ORTHOGONAL POLYNOMIALS AND SPECIAL FUNCTIONS

Part I

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Ana F. Loureiro (a.loureiro@kent.ac.uk)

University of Kent

Outline

- Part 1. Special Functions
  - Gamma, Digamma and Beta Function
  - Hypergeometric Functions and Hypergeometric Series
  - Confluent Hypergeometric Function
- Part 2. Orthogonal Polynomials
  - Main properties
    Recurrence relations, zeros, distribution of the zeros and so on and on....
  - Classical Orthogonal Polynomials
    Hermite, Laguerre, Bessel and Jacobi!!
  - Other notions of "classical orthogonal polynomials"
    How to identify this on the Askey Scheme?
  - Semiclassical Orthogonal Polynomials
    How do these link to Random Matrix Theory, Painlevé equations and so on?
- Part 3. Multiple Orthogonal Polynomials
  When the orthogonality measure is spread across a vector of measures?

1. Gamma function.

Introduced by Euler in 1729 in a letter to Golbach, the Gamma function arose to answer the question of finding a function mapping any nonnegative integer n to its factorial $n!$, that is

$$\begin{array}{cccl} \Gamma\Big|_{\mathbb{N}_0} : & \mathbb{N}_0 & \longrightarrow & \mathbb{N}_0 \\ & n & \longmapsto & \begin{cases} n! = n(n-1)\ldots 3\cdot 2\cdot 1 = \displaystyle\prod_{j=1}^{n} j & \text{if} \quad n \geq 1, \\ 0! = 1 & \text{if} \quad n = 0. \end{cases} \end{array}$$

$$\boxed{\Gamma(z) = \frac{1}{z}\prod_{j=1}^{+\infty}\left(\left(1+\frac{1}{j}\right)^{z}\left(1+\frac{z}{j}\right)^{-1}\right)} \tag{1}$$

valid for any z such that $z \neq 0, -1, -2, \ldots$. Observe that

$$\frac{1}{z}\prod_{j=1}^{n-1}\left(1+\frac{z}{j}\right)^{-1} = \frac{(n-1)!}{(z)_n} \quad \text{and} \quad \prod_{j=1}^{n-1}\left(1+\frac{1}{j}\right)^{z} = n^z$$

Therefore, we can conclude that the function $\Gamma(z)$ can be equivalently given by

$$\boxed{\Gamma(z) = \lim_{n\to+\infty}\frac{(n-1)!n^z}{(z)_n}.} \tag{2}$$

Euler's formula (1) gives

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{z}{z+1}\lim_{m\to\infty}\frac{(m+1)(z+1)}{m+1+z} = z$$

Hence we obtain the most remarkable property for the Gamma function:

$$\boxed{\Gamma(z+1) = z\Gamma(z) \quad \text{with} \quad \Gamma(1) = 1 = 0! \quad .} \tag{3}$$

Theorem

For $z > 0$, we have

$$\boxed{\Gamma(z) = \int_0^{+\infty} \mathrm{e}^{-t} t^{z-1} \mathrm{d}t.}$$

Proof. For $z > 0$ and for any positive integer n, let

$$\Pi(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} \mathrm{d}t = n^z \int_0^1 (1-\tau)^n \tau^{z-1} \mathrm{d}\tau$$

Repeated integration by parts gives

$$\begin{aligned} \Pi(z,n) &= n^z \frac{n(n-1)\ldots 2\cdot 1}{z(z+1)\ldots(z+n-1)} \int_0^1 \tau^{z+n-1} \mathrm{d}\tau \\ &= \frac{n(n-1)\ldots 2\cdot 1}{z(z+1)\ldots(z+n)} n^z = \frac{n! n^z}{(z)_n} \end{aligned}$$

so that

$$\Gamma(z) = \lim_{n\to+\infty} \Pi(z,n).$$

On the other hand, observe that $\lim_{n\to+\infty}\left(1-\frac{t}{n}\right)^n = \mathrm{e}^{-t}$, and

$$0 \le \mathrm{e}^{-t} - \left(1-\frac{t}{n}\right)^n \le \frac{t^2\mathrm{e}^{-t}}{n}.$$

Since

$$\begin{aligned}\int_0^{+\infty} \mathrm{e}^{-t} t^{z-1}\mathrm{d}t - \Gamma(z) &= \lim_{n\to+\infty} \int_0^n \left(\mathrm{e}^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\mathrm{d}t\Bigg) \\ &\le \lim_{n\to+\infty} \left(\frac{1}{n}\int_0^n t^{z+1}\mathrm{e}^{-t}\mathrm{d}t\right) \\ &< \frac{1}{n}\int_0^{+\infty} t^{z+1}\mathrm{e}^{-t}\mathrm{d}t \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

then

$$\Gamma(z) = \int_0^{+\infty} \mathrm{e}^{-t} t^{z-1}\mathrm{d}t.$$

$\square$

Integration by parts

$$\begin{aligned}\Gamma(z+1) &= \int_0^{+\infty} \mathrm{e}^{-t}\ t^z \mathrm{d}t = \left(\left(-\mathrm{e}^{-t}\right)\ t^z \right)\Big|_0^{+\infty} - \int_0^{+\infty} \left(-\mathrm{e}^{-t}\right) z t^{z-1} \mathrm{d}t \\ &= z \int_0^{+\infty} \mathrm{e}^{-t} t^{z-1} \mathrm{d}t = z\ \Gamma(z)\end{aligned}$$

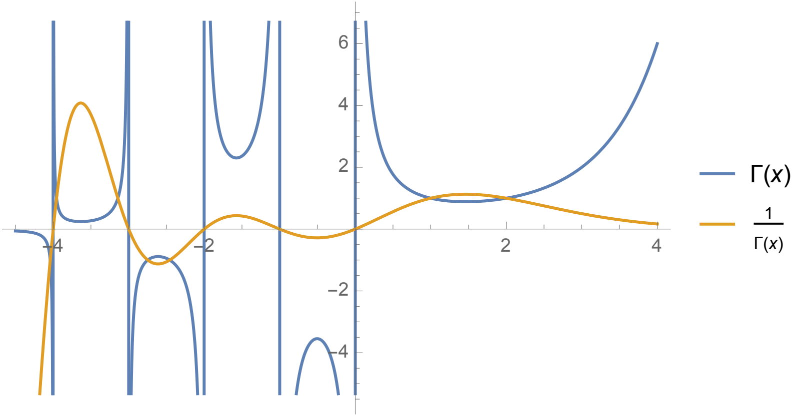
gives

$$\Gamma(z+1) = z\ \Gamma(z)$$

Remark. The Gamma function does not satisfy any differential equation with rational coefficients (Hölder, 1887).

Gamma Function: a plot

Out[15]=



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The reciprocal of the Gamma function has the product representation

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{z\gamma} \prod_{n=1}^{+\infty} \left(\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right)\right) \tag{4}$$

where

$$\gamma := \lim_{n\to+\infty} \left(\sum_{k=1}^{n-1} \frac{1}{k} - \log(n) \right) = 0.5772156649\ldots \tag{5}$$

and called the *Euler's constant*.

(Proof due to Schlömilch and Newman in 1848.)

We have the identity

$$\frac{(z)_n}{(n-1)!n^z} = z \exp\left(z\left\{\sum_{k=1}^{n-1}\frac{1}{k} - \log(n)\right\}\right)\left(\prod_{k=1}^{n-1}\left(1+\frac{z}{k}\right)\exp\left(-\frac{z}{k}\right)\right)$$

Observe that $\log\left(\left(1+\frac{z}{k}\right)\exp\left(-\frac{z}{k}\right)\right) = \mathcal{O}(k^{-2})$ and therefore the product $\prod_{k=1}^{n-1}\left(1+\frac{z}{k}\right)\exp\left(-\frac{z}{k}\right)$ converges uniformly in bounded sets as $n \to +\infty$.
Furthermore,

$$\sum_{k=1}^{n-1}\frac{1}{k} - \log(n) = \sum_{k=1}^{n-1}\int_k^{k+1}\left(\frac{1}{k}-\frac{1}{t}\right)\mathrm{d}t = \sum_{k=1}^{n-1}\int_k^{k+1}\left(\frac{t-k}{kt}\right)\mathrm{d}t$$

and $\int_k^{k+1}\left(\frac{t-k}{kt}\right)\mathrm{d}t = \mathcal{O}(k^{-2})$ so the sum converges as $n \to +\infty$. Hence, the result now follows due to

$$\Gamma(z) = \lim_{n\to+\infty}\frac{(n-1)!n^z}{(z)_n} \qquad \text{and} \qquad \gamma := \lim_{n\to+\infty}\left(\sum_{k=1}^{n-1}\frac{1}{k} - \log(n)\right)$$

It can be shown that for large values of x,

$$\Gamma(x) = \mathrm{e}^{-x} x^{x-\frac{1}{2}} \sqrt{2\pi}(1 + \mathcal{O}(1/x)),$$

and this is known as the *Stirling's asymptotic formula for the Gamma function*.

Reflection formula for the Gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}. \tag{6}$$

After a multiplication by z, we obtain a symmetric version of (6):

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)},$$

which can be generalised to

$$\Gamma(n+1+z)\Gamma(n+1-z) = (n!)^2 \frac{\pi z}{\sin(\pi z)} \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right), \quad n = 1, 2, \ldots.$$

As an immediate consequence, the choice of $z = \frac{1}{2}$ brings

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{7}$$

Quiz: What is the value for

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2} \mathrm{d}x \quad ?$$

Beta function

The *Beta function* or *Beta integral* is a function of two variables a and b and is only defined for $a > 0$ and $b > 0$ by

$$B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}\mathrm{d}s. \tag{8}$$

Theorem

The Beta function satisfies the following identities

$$\begin{aligned} B(a,b) &= \int_0^1 s^{a-1}(1-s)^{b-1}\mathrm{d}s \\ &= \int_0^{+\infty} u^{a-1}\left(\frac{1}{1+u}\right)^{a+b}\mathrm{d}u \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \end{aligned} \tag{9}$$

which are valid for $a, b > 0$.

The first equal sign is a consequence of the c.o.v. $u = \frac{s}{1-s}$.
For the 2nd equal sign, consider the product of $\Gamma(a)\Gamma(b)$:

$$\Gamma(a)\Gamma(b) = \int_0^{+\infty}\int_0^{+\infty} \mathrm{e}^{-(s+t)} s^{a-1} t^{b-1} \mathrm{d}s\mathrm{d}t.$$

Take the c.o.v. $s = xu$ and $t = x(1-u)$, whose Jacobian is

$$\frac{\partial(t,s)}{\partial(x,u)} = \det \begin{bmatrix} u & x \\ 1-u & -x \end{bmatrix} = -x$$

so that

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= \int_0^1 \int_0^{+\infty} \mathrm{e}^{-x} x^{a-1} u^{a-1} x^{b-1} (1-u)^{b-1} x \, \mathrm{d}x\mathrm{d}u \\
&= \left(\int_0^1 u^{a-1}(1-u)^{b-1}\mathrm{d}u\right)\left(\int_0^{+\infty} \mathrm{e}^{-x} x^{a+b-1}\mathrm{d}x\right) \\
&= \Gamma(a+b)B(a,b)
\end{aligned}$$

$\square$

We start by observing that for any integer $n > 1$, we have

$$\begin{aligned}\Gamma(2n) &= (2n-1)! = \frac{1}{2n}\prod_{k=0}^{n-1}(2k+1)(2k+2) = 2^{2n-1}\left(\tfrac{1}{2}\right)_n (n-1)! \\ &= \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma\left(n+\tfrac{1}{2}\right).\end{aligned}$$

More generally, for $z > 0$,

$$\frac{\Gamma(z)^2}{\Gamma(2z)} = B(z,z) = \int_0^1 s^{z-1}(1-s)^{z-1}\mathrm{d}s = 2\int_0^{\frac{1}{2}} s^{z-1}(1-s)^{z-1}\mathrm{d}s$$

and, with the c.o.v. $t = 4s(1-s)$, it follows

$$\frac{\Gamma(z)^2}{\Gamma(2z)} = 2\int_0^1 \left(\frac{t}{4}\right)^z \frac{1}{t\sqrt{1-t}}\mathrm{d}t = 2^{1-2z}B\left(z,\frac{1}{2}\right) = 2^{1-2z}\frac{\Gamma(z)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)}.$$

By analytic continuation, we obtain the **Legendre's duplication formula**:

$$\boxed{\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad \text{for} \quad 2z \neq 0,-1,-2,\ldots.} \qquad (10)$$

Definition. A series $\sum_{n\geq 0} c_n$ is called an *hypergeometric series* when $c_0 = 1$ and $\frac{c_{n+1}}{c_n}$ is a rational function in n (possibly complex valued).

Examples.

$$\sum_{n\geq 0} \frac{z^n}{n!} = \mathrm{e}^z \quad , \quad \sum_{n\geq 0} x^n = \frac{1}{1-x} \quad (\text{for } \ |x| < 1)$$

The property $c_0 = 1$ and $\frac{c_{n+1}}{c_n}$ is a rational function in n is satisfied if

$$\boxed{c_n = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \, \frac{x^n}{n!}}$$

where the symbol $(\alpha)_n$ is the *Pochhammer symbol*:

$$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1) = \prod_{\sigma=0}^{n-1}(\alpha+\sigma), \quad n \geq 1,$$

$$(\alpha)_0 := 1$$

Hence, we may represent an hypergeometric series by

$$\boxed{{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right) := \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}} \tag{11}$$

The previous examples...

$$\mathrm{e}^z = \sum_{n \geq 0} \frac{z^n}{n!} = {}_0F_0\left(; z\right) = {}_pF_p\left(\begin{matrix} a_1, \ldots, a_p \\ a_1, \ldots, a_p \end{matrix}; z\right)$$

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n = \sum_{n \geq 0} (1)_n \frac{x^n}{n!} = {}_1F_0\left(\begin{matrix} 1 \\ - \end{matrix}; x\right) \quad (\text{for} \quad |x| < 1).$$

Theorem. The series ${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right)$

- converges for all x if $p \leq q$;
- converges for $|x| < 1$ if $p = q + 1$;
- diverges for all $x \neq 0$ if $p > q + 1$ and the series does not *terminate*.

Proof. *Exercise.*

Case where $p = q+1$ and $|x| = 1$

Theorem. The series ${}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; x\right)$ with $|x| = 1$

- converges absolutely if $\Re(\sum_{k=1}^{q} b_k - \sum_{k=1}^{q+1} a_k) > 0$;
- converges conditionally if $x = \mathrm{e}^{i\theta} \neq 1$ and $-1 < \Re(\sum_{k=1}^{q} b_k - \sum_{k=1}^{q+1} a_k) \leq 0$;
- diverges if $\Re(\sum_{k=1}^{q} b_k - \sum_{k=1}^{q+1} a_k) \leq -1$ and the series does not *terminate*.

Proof. Observe that the nth term is

$$\frac{(a_1)_n \ldots (a_{q+1})_n}{(b_1)_n \ldots (b_q)_n n!} \sim \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_{q+1})} \; n^{(\sum_{k=1}^{q+1} a_k - \sum_{k=1}^{q} b_k)-1}$$

as $n \to +\infty$. Hence, the result.

For a positive integer m

$$(-m)_n := \begin{cases} \prod_{\sigma=0}^{n-1}(-m+\sigma) = (-1)^n \prod_{\sigma=0}^{n-1}(m-\sigma) = (-1)^n \dfrac{m!}{(m-n)!} & \text{if} \quad 0 \le n \le m, \\ 0 & \text{if} \quad n \ge m+1. \end{cases}$$

Hence,

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right)$$

is a *terminating series* if $\exists j \in \{1, \ldots, p\}$ s.t. $a_j = -m$ for some $m \in \mathbb{N}$.

So, we also require $b_j \neq -m$ for any $m \in \mathbb{N}$.

Examples.

- ${}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \dfrac{1}{(\alpha+1)_k} x^k$ is a polynomial

(Laguerre polynomials)

The Gauss series $\boxed{{}_2F_1\left({a, b \atop c}; z\right) := \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}}$ defined on the disk $|z| < 1$ and by analytic continuation elsewhere,
is the **Hypergeometric Function** (*aka* the **Gauss function**).

(see DLMF/Chapter15)

On the circle of convergence $|z| = 1$,

- converges absolutely when $\Re(c - a - b) > 0$.
- converges conditionally when $-1 < \Re(c - a - b) \leq 0$ and $z \neq 1$.
- diverges when $\Re(c - a - b) \leq -1$.

It is not defined when $c = 0, -1, -2, \ldots$.

The Gauss series $\boxed{{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}}$ defined on the disk $|z| < 1$ and by analytic continuation elsewhere,
is the **Hypergeometric Function** (*aka* the **Gauss function**).

(see DLMF/Chapter15)

Some properties:

- ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = {}_2F_1\left(\begin{matrix} b, a \\ c \end{matrix}; x\right)$
- ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 0\right) = 1$
- $\frac{\mathrm{d}}{\mathrm{d}x} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{ab}{c} \, {}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right)$ (prove this formally!)
- Diverges if $x = 1$ and $\Re(c - a - b) \leq 0$.

Since

$$\frac{(b)_k}{(c)_k} = \frac{1}{B(b,c-b)} B(b+k,c-b) = \frac{1}{B(b,c-b)} \int_0^1 s^{b+k-1}(1-s)^{c-b-1}\mathrm{d}s$$

then we can write

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{k\geq 0} \frac{(a)_k x^k}{k!} \frac{1}{B(b,c-b)} \int_0^1 s^{b+k-1}(1-s)^{c-b-1}\mathrm{d}s$$

The series $\sum_{k\geq 0} \frac{(a)_k x^k}{k!} s^{b+k-1}(1-s)^{c-b-1}$ converges uniformly with respect to $s \in (0,1)$. Therefore we can interchange the oder of integration and summation for b, c, x s.t. $\Re(b) > 1$, $\Re(c-b) > 1$ and $|x| < 1$, so that

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{1}{B(b,c-b)} \int_0^1 s^{b-1}(1-s)^{c-b-1} \underbrace{\sum_{k\geq 0} \frac{(a)_k (xs)^k}{k!}}_{(1-xs)^{-a}}$$

and we have

$$\boxed{ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 s^{b-1}(1-s)^{c-b-1}(1-xs)^{-a}\mathrm{d}s}$$

When $\Re(c-a-b)>0$, then

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} $$

If, in addition, $a=-n$ with $n\in\mathbb{N}$, then

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; 1\right) = \frac{(c-b)_n}{(c)_n} $$

(Chu-Vandermonde's formula)

Remark.

- When $\Re(c-a-b)<0$, then $\displaystyle\lim_{x\to 1^-}\frac{{}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right)}{(1-x)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$.
- When $c=a+b$, then $\displaystyle\lim_{x\to 1^-}\frac{{}_2F_1\left(\begin{smallmatrix} a,b \\ a+b \end{smallmatrix}; x\right)}{-\log(1-x)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$.

When $c = b - a + 1$, then

$$\begin{aligned}\lim_{x\to -1} {}_2F_1\left(\begin{matrix} a, b \\ b-a+1 \end{matrix}; x\right) &= \frac{\Gamma(1+b-a)}{\Gamma(b)\Gamma(1-a)} \int_0^1 (1-t^2)^{-a} t^{b-1} \mathrm{d}t \\ &= \frac{\Gamma(1+b-a)}{\Gamma(b)\Gamma(1-a)} \frac{1}{2} \int_0^1 (1-\zeta)^{-a} \zeta^{b/2-1} \mathrm{d}\zeta \\ &= \frac{1}{2} \frac{\Gamma(1+b-a)}{\Gamma(b)\Gamma(1-a)} B\left(\frac{b}{2}, 1-a\right)\end{aligned}$$

and we obtain the Kummer's result

$$\boxed{{}_2F_1\left(\begin{matrix} a, b \\ b-a+1 \end{matrix}; -1\right) = \frac{\Gamma(1+b-a)\Gamma(1+\frac{b}{2})}{\Gamma(b+1)\Gamma(1-a+\frac{b}{2})}}$$

Hypergeometric function: some identities

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{b-1} t^{c-b-1} (1 - x(1-t))^{-a} \mathrm{d}t \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} (1-x)^{-a} \int_0^1 (1-t)^{b-1} t^{c-b-1} (1 - \frac{x}{x-1} t)^{-a} \mathrm{d}t \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \left(\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\right)^{-1} (1-x)^{-a} {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{x}{x-1}\right)
\end{aligned}
$$

so that

$$
\boxed{{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{x}{x-1}\right)}
$$

- For fixed $x \in [-1, 1]$, the function

$$ {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) $$

is an entire function of a and b, and a meromorphic function in c with simple poles at $c = -n$, for $n = 0, 1, 2, \ldots$.

- For fixed $x \in [-1, 1]$, the function

$$ \frac{1}{\Gamma(c)} \, {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) $$

is an entire function of a, b and c.

The Gauss function ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)$ is a solution to the 2nd order differential equation

$$\boxed{x(1-x)y'' + (c-(a+b+1)x)y' - ab\ y = 0} \tag{12}$$

known as the **hypergeometric differential equation**.

(Here $y' := \frac{\mathrm{d}y}{\mathrm{d}x}$.)

It has three regular singular points at $0,\ 1$ and ∞. Why is that?

Given a differential equation of the form

$$y'' + p(x)y' + q(x)\ y = 0 \tag{13}$$

with $p : D \to \mathbb{C}$ and $p : D \to \mathbb{C}$.

A point $x = a \in D$ is

- a **regular point** of the differential equation (13) if both $p(x)$ and $q(x)$ are analytic at $x = a$;
- a **regular singular point** of the differential equation (13) if $p(x)$ and $q(x)$ are not analytic at $x = a$ but $(x-a)p(x)$ and $(x-a)^2q(x)$ are analytic at $x = a$;
- a **singular point** otherwise.

In the case of the hypergeometric equation

$$y'' + \underbrace{\frac{(c-(a+b+1)x)}{x(1-x)}}_{p(x)} y' - \underbrace{\frac{ab}{x(1-x)}}_{q(x)} \; y = 0$$

we see that $x = 0$ and $x = 1$ are two regular singular points of the equation.

To analyse the nature of the point at ∞, we consider the transformation $x \to \frac{1}{t}$, so that we have

$$\tilde{y}'' + \underbrace{\left(\frac{2}{t} - \frac{1}{t^2} p\left(\frac{1}{t}\right)\right)}_{\dfrac{1-(a+b)-(2-c)t}{t(1-t)}} \tilde{y}' + \underbrace{\frac{1}{t^4} q\left(\frac{1}{t}\right)}_{\dfrac{ab}{t^2(1-t)}} \tilde{y} = 0$$

which has $t = 0$ as a regular singular point and therefore $x = \infty$ is a regular singular point of the original equation.

All the other points are regular.

Finding a solution

Following the Frobenius method (1873), we seek a solution to the differential equation

$$(x-a)^2\left(y'' + p(x)y' + q(x)\ y\right) = 0$$

around a regular singular point $x = a$ by finding μ and an expression to the coefficients c_n in the expansion

$$y(x) = (x-a)^{\mu}\sum_{n=0}^{+\infty} c_n (x-a)^n$$

in which the series converges in a neighbourhood of $x = a$, therefore, defining an analytic function.

Assuming

$$(x-a)p(x) = \sum_{n\geq 0} p_n (x-a)^n \quad \text{and} \quad (x-a)^2 q(x) = \sum_{n\geq 0} q_n (x-a)^n$$

we insert the expression for $y(x)$ into the equation to then equate the coefficients.

By equating the coefficients of $(x-a)^{\mu}$, we obtain

$$\boxed{\mu(\mu-1)+\mu\ p_0+q_0=0} \quad \longleftarrow \quad \text{this is the 'indicial equation'}$$

which has two roots μ_1 and μ_2 (the so-called exponents)

For the regular singular point at ∞ we use the same procedure using the transformed equation after the c.o.v. $x \to \frac{1}{t}$.

In the case of the hypergeometric differential equation

$$\boxed{x(1-x)y''+(c-(a+b+1)x)y'-ab\ y=0}$$

we have

reg. sing. point	1st exponent	2nd exponent
$x=0$	$\mu_1=0$	$\mu_2=1-c$
$x=1$	$\mu_1=0$	$\mu_2=c-a-b$
$x=\infty$	$\mu_1=a$	$\mu_2=b$

and this leads to

Hence, for

$$\boxed{x(1-x)y'' + (c-(a+b+1)x)y' - ab\ y = 0}$$

we have the following sets of fundamental solutions

- near $x = 0$:

$$y_1(x) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) \quad , \quad y_2(x) = x^{1-c}{}_2F_1\left(\begin{matrix} b-c+1, b \\ 2-c \end{matrix}; x\right)$$

- near $x = 1$:

$$y_1(x) = {}_2F_1\left(\begin{matrix} a, b \\ a+b+1-c \end{matrix}; 1-x\right), \ y_2(x) = (1-x)^{c-a-b}{}_2F_1\left(\begin{matrix} c-b, c-a \\ c-a-b+1 \end{matrix}; 1-x\right)$$

- near $x = \infty$:

$$y_1(x) = x^{-a}{}_2F_1\left(\begin{matrix} a, a-c+1 \\ a-b+1 \end{matrix}; \frac{1}{x}\right) \quad , \quad y_2(x) = x^{-b}{}_2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix}; \frac{1}{x}\right)$$

another characterisation of the hypergeometric differential equation

Theorem. Any homogeneous linear differential equation of 2nd order with at most three singularities (including perhaps one at infinity) which are regular singular points can be transformed into the hypergeometric equation.

Confluent Hypergeometric Functions (*aka* Kummer Functions)

(see DLMF/ Ch.13)

Consider the hypergeometric function ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; \frac{x}{b}\right)$ which is a solution to a differential equation that has the point $x = b$ as a regular singular point.

Now, by taking the limit as $b \to +\infty$, we obtain a new function

$$M(a, c; x) := \lim_{b \to +\infty} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; \frac{x}{b}\right)$$

which obviously results in

$$M(a, c; x) := {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}; x\right)$$

(again with $c \neq 0, -1, -2, \ldots$)

Confluent Hypergeometric Functions and corresponding diff eq.

Applying the same procedure to the hypergeometric differential equation

$$x(1-x)y'' + (c - (a+b+1)x)y' - ab\ y = 0$$

(*i.e.* changing $x \to x/b$ and then taking the limit as $b \to +\infty$) brings the so-called

confluent differential equation $\longrightarrow$ $\boxed{xy'' + (c-x)y' - a\ y = 0}$

(*aka Kummer's differential equation*)

Remarks.

- The point $x = 0$ is a regular singular point of the confluent equation.
- The limiting process we have taken merges the two regular singular at $x = b$ and $x = \infty$ in the hypergeometric diff. eq. into a single one at ∞. This point $x = \infty$ is a singular point of the confluent equation which is not regular.
  So, one cannot expect convergent series in terms of powers of $1/x$!!!
- A solution to the confluent equation is $M(a, c; x) := {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}; x\right)$
- A 2nd (independent) solution can be obtained by the same confluent process and corresponds to $x^{1-c}M(a-c+1, 2-c; x)$

Confluent Hypergeometric Functions: an integral representation

We have seen that

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 s^{a-1}(1-s)^{c-a-1}(1-xs)^{-b}\mathrm{d}s $$

provided that $\Re(a) > 0$ and $\Re(c-a) > 0$.

Since

$$ M(a, c; x) := \lim_{b\to\infty} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; \frac{x}{b}\right) $$

then it follows

$$ \boxed{M(a, c; x) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 s^{a-1}(1-s)^{c-a-1}\mathrm{e}^{xs}\mathrm{d}s} $$

for $\Re(a) > 0$ and $\Re(c-a) > 0$.

Quiz: prove the latter identity!

And from this we obtain

$$ \boxed{M(a, c; x) := \frac{\Gamma(c)\ \mathrm{e}^{x}}{\Gamma(a)\Gamma(c-a)} \int_0^1 (1-t)^{a-1}t^{c-a-1}\mathrm{e}^{-xt}\mathrm{d}t} $$

In the integral in the representation

$$\boxed{M(a, c; x) := \frac{\Gamma(c)\,\mathrm{e}^{x}}{\Gamma(a)\Gamma(c-a)} \int_0^1 (1-t)^{a-1} t^{c-a-1} \mathrm{e}^{-xt} \mathrm{d}t}$$

we recognise a Laplace integral on a bounded interval.

So it makes sense to make use of

Watson's Lemma. Suppose that

(a) $f(t)$ is a (real or complex) function of $t > 0$ with a finite number of discontinuities;

(b) $f(t) \sim t^{\lambda-1} \sum_{n \geq 0} a_n t^n$ as $t \to 0^+$ with $\Re\lambda > 0$;

(c) $F(x) = \int_0^\infty f(t)\mathrm{e}^{-xt}\mathrm{d}t$ is convergent for sufficient large values of x

then

$$F(x) \sim \sum_{n \geq 0} \Gamma(n+\lambda) \frac{a_n}{x^{n+\lambda}} \quad \text{as} \quad x \to \infty,$$

provided that $|\arg x| < \pi/2$ when $z^{n+\lambda}$ has its principal value.

Indeed, we have

$$(1-t)^{a-1} = \sum_{n\geq 0} \frac{(1-a)_n}{n!} t^n$$

and

$$\int_0^1 \frac{(1-a)_n}{n!} t^{n+c-a-1} \mathrm{e}^{-xt} \mathrm{d}t = \frac{(1-a)_n x^{-n-c+a}}{n!} \int_0^x s^{n+c-a-1} \mathrm{e}^{-s} \mathrm{d}t$$

so that

$$\boxed{M(a, c; x) \sim \frac{\Gamma(c)\ \mathrm{e}^x x^{a-c}}{\Gamma(a)} \sum_{n\geq 0} \frac{(c-a)_n (1-a)_n}{n!} x^{-n}} \quad \text{as} \quad x \to \infty,$$

and this is valid in the sector $|\arg x| < \pi/2$.

The general solution of the confluent differential equation

$$xy'' + (c - x)y' - a\ y = 0$$

can be written as

$$y(x) = A\ M(a, c; x) + B\ x^{1-c} M(a - c + 1, 2 - c; x)$$

assuming $c \neq 0, -1, \ldots$.

- Both functions are analytic at 0, producing two independent solutions;
- As $x \to \infty$, the general solution presented above

$$y(x) \sim \left(A \frac{\Gamma(c)}{\Gamma(a)} + B \frac{\Gamma(2-c)}{\Gamma(a-c+1)} \right) \mathrm{e}^x x^{a-c} \sum_{n \geq 0} \frac{(c-a)_n (1-a)_n}{n!} x^{-n}$$

- When A and B are chosen such that $A\frac{\Gamma(c)}{\Gamma(a)} + B\frac{\Gamma(2-c)}{\Gamma(a-c+1)} = 0$ (which is possible), this does not mean that the function will vanish. Rather, we expect the solution to be of lower order in terms of behaviour (not behaving as $\mathrm{e}^x \times$ (algebraic function).

The confluent/Kummer function of 2nd kind

By taking $A = \dfrac{\Gamma(1-c)}{\Gamma(a-c+1)}$ and $B = \dfrac{\Gamma(c-1)}{\Gamma(a)}$, we obtain

$$\boxed{U(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} M(a,c;x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} M(a-c+1, 2-c; x)}$$

which has a meaning for all values of x, a and c with exception at the point $x = 0$ (where U is in general singular).

Observe that

- if we seek to the confluent equation of the form $v(x) = \int_\alpha^\beta \mathrm{e}^{-xt}\phi(t)\mathrm{d}t$ for some integrable function $\phi(t)$, then we obtain $\phi(t) = \tilde{A}t^{a-1}(1+t)^{c-a-1}$ and that $t\phi(t) \underset{t\to 0}{\longrightarrow} 0$ (if $a > 0$) so that $\phi'(t)$ is integrable and we have

$$U(a,c;x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{c-a-1}\mathrm{e}^{xt}\mathrm{d}t, \quad \text{with} \quad a, x > 0.$$

- Using Watson's Lemma,

$$U(a,c;x) \sim x^{-a} \sum_{n\geq 0} \frac{(a-c+1)_n (a)_n}{n!} (-x)^{-n} \quad \text{as} \quad x \to \infty,$$

in the sector $|\arg x| < 3\pi/2$.

The confluent/Kummer function of 2nd kind (cont.)

- $U(a, c; x) = \dfrac{1}{\Gamma(a)} \displaystyle\int_0^\infty t^{a-1}(1+t)^{c-a-1}\mathrm{e}^{xt}\mathrm{d}t, \quad \text{with} \quad a, x > 0.$
- $U(a, c; x) \sim x^{-a} \displaystyle\sum_{n \geq 0} \frac{(a-c+1)_n (a)_n}{n!}(-x)^{-n} \quad \text{as} \quad x \to \infty,$
  in the sector $|\arg x| < 3\pi/2$.
- The U-function also satisfies the functional equation:

$$U(a, c; x) = x^{1-c}U(a-c+1, 2-c; x).$$

- $M(a, a; x) = \mathrm{e}^x$
- Laguerre polynomials: $L_n(x; \alpha) = M(-n, \alpha + 1; x)$
- incomplete Gamma Functions

$$\gamma(a, x) = \int_0^x t^{a-1} \mathrm{e}^{-t} \mathrm{d}t = a^{-1} x^a M(a, a + 1; -x)$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \mathrm{e}^{-t} \mathrm{d}t = x^a \mathrm{e}^{-x} U(1, a + 1; x) = \mathrm{e}^{-x} U(1 - a, 1 - a; x)$$

- error functions

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \mathrm{d}t = x M(\tfrac{1}{2}, \tfrac{3}{2}; -x^2)$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2} \mathrm{d}t = \mathrm{e}^{-x^2} U(\tfrac{1}{2}, \tfrac{1}{2}; x^2)$$

- **Bessel functions**

$$J_\nu(x) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu \mathrm{e}^{ix} M(\nu + \tfrac{1}{2}, 2\nu + 1, 2ix)$$

$$K_\nu(x) = \sqrt{\pi}\, (2x)^\nu\, \mathrm{e}^{-x} U(\nu + \tfrac{1}{2}, 2\nu + 1, 2x)$$

The study of these functions started with Bessel (1824). Consider the expansion (on a Laurent series)

$$\exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right)=\sum_{n=-\infty}^{n=+\infty} J_n(z)t^n \ .$$

The substitution $t \to -\frac{1}{t}$ implies

$$J_{-n}(z)=(-1)^n J_n(z),\ \forall n\in\mathbb{Z},$$

so that

$$\exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right)=J_0(z)+\sum_{n=0}^{n=+\infty}\left(t^n+(-1)^n t^{-n}\right)J_n(z)$$

Besides,

$$J_0(z) + \sum_{n=0}^{n=+\infty} \left(t^n + (-1)^n t^{-n}\right) J_n(z) = \exp\left(\frac{z}{2}\left(t - \frac{1}{t}\right)\right)$$

$$= \exp\left(\frac{zt}{2}\right)\ \exp\left(-\frac{z}{2t}\right) = \left(\sum_{n=0}^{+\infty} \frac{(z/2)^n t^n}{n!}\right)\left(\sum_{n=0}^{+\infty} \frac{(-z/2)^n t^{-n}}{n!}\right)$$

implies (using the Cauchy product of two series)

$$\boxed{J_n(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!}, \quad n = 0, 1, 2, \ldots}$$

and the latter series is convergent for any x and any integer $n \in \mathbb{Z}$. Moreover,

$$|J_n(z)| \leq \frac{|z/2|^n}{n!} \exp(z^2/4), \quad n \geq 0.$$

Bessel functions: J_n (cont.)

A differentiation of $\exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{n=+\infty} J_n(z)t^n$ with respect to t (after justification) implies

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z).$$

A differentiation of $\exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{n=+\infty} J_n(z)t^n$ with respect to z (after justification) implies

$$J_{n-1}(z) - J_{n+1}(z) = 2\frac{\mathrm{d}}{\mathrm{d}z} J_n(z).$$

From the previous two relations, we conclude that $y(z) = J_n(z)$ is a solution to the 2nd order differential equation

$$\boxed{y'' + \frac{1}{z}y' + \left(1 - \frac{n^2}{z^2}\right) y = 0}$$

Allowing the integer n to be replaced by an arbitrary parameter ν in the latter diff. eq. leads to the so-called

$$\text{Bessel's differential equation} \longrightarrow \boxed{y'' + \frac{1}{z}y' + \left(1 - \frac{\nu^2}{z^2}\right) y = 0}$$

or, equivalently,

$$\text{Bessel's differential equation} \longrightarrow \boxed{z^2 y'' + zy' + \left(z^2 - \nu^2\right) y = 0}$$

Bessel's differential equation (cont.)

Bessel's differential equation $\longrightarrow$ $\boxed{z^2 y'' + zy' + \left(z^2 - \nu^2\right) y = 0}$

Remarks.

- The point $z = 0$ is a regular singular point, with $\mu^2 - \nu^2 = 0$ as indicial equation, whose roots are $\mu = \pm\nu$. Frobenius method allows to conclude that the general solution to the Bessel's diff eq. can be written as

$$y(z) = A\ J_\nu(z) + B\ J_{-\nu}(z)$$

  when $\nu \notin \mathbb{Z}$, where

$$J_\nu(z) = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} {}_0F_1\left(\frac{-}{\nu + 1}; -\frac{z^2}{4}\right)$$

- When $\nu = -n$ the two solutions described above are not independent. This case requires further analysis and will give rise to the Y-Bessel function.

A substitution of z by iz in the Bessel's differential gives the

modified Bessel's differential equation $\longrightarrow$ $\boxed{z^2 y'' + zy' + \left(z^2 + \nu^2\right) y = 0}$

which has the pair of modified Bessel functions

$$I_\nu(z) \quad \text{and} \quad K_\nu(z)$$

as independent solutions.

They admit the following series representation

$$I_\nu(z) = i^{-\nu} J_\nu(ix) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} {}_0F_1\left({- \atop \nu+1}; \frac{z^2}{4} \right)$$

and

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu} - I_\nu}{\sin(\pi\nu)}.$$

Clearly,

$$I_{-n}(z) = I_n(z) \quad \text{and} \quad K_{-\nu}(z) = K_\nu(z)$$

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  Comments: See Chapters. 1, 5 and 9

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