = 2, \quad p(3) = 3, \quad p(4) = 5,$$

then we find $p(5) = p(4) + p(3) - p(0) = 7$, $p(6) = p(5) + p(4) - p(1) = 11$, and so on.

I will give a brief sketch of the proof.

Step 1: The generating function.

$$\sum_{n\geq 0} p(n) x^n = \prod_{k\geq 1} (1 - x^k)^{-1}.$$

For on the right, we have the product of geometric series $1 + x^k + x^{2k} + \cdots$, and the coefficient of x^n is the number of ways of writing $n = \sum k a_k$, which is just $p(n)$.

Step 2: The inverse of the generating function. We need to find

$$\prod_{k\geq 1}(1-x^k).$$

The coefficient of x^n in this product is obtained from the expressions for n as a sum of *distinct* positive integers, where sums with an even number of terms contribute $+1$ and sums with an odd number contribute -1. For example,

$$9 = 8+1 = 7+2 = 6+3 = 5+4 = 6+2+1 = 5+3+1 = 4+3+2,$$

so there are four sums with an even number of terms and four with an odd number of terms, and so the coefficient is zero.

Step 3: Pentagonal numbers appear. It turns out that the following is true:

> The numbers of expressions for n as the sum of an even or an odd number of distinct positive integers are equal for all n except those of the form $k(3k \pm 1)/2$, for which the even expressions exceed the odd ones by one if k is even, and *vice versa* if k is odd.

This requires some ingenuity, and I do not give the proof here.

This shows that the expression in Step 2 is equal to

$$1 + \sum_{k\geq 1}(-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right),$$

and we immediately obtain the required recurrence relation.

Exercises

1. Suppose that R is a field. Show that $R[[x]]$ has a unique maximal ideal, consisting of the formal power series with constant term zero. Describe all the ideals of $R[[x]]$.

2. Suppose that $A(x)$, $B(x)$ and $C(x)$ are the exponential generating functions of sequences (a_n), (b_n) and (c_n) respectively. Show that $A(x)B(x) = C(x)$ if and only if

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

3. (a) Let (a_n) be a sequence of integers, and (b_n) the sequence of partial sums of (a_n) (in other words, $b_n = \sum_{i=0}^{n} a_i$). Suppose that the generating function for (a_n) is $A(x)$. Show that the generating function for (b_n) is $A(x)/(1-x)$.

(b) Let (a_n) be a sequence of integers, and let $c_n = na_n$ for all $n \geq 0$. Suppose that the generating function for (a_n) is $A(x)$. Shos that the generating function for (c_n) is $x(\mathrm{d}/\mathrm{d}x)A(x)$. What is the generating function for the sequence $(n^2 a_n)$?

(c) Use the preceding parts of this exercise to find the generating function for the sequence whose nth term is $\sum_{i=1}^{n} i^2$, and hence find a formula for the sum of the first n squares.