ORTHOGONAL POLYNOMIALS AND SPECIAL FUNCTIONS

Part 2

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Outline

- Part 2. Orthogonal Polynomials: an introduction
  - Main properties
    Recurrence relations, zeros, distribution of the zeros and so on and on....
  - Classical Orthogonal Polynomials
    Hermite, Laguerre, Bessel and Jacobi!!
  - Other notions of "classical orthogonal polynomials"
    How to identify this on the Askey Scheme?
  - Semiclassical Orthogonal Polynomials
    How do these link to Random Matrix Theory, Painlevé equations and so on?
- Part 3. Multiple Orthogonal Polynomials
  When the orthogonality measure is spread across a vector of measures?

Let $\mathscr{P}$ be the vector space of polynomials $\mathscr{P}$ defined as

$$\mathscr{P} = \bigcup_{n=0}^{+\infty} \mathscr{P}_n$$

where $\mathscr{P}_n$ represents the finite dimensional vector space of polynomials of degree $\leq n$ with complex coefficients.

Consider a sequence of polynomials

$$\boxed{\{P_n\}_{n\geq 0} \subset \mathscr{P} \quad \text{such that} \quad \deg P_n(x) = n}$$

- Clearly $\{P_n\}_{n\geq 0}$ forms a basis for the vector space of polynomials $\mathscr{P}$ of complex coefficients.
- It is a **monic polynomial sequence** if $\deg(P_n - x^n) < n$

Each $\boxed{\{P_n\}_{n\geq 0}\subset \mathscr{P} \quad \text{such that} \quad \deg P_n(x)=n}$ can be defined via

- a terminating series of the form

$$P_n(x)=\sum_{k=0}^{n} c_{n,k}\ (x-a)^k,\ n\geq 0,$$

  or of the form

$$P_n(x)=\sum_{k=0}^{n} c_{n,k}\ (x-a)_k,\ n\geq 0,$$

  or in any other polynomial basis expansion. In particular, we can consider...

- a **structural relation**, which is basically the Euclidean division of $P_{n+1}(x)$ by $P_n(x)$ and this means there exist coefficients β_n and $\chi_{n,j}$ with $j\in\{0,1,\dots,n-1\}$ such that

$$P_{n+1}(x)=(x-\beta_n)P_n(x)-\sum_{j=0}^{n-1}\chi_{n,j}P_j(x). \qquad (1)$$

Preliminaries 1

Each $\boxed{\{P_n\}_{n\geq 0} \subset \mathscr{P} \quad \text{such that} \quad \deg P_n(x) = n}$ can be also defined via

- a **generating function** of exponential type

$$\Psi(x,t) = \sum_{n\geq 0} P_n(x) \frac{t^n}{n!}$$

  or of horizontal type

$$\Psi(x,t) = \sum_{n\geq 0} P_n(x) t^n.$$

- a lowering/raising operator $\mathscr{O}$ and a function $f(x)$ such that

$$f(x) P_n(x) = \rho_n \ \mathscr{O}^n \Big(f(x) \Big)$$

  where $\mathscr{O}^{n+1}\Big(f(x)\Big) := \mathscr{O}\left(\mathscr{O}^n\Big(f(x)\Big)\right)$ and $\mathscr{O}^0\Big(f(x)\Big) := f(x)$ and $\rho_n \neq 0$ is a normalization constant.
- a **differential-difference equation**
- *etc.*

Let μ be a **positive Borel measure** with support S defined on $\mathbb{R}$ for which **moments** of all orders exist, *i.e.* ,

$$\mu_n = \int_S x^n \mathrm{d}\mu(x) \ < \ \infty, \quad n = 0, 1, 2, \dots \ .$$

Definition

A sequence of polynomials $\{P_n\}_{n \geq 0}$ with $\deg P_n = n$ is orthogonal w.r.t. the measure μ if

$$\int_S P_k(x) P_n(x) \mathrm{d}\mu(x) = N_n \ \delta_{n,k} \quad n, k = 0, 1, 2, \dots \ .$$

where S is the support of μ and N_n is the square of the weighted L^2-norm of P_n given by

$$N_n = \int_S (P_n(x))^2 \mathrm{d}\mu(x) > 0.$$

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Lemma

A sequence of polynomials $\{P_n\}_{n\geq 0}$, with $P_n(x) = k_n x^n + \dots$ terms of lower degree, is orthogonal w.r.t. the measure μ iff

$$\boxed{\int_S x^k P_n(x)\mathrm{d}\mu(x) = N_n (k_n)^{-1} \ \delta_{n,k}} \quad \textit{if } n \textit{ and } k \textit{ are integers s.t.} \quad \boxed{0 \leq k \leq n}.$$

where S is the support of μ and N_n is the square of the weighted L^2-norm of P_n given by

$$(k_n)^{-1} N_n = \int_S x^n P_n(x)\mathrm{d}\mu(x) > 0.$$

Proof. Exercise.

When the measure μ is absolutely continuous, there exists a locally integrable function $w(x)$ defined on (a,b), (*i.e.* $w(x)$ is Lebesgue integrable over every compact subset K of (a,b)) with distributional derivative $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$ where the **moments** of all orders exist, *i.e.* ,

$$\mu_n = \int_a^b x^n w(x)\mathrm{d}x \ < \ \infty, \quad n = 0, 1, 2, \dots \ .$$

In this case, the orthogonality conditions become

$$\int_a^b P_k(x)P_n(x)w(x)\mathrm{d}x = N_n \ \delta_{n,k} \quad n, k = 0, 1, 2, \dots \ .$$

where (a,b) is the support of $w(x)$ and N_n

$$\int_a^b (P_n(x))^2 w(x)\mathrm{d}x = N_n > 0.$$

1. *Chebyshev polynomials*: $\{T_n\}_{n\geq 0}$ defined by $T_n(x)=\cos(n\theta)$, where $x=\cos(\theta)$, with $\theta\in(0,\pi)$. We have

$$\begin{aligned}\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}\mathrm{d}x &= \int_0^{\pi}\cos(n\theta)\cos(m\theta)\mathrm{d}\theta\\ &= \int_0^{\pi}\frac{\cos((n+m)\theta)+\cos((m-n)\theta)}{2}\mathrm{d}\theta\\ &= \begin{cases} N_n & \text{if} \quad m=n\geq 0\\ 0 & \text{if} \quad m\neq n\geq 0.\end{cases}\end{aligned}$$

where

$$N_n=\begin{cases}\pi & \text{if} \quad n=0,\\ \pi/2 & \text{if} \quad n\geq 1.\end{cases}$$

2. *Laguerre polynomials*: $\{L_n(\cdot;\alpha)\}_{n\geq 0}$ defined by

$$L_n(x;\alpha) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-1)^k}{(\alpha+1)_k}\binom{n}{k}x^k$$
$$= \frac{(\alpha+1)_n}{n!}M(-n,\alpha+1;x),\ \ n\geq 0.$$

For each $\alpha > -1$, $\{L_n(x;\alpha)\}_{n\geq 0}$ satisfies the orthogonality relations

$$\int_0^{+\infty} L_n(x)L_m(x)\mathrm{e}^{-x}x^{\alpha}\mathrm{d}x = \begin{cases} \frac{\Gamma(n+1+\alpha)}{n!} & \text{if} \quad m=n \text{ and } \ n\geq 0, \\ 0 & \text{if} \quad m\neq n. \end{cases}$$

Exercise: Prove the latter identity.

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Exercise: Prove the latter identity.

Hint. Start by showing $\displaystyle\int_0^{+\infty} x^m L_n(x)\mathrm{e}^{-x}x^{\alpha}\mathrm{d}x = \frac{\Gamma(\alpha+1)\Gamma(m+\alpha+1)(-m)_n}{\Gamma(n+\alpha+1)}$.

3. *Charlier polynomials*: $\{C_n(x;\alpha)\}_{n\geq 0}$ depending on a parameter α defined by

$$C_n(x;\alpha) = n!\ L_n(\alpha; x-n),\ \ n \geq 0,$$

is a polynomial sequence with $\deg C_n(x;\alpha) = n$.

It is an orthogonal polynomial sequence, because it satisfies the (discrete) orthogonal relation

$$\sum_{x=0}^{+\infty} C_n(x;\alpha) C_m(x;\alpha) \frac{\alpha^x}{x!} = \begin{cases} \mathrm{e}^{\alpha}\alpha^n n! \neq 0 & \text{if} \quad m = n \text{ and } \ n \geq 0, \\ 0 & \text{if} \quad m \neq n, \end{cases}$$

under the assumption that $\alpha > 0$.

If the weight function $w(x)$ is discrete so that $w(x_k) > 0$ are the values of the weight at the distinct points $x_k,\ k = 0, 1, \ldots, M$ for $M \in \mathbb{N} \cup \{\infty\}$, then the orthogonality relations read as

$$\sum_{k=0}^{M} P_n(x_k) P_m(x_k) w(x_k) = N_n \delta_{n,m},\ \ n, m \geq 0.$$

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More generally, we can make use of the theory of distributions to define the Borel measures and further extend the orthogonality notion to the non-positive definite sense.

For that, we define...

Without entering into further details...

Consider a moment linear functional

$$\begin{array}{rcccl} \mathscr{L} & : & \mathscr{P} & \longrightarrow & \mathbb{R} \text{ (or } \mathbb{C}) \\ & & p(x) & \longmapsto & \langle \mathscr{L}, p(x) \rangle \end{array}$$

so, $\mathscr{L}$ is an element of the **dual space of** $\mathscr{P}$, denoted by $\mathscr{P}'$.

The duality pairing between a moment linear functional (or distribution) $\mathscr{L}$ in $\mathscr{P}'$ and any polynomial (in $\mathscr{P}$) will be denoted by angle brackets

$$\begin{array}{ccc} \mathscr{L}' \times \mathscr{L} & \longrightarrow & \mathbb{R} \text{ (or } \mathbb{C}) \\ (\mathscr{L}, p(x)) & \longmapsto & \langle \mathscr{L}, p(x) \rangle \end{array}$$

For instance, any locally integrable function ϕ defined on a set U yields a moment linear functional on $\mathscr{P}'$ – that is, an element of $\mathscr{P}'$ – denoted here by $\mathscr{L} := \mathscr{L}_\phi$ whose value on the space of polynomials is

$$\langle \mathscr{L}, p(x) \rangle = \int_U p(x) \cdot \phi(x) \mathrm{d}x$$

Moment linear functionals

Operations on the dual space $\mathscr{P}'$:

- are defined by means of the transpose operator, ${}^t\mathscr{L}$;
- if $\mathscr{O}$ is a continuous linear operator defined on $\mathscr{P}$, then ${}^t\mathscr{L}$ is defined by duality via

$$< {}^t\mathscr{O}\mathscr{L}, p(x) >=< \mathscr{L}, \mathscr{O}p(x) >, \quad \text{for any} \quad p \in \mathscr{P}.$$

- If

$$\langle \mathscr{L}, p(x) \rangle = \int_U p(x) \cdot \phi(x) \mathrm{d}x$$

then

$$< {}^t\mathscr{O}\mathscr{L}, p(x) >= \int_U p(x) \cdot \left({}^t\mathscr{O}\phi(x) \right) \mathrm{d}x = \int_U (\mathscr{O}p(x)) \cdot \phi(x) \mathrm{d}x$$

- For instance, given a polynomial $g(x)$ and a linear functional $\mathscr{L}$, we define:

$$< g(x)\mathscr{L}, p(x) >=< \mathscr{L}, g(x)p(x) >, \quad \text{for any} \quad p \in \mathscr{P};$$

$$< D\mathscr{L}, p(x) >= - < \mathscr{L}, Dp(x) >, \quad \text{for any} \quad p \in \mathscr{P} \quad \text{with} \quad Dp(x) := p'(x);$$

So, with some abuse of notation

$$< \mathscr{L}', p(x) >:= - < \mathscr{L}, p'(x) >$$

Lemma

A linear functional is uniquely defined by its **sequence of moments** $\{\mu_n\}_{n\geq 0}$*, which are given by*

$$\mu_n := <\mathscr{L}, x^n>, \ n \geq 0.$$

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Example of application of the operations. We have

$$(DxD - \alpha D)(x^{\alpha}\mathrm{e}^{-x}) = (x - (\alpha+1))(x^{\alpha}\mathrm{e}^{-x}).$$

So, if

$$\langle \mathscr{L}, p(x) \rangle = \int_0^{+\infty} p(x) \left(x^{\alpha}\mathrm{e}^{-x}\right) \mathrm{d}x$$

then

$$(DxD - \alpha D)\mathscr{L} = (x - (\alpha+1))\mathscr{L}$$

which implies

$$\begin{aligned} &\langle (x-(\alpha+1))\mathscr{L}, x^n \rangle \\ &= \langle (DxD - \alpha D)\mathscr{L}, x^n \rangle \\ &= \langle \mathscr{L}, (DxD + \alpha D)x^n \rangle \\ &= \langle \mathscr{L}, n(n+\alpha)x^{n-1} \rangle \end{aligned} \quad \Rightarrow \quad \mu_{n+1} - (\alpha+1)\mu_n = n(n+\alpha)\mu_{n-1}$$

Remarks. Given a polynomial $p(x)$ and a moment linear functional $\mathscr{L}$, then

1. For any coefficients a and b and polynomials $f(x)$ and $g(x)$, we have

$$< \mathscr{L}, af(x) + bg(x) >= a < \mathscr{L}, f(x) > +b < \mathscr{L}, g(x) > .$$

2. The image of the null polynomial is zero: $< \mathscr{L}, 0 >= 0$.
3. If $\mathscr{L} = 0$, then $< \mathscr{L}, P_n(x) >= 0$.
4. $< \mathscr{L}, P_n(x) >= 0$ does not imply (in general) that $\mathscr{L} = 0$.

Example.

$$\int_0^\infty \mathrm{e}^{-x^{1/4}} \sin(x^{1/4})\ x^n \mathrm{d}x = 0,\ n \geq 0,$$

(and therefore $\int_0^\infty \mathrm{e}^{-x^{1/4}} \sin(x^{1/4})\ f(x) \mathrm{d}x = 0$, for any polynomial $f(x)$).

In fact,

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-x^{1/4}} \sin(x^{1/4})\ x^n \mathrm{d}x \\ &= -2i \int_0^{+\infty} u^{4n+3} \left(\mathrm{e}^{-(1+i)u} - \mathrm{e}^{-(1-i)u} \right) \mathrm{d}u = \frac{2i(4n+3)!}{(1+i)^{4n+4}} + \frac{2i(4n+3)!}{(1-i)^{4n+4}} = 0 \end{aligned}$$

Definition

A polynomial sequence $\{P_n\}_{n\geq 0}$ is said to be orthogonal if there exists a linear functional $\mathscr{L}$ such that

$$\langle \mathscr{L}, P_n P_k \rangle = N_n \delta_{n,k} \ , \text{ with } \ N_n \neq 0.$$

with $N_n \neq 0$ for any $n \geq 0$. In this case we say that $\{P_n\}_{n\geq 0}$ is an **orthogonal polynomial sequence (OPS)** for $\mathscr{L}$.

- Equivalently, $\{P_n\}_{n\geqslant 0}$ is an OPS for $\mathscr{L}$ iff

$$\langle \mathscr{L}, x^m P_n \rangle = \begin{cases} 0 & \text{if } \ n > m \geq 0, \\ N_n & \text{if } \ n = m, \text{ for } \ n \geq 0. \end{cases}$$

When $N_n = 1$ for all $n \geq 0$, then $\{P_n\}_{n\geqslant 0}$ is an **orthonormal** sequence for $\mathscr{L}$.

Lemma

Let $\{P_n\}_{n\geq 0}$ be an OPS for $\mathscr{L}$. Any polynomial $\pi(x)$ of degree $m \geq 0$ can be expanded on the basis $\{P_n\}_{n\geq 0}$ of $\mathscr{P}$

$$\pi(x) = \sum_{k=0}^{m} c_k P_k(x)$$

and the coefficients are given by

$$c_k = \frac{< \mathscr{L}, \pi(x) P_k(x) >}{< \mathscr{L}, P_k^2(x) >}, \; k = 0, 1, \ldots m.$$

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Questions:

- Given a linear functional, is it possible to always find an OPS for it? If not, which necessary and/or sufficient conditions that a linear functional needs to fulfil?
- If an OPS for a certain linear functional exists, is it unique?

Corollary

Suppose that $\{P_n\}_{n\geq 0}$ *is an OPS for* $\mathscr{L}$*. If* $\{Q_n\}_{n\geq 0}$ *is also an OPS for* $\mathscr{L}$*, then there are constants* $c_n \neq 0$*, with* $n \geq 0$*, such that*

$$Q_n(x) = c_n P_n(x),\ \ n \geq 0.$$

Proof. Exercise.

- So, an OPS $\{P_n\}_{n\geq 0}$ for $\mathscr{L}$ is uniquely determined if we fix a condition for the leading coefficient, that is, the coefficient of x^n in $P_n(x)$.
- We will mainly consider **monic OPS**s (unless said otherwise)
- The corresponding **orthonormal polynomial sequence** of an OPS $\{P_n\}_{n\geq 0}$ is

$$p_n(x) = \left(< \mathscr{L}, P_n^2(x) >\right)^{-1/2} P_n(x),\ \ n \geq 0.$$

- If $\{P_n\}_{n\geq 0}$ is an OPS for $\mathscr{L}$, then it also is an OPS for any multiple of $\mathscr{L}$, that is, it is also an OPS for $\widetilde{\mathscr{L}} = c\,\mathscr{L}$ for any fixed constant $c \neq 0$

Theorem

A necessary and sufficient condition for existence of an OPS $\{P_n\}_{n\geq 0}$ for a given linear functional $\mathscr{L}$ is that

$$\Delta_n(\mathscr{L}) := \det[\mu_{j+k}]_{0\leq j,k\leq n} = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{vmatrix} \neq 0, \quad \textit{for all} \quad n\geq 0.$$

The determinant $\Delta_n(\mathscr{L})$ is known as the **Hankel determinant**.

Proof. Suppose that $\{P_n\}_{n\geq 0}$ is an OPS for $\mathscr{L}$. For any $n\geq 0$, $\exists c_{n,k}$ so that $P_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$ and this expansion is unique. The linearity of the linear functional $\mathscr{L}$ allows to express

$$< \mathscr{L}, x^m P_n(x) > = \sum_{k=0}^{n} c_{n,k} < \mathscr{L}, x^{k+m} > = \sum_{k=0}^{n} c_{n,k}\ \mu_{k+m}.$$

On the other hand we also have

$$< \mathscr{L}, x^m P_n(x) > = \begin{cases} 0 & \text{if} \quad m \leq n, \\ K_n = < \mathscr{L}, x^n P_n(x) > \neq 0 & \text{if} \quad m = n. \end{cases}$$

This information can be summarised in the following system of equations:

$$\begin{bmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{bmatrix} \begin{bmatrix} c_{n,0} \\ c_{n,1} \\ \vdots \\ c_{n,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ K_n \end{bmatrix}. \tag{2}$$

with $K_n = < \mathscr{L}, x^n P_n(x) >$.

Since the system has always a unique solution, then $\Delta_n(\mathscr{L}) \neq 0$, for any $n \geq 0$.

Conversely, if $\Delta_n(\mathscr{L}) \neq 0$, for any $n \geq 0$, the system (2) has a unique nonzero solution which is obtained for any given $K_n \neq 0$, for all $n \geq 0$. Therefore for each $n \geq 0$, a polynomial $P_n(x)$ exists. Moreover, an application of Cramer's rule to the system (2) yields

$$c_{n,n} = \frac{\Delta_{n-1}\ K_n}{\Delta_n} \neq 0,\ n \geq 1.$$

For $n = 0$, we have $c_{0,0} = K_0/\Delta_0$, as we have defined $\Delta_{-1} := 0$. $\square$

Exercise 1. Show that if $\{P_n\}_{n\geqslant 0}$ is a monic OPS for $\mathscr{L}$, then

$$P_n(x) = (\Delta_{n-1})^{-1} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

Exercise 1. Show that if $\{P_n\}_{n\geqslant 0}$ is a monic OPS for $\mathscr{L}$, then

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Exercise 2. Let $\{\phi_n\}_{n\geqslant 0}$ a monic polynomial sequence. What is the relation between the polynomials $Q_n(x)$ and $P_n(x)$ if

$$Q_n(x) = (\Delta_{n-1})^{-1} \begin{vmatrix} \widetilde{\mu}_{0,0} & \widetilde{\mu}_{0,1} & \cdots & \widetilde{\mu}_{0,n} \\ \widetilde{\mu}_{1,0} & \widetilde{\mu}_{1,1} & \cdots & \widetilde{\mu}_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{\mu}_{n-1,0} & \widetilde{\mu}_{n-1,1} & \cdots & \widetilde{\mu}_{n-1,n} \\ \phi_0(x) & \phi_1(x) & \cdots & \phi_n(x) \end{vmatrix},$$

with $\widetilde{\mu}_{i,j} = \mathscr{L}[x^i \phi_j(x)],\ i,j \geqslant 0$.

Theorem

A monic polynomial sequence $\{P_n\}_{n\geq 0}$ is orthogonal for a linear functional Ł if and only if there exist constants β_n and $\gamma_{n+1} \neq 0$ for $n \geq 0$ so that

$$\begin{aligned} &P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \;\; n \geq 0, \\ &P_0(x) = 1 \qquad \text{and} \qquad P_1(x) = x - \beta_0. \end{aligned} \tag{3}$$

In this case, we have

$$\beta_n = \frac{\langle \mathscr{L}, xP_n^2 \rangle}{\langle \mathscr{L}, P_n^2 \rangle} \qquad \text{and} \qquad \gamma_{n+1} = \frac{\langle \mathscr{L}, P_{n+1}^2 \rangle}{\langle \mathscr{L}, P_n^2 \rangle} \neq 0,\; n \in \mathbb{N}$$

Proof. ($\Rightarrow$) Suppose $\{P_n\}_{n\geq 0}$ is a monic OPS for $\mathscr{L}$. Since $\deg P_n(x) = n$ then

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \sum_{j=0}^{n-1} \chi_{n,j} P_j(x). \tag{4}$$

so that

$$< \mathscr{L}, xP_n(x)P_k(x) > = < \mathscr{L}, P_{n+1}(x)P_k(x) > + \beta_n < \mathscr{L}, P_n(x)P_k(x) > \\ + \sum_{j=0}^{n-1} \chi_{n,j} < \mathscr{L}, P_j(x)P_k(x) > .$$

From the orthogonality conditions, we obtain

$$\beta_n = \frac{< \mathscr{L}, xP_n^2(x) >}{< \mathscr{L}, P_n^2(x) >}, \quad \chi_{n,n-1} = \frac{< \mathscr{L}, xP_{n-1}(x)P_n(x) >}{< \mathscr{L}, P_{n-1}^2(x) >} \neq 0, \; n \geq 1,$$

and

$$\chi_{n,j} = \frac{< \mathscr{L}, xP_j(x)P_n(x) >}{< \mathscr{L}, P_j^2(x) >} = 0 \quad \text{for} \quad j = 0, 1, \ldots n-2 \; \text{ and } \; n \geq 2.$$

Consequently, the structural relation (4) can be written as in (3), with

$$\gamma_{n+1} = \chi_{n+1,n} \neq 0, \; n \geq 0.$$

($\Leftarrow$) Let β_n and $\gamma_{n+1} \neq 0$ and $\{P_n\}_{n\geq 0}$ be such that

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \; n \geq 1, \tag{5}$$

Since a linear functional is uniquely determined by its sequence of moments, it can be inductively defined by

$$< \mathscr{L}, 1 >= \mu_0 \neq 0, \quad < \mathscr{L}, P_n(x) >= 0, \; n \geq 0. \tag{6}$$

Hence, $< \mathscr{L}, P_1(x) >= \mu_1 - \beta_0\mu_0$ implies $\mu_1 = \beta_0\mu_0$.
Next, $< \mathscr{L}, P_2(x) >= \mu_2 - (\beta_0 + \beta_1)\mu_1 + (\beta_0\beta_1 - \gamma_1)\mu_0$ gives μ_2 and so on.

Now, (5) implies $< \mathscr{L}, 1 >= \mu_0 \neq 0$ and

$$< \mathscr{L}, xP_n(x) >= 0, \; n \geq 1, \qquad < \mathscr{L}, x^2 P_n(x) >= 0, \; n \geq 2.$$

and, by induction, $< \mathscr{L}, x^k P_n(x) >= 0$, for any $k = 0, \ldots n-1$ and $n \geq 1$, whilst

$$< \mathscr{L}, x^n P_n(x) >= \gamma_n < \mathscr{L}, x^{n-1} P_{n-1}(x) >, \text{ for any } n \geq 1. \qquad \square$$

A 2nd order recurrence relation for an OPS: remarks

- Proof does not give explicit information about measure or support.
- Measure representation for the linear functional need not be unique and depends on Hamburger moment problem
- Can be traced back to earlier work on continued fractions with a rudimentary form given by Stieltjes in 1894;
- Also appears in books by Wintner [1929] and Stone [1932].
- Often referred to as Favard's theorem but was in fact independently discovered by Favard, Shohat and Natanson around 1935. We nowadays often call it the *spectral theorem*.

Let $\{P_n\}_{n\geq 0}$ be orthogonal for $\mathscr{L}$ satisfying

$$P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \;\; n \geq 0,$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x - \beta_0$.

- $\{P_n\}_{n\in\mathbb{N}}$ is real if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} \in \mathbb{R}-\{0\}$ and all the moments of $\mathscr{L}$ are real.

- $\mathscr{L}$ is positive-definite if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$ and this implies $\Delta_{n+1}(u_0) > 0$. Consequently,

$$\langle \mathscr{L}, x^{2n}\rangle \; > \; 0 \quad \text{and} \quad \langle \mathscr{L}, x^{2n+1}\rangle \in \mathbb{R}.$$

Exercise. Show the latter condition on the moments for $\mathscr{L}$.

- $\mathscr{L}$ is negative definite if and only if it is real and $\Delta_{4n+1}(u_0) < 0$, $\Delta_{4n+2}(u_0) < 0$, $\Delta_{4n+3}(u_0) > 0$, $\Delta_{4n+4}(u_0) > 0$

Let $\{P_n\}_{n\geq 0}$ be orthogonal for $\mathscr{L}$ satisfying

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \ n \geq 0,$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x - \beta_0$.

If $\widetilde{P}_n(x) = a^{-n}P_n(ax+b)$ with $a \neq 0$, then $\{\widetilde{P}_n\}_{n\geq 0}$ is also orthogonal and satisfies

$$\widetilde{P}_{n+2}(x) = \left(x - \frac{\beta_{n+1} - b}{a}\right)\widetilde{P}_{n+1}(x) - \frac{\gamma_{n+1}}{a^2}\widetilde{P}_n(x), \ n \geq 0,$$

with initial conditions $\widetilde{P}_0(x) = 1$ and $\widetilde{P}_1(x) = x - \frac{\beta_0 - b}{a}$.

When an OPS $\{B_n\}_{n\geq 0}$ is not monic, there exists a corresponding monic OPS $\{P_n\}_{n\geq 0}$ so that $B_n(x) = k_n P_n(x)$, for all $n \geq 0$. As an OPS, $\{B_n\}_{n\geq 0}$ satisfies a second order recurrence relation. So, assuming that (3) holds, then $\{B_n\}_{n\geq 0}$ is such that

$$B_{n+1}(x) = (a_n x - b_n) B_n(x) - c_n B_{n-1}(x), \ n \geq 1 \tag{7}$$

where

$$a_n = \frac{k_{n+1}}{k_n}, \quad b_n = \frac{k_{n+1}}{k_n}\beta_n \quad \text{and} \quad c_n = \frac{k_{n+1}}{k_{n-1}}\gamma_n, \ n \geq 0, \tag{8}$$

under the assumption that $c_0 = 0$.

Exercise 2. Show that if $\{P_n\}_{n\geqslant 0}$ is a monic OPS for $\mathscr{L}$, then $P_n(x)$ is the characteristic polynomial of the matrix tri-diagonal A_n given by:

$$A_n = \begin{bmatrix} \beta_0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \gamma_1 & \beta_1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{n-2} & \beta_{n-2} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_{n-1} & \beta_{n-1} \end{bmatrix}, \ n \geqslant 0.$$

Quiz 1: What is the relation between the zeros of $P_n(x)$ and the eigenvalues of A_n?

Quiz 2: Can an OPS have complex zeros?

Suppose

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \;\; n \geq 0,$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x - \beta_0$ and assume $\gamma_n > 0$.

If $\quad B_n(x) = k_n P_n(x) \quad$ with $k_{n-1}/k_n = \sqrt{\gamma_n}$. Then B_n satisfies

$$xB_n(x) = \sqrt{\gamma_n} B_{n+1}(x) + \beta_n B_n(x) + \sqrt{\gamma_{n-1}} B_{n-1}(x), \;\; n \geq 0,$$

and we have

$$\left(\begin{bmatrix} \beta_0 & \sqrt{\gamma_1} & \cdots & 0 & 0 \\ \sqrt{\gamma_1} & \beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{n-1} & \sqrt{\gamma_{n-1}} \\ 0 & 0 & \cdots & \sqrt{\gamma_{n-1}} & \beta_n \end{bmatrix} - x\, I_{n+1} \right) \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{n-1}(x) \\ B_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\sqrt{\gamma_n} B_{n+1}(x) \end{bmatrix}$$

So we have

$$\left(\underbrace{\begin{bmatrix} \beta_0 & \sqrt{\gamma_1} & \cdots & 0 & 0 \\ \sqrt{\gamma_1} & \beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{n-1} & \sqrt{\gamma_{n-1}} \\ 0 & 0 & \cdots & \sqrt{\gamma_{n-1}} & \beta_n \end{bmatrix}}_{\mathbf{J_n}} - x\ I_{n+1} \right) \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{n-1}(x) \\ B_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\sqrt{\gamma_n} B_{n+1}(x) \end{bmatrix}$$

and J_n is a truncated Jacobi matrix, whose eigenvalues are the zeros of $B_n(x)$ (as well as those of $P_n(x)$)

So we have

$$\left(\underbrace{\begin{bmatrix} \beta_0 & \sqrt{\gamma_1} & \cdots & 0 & 0 \\ \sqrt{\gamma_1} & \beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{n-1} & \sqrt{\gamma_{n-1}} \\ 0 & 0 & \cdots & \sqrt{\gamma_{n-1}} & \beta_n \end{bmatrix}}_{\mathbf{J_n}} - x\ I_{n+1} \right) \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{n-1}(x) \\ B_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\sqrt{\gamma_n} B_{n+1}(x) \end{bmatrix}$$

and J_n is a truncated Jacobi matrix, whose eigenvalues are the zeros of $B_n(x)$ (as well as those of $P_n(x)$)

therefore

<u>all the zeros of $B_n(x)$ are simple and real.</u>

Theorem
Let $\{P_n(x)\}_{n\geq 0}$ be an OPS (for some linear functional $\mathscr{L}$) satisfying the recurrence relation (3) *with $\gamma_{n+1} \neq 0,\ n \geq 0$. Then,*

$$\frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y} = (\gamma_0\gamma_1\dots\gamma_n)\sum_{k=0}^{n}\frac{P_k(x)P_k(y)}{\gamma_0\gamma_1\cdots\gamma_k},\ n\geq 0, \tag{9}$$

under the assumption where $\gamma_0 := 1$.

Proof. Exercise.

Observe that if we take the limit as $y \to x$ in (9), then we obtain the confluent version

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) = (\gamma_0\gamma_1\dots\gamma_n)\sum_{k=0}^{n}\frac{P_k^2(x)}{\gamma_0\gamma_1\cdots\gamma_k},\ n\geq 0, \tag{10}$$

Under the assumption that $\gamma_n > 0$, then

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) = (\gamma_0\gamma_1\ldots\gamma_n)\sum_{k=0}^{n}\frac{P_k^2(x)}{\gamma_0\gamma_1\cdots\gamma_k}, \; n \geq 0, \qquad (11)$$

implies that (see [Chihara, §5.1])

- all the zeros of $P_n(x)$ are simple and real. (Exercise)
- $P_n(x)$ and $P_{n+1}(x)$ do not have common zeros. (Exercise)
- Between two consecutive zeros of $P_{n+1}(x)$ there exist exactly one zero of $P_n(x)$, *i.e.*, the zeros of P_n and P_{n+1} separate each other (interlacing propperty). (Exercise)

Under the assumption that $\gamma_n > 0$, then

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) = (\gamma_0\gamma_1 \dots \gamma_n) \sum_{k=0}^{n} \frac{P_k^2(x)}{\gamma_0\gamma_1 \cdots \gamma_k}, \ n \geq 0, \tag{11}$$

implies that (see [Chihara, §5.1])

- all the zeros of $P_n(x)$ are simple and real. (Exercise)
- $P_n(x)$ and $P_{n+1}(x)$ do not have common zeros. (Exercise)
- Between two consecutive zeros of $P_{n+1}(x)$ there exist exactly one zero of $P_n(x)$, *i.e.*, the zeros of P_n and P_{n+1} separate each other (interlacing propperty). (Exercise)

Let us consider the set of all zeros $\{x_{n,k}\}_{k=1}^{n}$ of $P_n(x)$ ordered so that

$$x_{n,1} < \cdots < x_{n,k} < x_{n,k+1} < \cdots < x_{n,n}$$

Definition. Let $E \subset (-\infty, +\infty)$. A moment linear functional $\mathscr{L}$ is said to be **positive-definite on** E iff $\langle \mathscr{L}, p(x) \rangle > 0$ for every real polynomial $p(x) \geq 0$ with $x \in E$ that does not vanish identically on E.
The set E is called a **supporting set for** $\mathscr{L}$.

Theorem. If $\mathscr{L}$ is positive-definite on E and E is an infinite set, then $\mathscr{L}$ is positive-definite on every set containing E and also on every dense subset of E.

Proof. See [Chihara,p.27].

Theorem. If E is a supporting interval for a positive-definite $\mathscr{L}$, then all the zeros of $P_n(x)$ are located in the interior of E.

Proof. Since $<\mathscr{L}, P_n(x)>=0$ (by orthogonality), then $P_n(x)$ must change sign at least once in the interior of E.

So, $\exists$ zero of odd multiplicity on located in the interior of E.

Let $z_1, \dots, z_j$ denote the distinct zeros of odd multiplicity in the interior of E and set

$$\rho(x) = (x - z_1) \cdots (x - z_j)$$

Then $\rho(x)P_n(x) \geq 0$ for $x \in E$ which implies $\langle \mathscr{L}, \rho(x)P_n(x) \rangle > 0$ and this contradicts the orthogonality conditions, unless $k = n$. $\square$

Regarding the set $\{x_{n,k}\}_{k=1}^{n}$ of all zeros of $P_n(x)$ s.t.

$$x_{n,1} < \cdots < x_{n,k} < x_{n,k+1} < \cdots < x_{n,n}$$

- For each $k \geq 1$, the sequence $\{x_{n,k}\}_{n=k}^{+\infty}$ is a decreasing sequence:

$$x_{k,k} > x_{k+1,k} > x_{k+2,k} > \ldots > x_{n+k,k} > \ldots \ ,$$

  and the limit $\qquad \zeta_i = \lim_{n\to\infty} x_{n,i}, \quad (i = 1, 2, \ldots) \quad$ exists.
- For each $k \geq 1$, the sequence $\{x_{n,n-k+1}\}_{n=k}^{+\infty}$ is an increasing sequence:

$$x_{k,1} < x_{k+1,2} < x_{k+2,3} < \ldots < x_{n+k,n+1} < \ldots \ ,$$

  and the limit $\qquad \eta_j = \lim_{n\to\infty} x_{n,n-j+1}, \quad (j = 1, 2, \ldots) \quad$ exists.

Zeros of an OPS

Regarding the set $\{x_{n,k}\}_{k=1}^{n}$ of all zeros of $P_n(x)$ s.t.

$$x_{n,1} < \cdots < x_{n,k} < x_{n,k+1} < \cdots < x_{n,n}$$

- For each $k \geq 1$, the sequence $\{x_{n,k}\}_{n=k}^{+\infty}$ is a decreasing sequence:

  $$x_{k,k} > x_{k+1,k} > x_{k+2,k} > \ldots > x_{n+k,k} > \ldots \ ,$$

  and the limit $\quad \zeta_i = \lim_{n\to\infty} x_{n,i}, \quad (i = 1, 2, \ldots) \quad$ exists.
- For each $k \geq 1$, the sequence $\{x_{n,n-k+1}\}_{n=k}^{+\infty}$ is an increasing sequence:

  $$x_{k,1} < x_{k+1,2} < x_{k+2,3} < \ldots < x_{n+k,n+1} < \ldots \ ,$$

  and the limit $\quad \eta_j = \lim_{n\to\infty} x_{n,n-j+1}, \quad (j = 1, 2, \ldots) \quad$ exists.

The closed interval $[\zeta_1, \eta_1]$, called the **true interval of orthogonality**, is:

- the smallest closed interval that contains all the zeros of all P_n;
- the smallest closed interval that is a supporting set for $\mathscr{L}$.

Definition. A polynomial sequence $\{S_n(x)\}_{n\geq 0}$ is called **symmetric** whenever

$$S_n(-x) = (-1)^n S_n(x),\ n \geq 0.$$

This means that $\exists$ $\{R_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ s.t.

$$S_{2n}(x) = R_n(x^2) \quad \text{and} \quad S_{2n+1}(x) = xQ_n(x^2),\ n \geq 0.$$

Proof. Exercise.

Definition. A linear functional $\mathscr{L}$ is called **symmetric** when
$\mathscr{L}[x^{2n+1}] = 0, \quad n \geq 0.$

For a symmetric $\mathscr{L}$, we have

$$\langle \mathscr{L}, p(-x) \rangle = \langle \mathscr{L}, p(x) \rangle, \quad \text{for any polynomial} \quad p(x).$$

Proposition. Let $\{P_n(x)\}_{n\geq 0}$ be the monic OPS for $\mathscr{L}$. The following are equivalent:

(a) $\mathscr{L}$ is symmetric.

(b) $\{P_n(x)\}_{n\geq 0}$ is symmetric, that is, $P_n(-x) = (-1)^n P_n(x),\ n \geq 0$.

(c) There exist a sequence of coefficients $\gamma_n \neq 0$ for $n \geq 1$, so that $\{P_n(x)\}_{n\geq 0}$ satisfies

$$P_{n+1}(x) = xP_n(x) - \gamma_n P_{n-1}(x)$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x$.

Proposition. Let $\{P_n(x)\}_{n\geq 0}$ be the monic OPS for $\mathscr{L}$. The following are equivalent:

(a) $\mathscr{L}$ is symmetric.

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$$P_{n+1}(x) = xP_n(x) - \gamma_n P_{n-1}(x)$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x$.

Hence, for a symmetric OPS $\{S_n(x)\}_{n\geq 0}$, then the two components of its quadratic decomposition

$$S_{2n}(x) = R_n(x^2) \quad \text{and} \quad S_{2n+1}(x) = xQ_n(x^2),\ n \geq 0.$$

are also orthogonal and they respectively satisfy

$$R_{n+1} = (x - (\gamma_{2n} + \gamma_{2n+1}))R_n(x) - \gamma_{2n}\gamma_{2n-1}R_{n-1}(x)$$
$$Q_{n+1} = (x - (\gamma_{2n+1} + \gamma_{2n+2}))Q_n(x) - \gamma_{2n}\gamma_{2n+1}Q_{n-1}(x)$$

In case $\mathscr{L}$ admits an integral representation via a weight function $W(x)$ on the interval (a,b), that is,

$$< \mathscr{L}, f(x) >= \int_a^b f(x)W(x)\mathrm{d}x, \quad \text{for any} \quad f \in \mathscr{P},$$

then $a = -b$ and $W(-x) = W(x)$ for $x \in (0,b)$.

In this case $\{S_n(x)\}_{n\geq 0}$ is an OPS for

$$< \widehat{\mathscr{L}}, f(x) >= \int_0^{b^2} f(x)\widehat{W}(x)\mathrm{d}x, \quad \text{for any} \quad f \in \mathscr{P},$$

with

$$\widehat{W}(x) = \frac{W(\sqrt{x}) + W(-\sqrt{x})}{2\sqrt{x}}$$

The (monic) Laguerre polynomials $\{\hat{L}_n(x;\alpha)\}_{n\geq 0})$ are the orthogonal polynomial components of the so-called generalised Hermite polynomials $\{S_n(x;\alpha)\}_{n\geq 0})$, which are symmetric:

$$S_{2n}(x;\alpha) = \hat{L}_n(x^2;\alpha) \qquad \text{and} \qquad S_{2n+1}(x;\alpha) = x\hat{L}_n(x^2;\alpha+1)$$

Here $\{S_n(x;\alpha)\}_{n\geq 0})$ satisfies the orthogonality relation

$$\int_{-\infty}^{+\infty} S_m(x;\alpha)S_n(x;\alpha)|x|^{2\alpha+1}\mathrm{e}^{-x^2}\mathrm{d}x = K_n\delta_{n,m}$$

whilst

$$\int_0^{+\infty} L_m(x;\alpha)L_n(x;\alpha)x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x = K_n\delta_{n,m}$$

where it was assumed that $\alpha > -1$.
The particular case where $\alpha = -\frac{1}{2}$, brings the well known relation between Hermite and Laguerre polynomials.
Furthermore,

- Hermite and Laguerre are examples of *classical orthogonal polynomials.*
- Generalised Hermite ($\alpha \neq -1/2$) is an example of a *semiclassical orthogonal polynomial sequence.*

- T. S. Chihhara, introduction to Orthogonal Polynomials, Dover Publ. (reprinted version of 1978)
- M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge Univ. Press, 2009
- G. Szegő, Orthogonal Polynomials, 4th ed., AMS Colloquium Publ. 23, AMS, 1975.

A special collection of orthogonal polynomial sequences is the so-called **classical polynomials**, which has been tremendously applied in several areas.

Definition. An OPS $\{P_n\}_{n\geqslant 0}$ for $\mathscr{L}$ is **classical** when the sequence of derivatives $\{Q_n(x)\}_{n\geqslant 0}$ defined by

$$Q_n(x) := \frac{1}{n+1} P'_{n+1}(x), \quad n \geqslant 0, \tag{12}$$

is also orthogonal. In this case, the corresponding moment linear functional $\mathscr{L}$ is said to be a **classical**.

Collectively, the classical polynomials share a number of properties.

Theorem. Let $\{P_n\}_{n\geqslant 0}$ be a monic OPS for $\mathscr{L}$. The following are equivalent:

(a) $\{Q_n(x) := \frac{1}{n+1}\, P'_{n+1}(x)\}_{n\geqslant 0}$ is a monic OPS (Hahn's property)

(b) $\exists$ polynomials Φ, Ψ with $\deg \Phi \leq 2$ and $\deg \Psi = 1$ s.t.

$$D\left(\Phi(x)\mathscr{L}\right) + \Psi(x)\mathscr{L} = 0 \qquad \text{(Pearson equation)}$$

subject to $\Psi(0) - \frac{n}{2}\Phi''(0) \neq 0$ for any $n \geq 0$.

(c) $\exists$ polynomials Φ, Ψ with $\deg \Phi \leq 2$ and $\deg \Psi = 1$ and constants λ_n s.t.

$$\Phi(x)\frac{\mathrm{d}^2 P_n}{\mathrm{d}x^2} - \Psi(x)\frac{\mathrm{d}P_n}{\mathrm{d}x} = \lambda_n P_n(x) \qquad \text{(Bochner's equation)}$$

(d) $\exists$ polynomial Φ with $\deg \Phi \leq 2$ and nonzero constants ζ_n s.t.

$$P_n(x)W(x) = \zeta_n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(\Phi^n(x)W(x)\Big), \qquad \text{(Rodrigues' formula)}$$

(a) $\Rightarrow$ (b) and (c)

The dual sequence $\{u_n\}_{n\geqslant 0}$ of $\{P_n\}_{n\geqslant 0}$ is given by

$$u_n = (< u_0, x^n P_n >)^{-1} P_n(x) u_0, \quad \text{where} \quad \mathscr{L} = u_0.$$

Likewise the orthogonality of $\{Q_n\}_{n\geqslant 0}$ implies that its corresponding dual sequence $\{v_n\}_{n\geqslant 0}$ is given by

$$v_n = (< v_0, x^n Q_n >)^{-1} Q_n(x) v_0.$$

Besides, the relation $Q_n(x) := \frac{1}{n+1} P'_{n+1}(x)$ implies

$$v'_n = -(n+1) u_{n+1}, \; n \geq 0,$$

so that, we have

$$(Q_n(x) v_0)' = -\lambda_{n+1} P_{n+1}(x) u_0, \; n \geq 0,$$

that is,

$$Q_n(x) v'_0 + Q'_n(x) v_0 = -\lambda_{n+1} P_{n+1}(x) u_0, \; n \geq 0, \tag{13}$$

where

$$\lambda_n = (n+1) \frac{< v_0, x^n Q_n(x) >}{< u_0, x^{n+1} P_{n+1}(x) >} \neq 0 \;, \; n \geq 0.$$

With $n=0$, (13) brings

$$v_0' = -\Psi(x)u_0 \quad \text{with} \quad \Psi(x) = \lambda_1 P_1(x) \tag{14}$$

which implies that (13) becomes

$$Q_n'(x)v_0 = -\Big(\lambda_{n+1}P_{n+1}(x) - \Psi(x)Q_n(x)\Big)u_0,\ n \geq 1.$$

For $n=1$, the latter reads

$$v_0 = \Phi(x)u_0 \quad \text{with} \quad \Phi(x) = -\Big(\lambda_2 P_2(x) - \lambda_1 P_1(x) Q_1(x)\Big) \tag{15}$$

and $\deg\Phi \leq 2$. After a single differentiation of the latter identity, we prove (a)$\Rightarrow$(b), because of (14).

Now, inserting (14) and (15) in the equality (13) brings

$$-Q_n(x)\Psi(x)u_0 + Q_n'(x)\Phi(x)u_0 = -\lambda_{n+1}P_{n+1}(x)u_0,\ n \geq 0.$$

Since $\{P_n\}_{n\geq 0}$ is orthogonal for u_0, we have that $f(x)u_0 = 0 \ \Leftrightarrow\ f(x) = 0$ for any polynomial $f(x)$. Consequently, we obtain

$$-Q_n(x)\Psi(x) + Q_n'(x)\Phi(x) = -\lambda_{n+1}P_{n+1}(x),\ n \geq 0.$$

Using the definition of $Q_n(x) = \frac{1}{n+1}P_{n+1}'(x)$, we prove (a)$\Rightarrow$(c).

$(c) \Rightarrow (b)$
Bochner's differential equation implies

$$0 = < u_0, \Phi(x)P_n''(x) - \Psi(x)P_n'(x) > = < \left((\Phi(x)u_0)' + \Psi(x)u_0\right)', P_n >, \; n \geq 0.$$

Since the latter is valid for any $n \geq 0$ and $\{P_n\}_{n\geq 0}$ is orthogonal, then

$$\left((\Phi(x)u_0)' + \Psi(x)u_0\right)' = 0$$

and this implies

$$(\Phi(x)u_0)' + \Psi(x)u_0 = 0$$

$(b) \Rightarrow (a)$

$$\begin{aligned} 0 &= < (\Phi(x)u_0)' + \Psi(x)u_0, x^k P_{n+1} > = < u_0, -\Phi(x)\left(x^k P_{n+1}\right)' + \Psi(x)x^k P_{n+1} > \\ &= < u_0, -x^k \Phi(x)P_{n+1}'(x) + (-k\Phi(x) + x\Psi(x))\, x^{k-1} P_{n+1} > \end{aligned}$$

Hence

$$(n+1) < \Phi(x)u_0, x^k Q_n(x) > = < u_0, \underbrace{(-k\Phi(x) + x\Psi(x))\, x^{k-1}}_{\text{degree } \leq k+1} P_{n+1}(x) >$$

$(d) \Rightarrow (b)$:The particular choice of $n=1$ in the Rodrigues formula corresponds to Pearson equation.

$(c) \Rightarrow (d)$
From the Bochner's differential equation, and on account of the Pearson equation, we can write

$$\left(P_n'(x)\Phi(x)u_0\right)' = \lambda_n P_n(x)u_0$$

Similarly, we deduce that there are coefficients $\zeta_{k,n}$ such that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}P_{n+k}(x)\right)\Phi^k(x)u_0\right)' = \zeta_{k,n}P_k(x)u_0.$$

Now Rodrigues formula is obtained from the latter by setting $n=0$. $\square$

Proposition.
If $\{P_n\}_{n\geq 0}$ is classical, then so is $\{Q_n\}_{n\geq 0}$ with $Q_n(x) = \frac{1}{n+1} P'_{n+1}(x)$ and it satisfies

$$\Phi(x) Q''_n(x) - (\Psi(x) - \Phi'(x))\, Q'_n(x) = (\chi_{n+1} + \Psi'(0)) Q_n(x), \;\; n \geqslant 0. \qquad (16)$$

where Φ and Ψ are polynomials such that $\deg \Phi \leqslant 2$, $\deg(\Psi) = 1$ and Φ monic, and

$$\chi_0 = 0 \quad \text{and} \quad \chi_n = n\left(\Psi'(0) - \frac{\Phi''(0)}{2}(n-1)\right) \neq 0 \quad \text{for} \quad n \geq 1.$$

Proof. As $\{P_n\}_{n\geq 0}$ is classical, then Bochner's differential equation holds. We differentiate both sides of the equation w.r.t. x and then replace $P'_{n+1}(x) = (n+1) Q_n(x)$ to get (16).

Since $\{Q_n\}_{n\geq 0}$ is orthogonal and satisfies (16), we conclude that $\{Q_n\}_{n\geq 0}$ is classical. $\square$

More generally, we have:

Corollary. If $\{P_n\}_{n\geq 0}$ is classical, then for each $k \geq 1$, the sequence of kth derivatives

$$\{P_n^{[k]}(x) := \frac{1}{(n+1)_k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_{n+k}(x)\}_{n\geq 0}$$

is an OPS and also classical.

Proof. After the previous characterisation Theorem for classical polynomials and the latter Proposition, the result follows by induction. $\square$

Highlights. If $\{P_n\}_{n\geq 0}$ is classical (and orthogonal w.r.t. $\mathscr{L}$), then

$$\{P_n^{[k]}(x) := \frac{1}{(n+1)_k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_{n+k}(x)\}_{n\geq 0}$$

is classical and orthogonal w.r.t. the linear functional

$$\mathscr{L}^{[k]} = \Phi^k(x)\mathscr{L}$$

- The characterisation via the Pearson equation is due to J.L. Geronimus (1940).
- In 1929, S. Bochner studied all the solutions of the differential equation

$$\Phi(x)\frac{\mathrm{d}^2 P_n}{\mathrm{d}x^2} - \Psi(x)\frac{\mathrm{d}P_n}{\mathrm{d}x} = \lambda_n P_n(x)$$

  under the restrictions of $\deg\Phi \leq 2$ and $\deg\Psi = 1$. These consisted of essentially 5 distinct families of polynomials, up to a change of variable, which are the four families of classical polynomials (Hermite, Laguerre, Bessel and Jacobi) and the sequence $\{x^n\}_{n\geq 0}$ (which is not orthogonal). At that time, Bessel polynomials were disregarded as these are not orthogonal with respect to a positive definite linear functional.
- In 1935, W. Hahn observed that all the classical families of Hermite, Laguerre, Bessel and Jacobi polynomials are such that the sequence of its derivatives is also orthogonal. Moreover, he showed this as a necessary and sufficient condition. A year later, Hahn has shown (with an extremely short proof) that in fact it is a necessary and sufficient condition for an OPS to be orthogonal that the sequence of the kth derivatives is an OPS for some $k \geq 1$.

Proposition. Suppose $\{P_n\}_{n\geq 0}$ is classical and therefore assumed to satisfy

$$\Phi(x)P_n''(x) - \Psi(x)P_n'(x) = \chi_n P_n(x)$$

Then $\widetilde{P}_n(x) := a^{-n}P_n(ax+b)$ satisfies

$$\widetilde{\Phi}(x)\widetilde{P}_n''(x) - \widetilde{\Psi}(x)\widetilde{P}_n'(x) = \widetilde{\chi}_n \widetilde{P}_n(x)$$

where

$$\widetilde{\Phi}(x) = a^{-t}\Phi(ax+b),\ \widetilde{\Psi}(x) = a^{1-t}\Psi(ax+b),\ \text{and}\ \ \widetilde{\chi}_n = a^2\chi_n\ \ \text{with}\ \ t = \deg\Phi.$$

Proof.
The result is a mere consequence of the change of variable $x \to ax+b$. $\square$

Proposition. Suppose $\{P_n\}_{n\geq 0}$ is classical and therefore assumed to satisfy

$$\Phi(x)P_n''(x) - \Psi(x)P_n'(x) = \chi_n P_n(x)$$

Then $\widetilde{P}_n(x) := a^{-n}P_n(ax+b)$ satisfies

$$\widetilde{\Phi}(x)\widetilde{P}_n''(x) - \widetilde{\Psi}(x)\widetilde{P}_n'(x) = \widetilde{\chi}_n \widetilde{P}_n(x)$$

where

$$\widetilde{\Phi}(x) = a^{-t}\Phi(ax+b),\ \widetilde{\Psi}(x) = a^{1-t}\Psi(ax+b),\ \text{and}\ \ \widetilde{\chi}_n = a^2\chi_n\ \ \text{with}\ \ t = \deg\Phi.$$

Proof.

The result is a mere consequence of the change of variable $x \to ax+b$. □

The classical character is invariant under any affine transformation

$$\begin{array}{rccc} T: & \mathscr{P} & \longrightarrow & \mathscr{P} \\ & p(x) & \longmapsto & (h_a \circ \tau_{-b})\, p(x) := p(ax+b) \end{array}$$

with $a \in \mathbb{C}^*, b \in \mathbb{C}$, because T is an isomorphism preserving the orthogonality.

Classical polynomials - an equivalence relation

The transformed classical polynomials

$$\widetilde{P}_n(x) := a^{-n}\,(TP_n)(x) := a^{-n} P_n(ax+b),$$

orhtogonal w.r.t. the classical linear functional $\widetilde{\mathscr{L}} = \left(h_{a^{-1}} \circ \tau_{-b}\right)\mathscr{L}$ satisfying

$$D\left(\widetilde{\Phi}\,\widetilde{u}_0\right) + \widetilde{\Psi}\,\widetilde{u}_0 = 0,$$

with $\widetilde{\Phi}(x) = a^{-t}\,\Phi(ax+b)$, $\widetilde{\Psi}(x) = a^{1-t}\,\Psi(ax+b)$, where $t = \deg(\Phi) \leqslant 2$

Therefore it appears to be natural to define the following equivalence relation

$$\forall\, u, v \in \mathscr{P}', \quad u \sim v \quad \Leftrightarrow \quad \exists\, a \in \mathbb{C}^*, b \in \mathbb{C} \,:\, u = \left(h_{a^{-1}} \circ \tau_{-b}\right) v\,.$$

or, equivalently,

$$\{P_n\}_{n\geq 0} \sim \{B_n\}_{n\geq 0} \quad \Leftrightarrow \quad \exists\, a \in \mathbb{C}^*, b \in \mathbb{C} \,:\, B_n(x) = a^{-n} P_n(ax+b)\,.$$

where

$$\langle \tau_{-b} u, f(x)\rangle = \langle u, \tau_b f(x)\rangle = \langle u, f(x-b)\rangle$$
$$\langle h_a u, f(x)\rangle = \langle u, h_a f(x)\rangle = \langle u, f(ax)\rangle$$

As a result, there are **four equivalence classes**, determined by the nature of Φ (monic), which are:

- Hermite polynomials when $\deg \Phi = 0$;

  We will take $\Phi(x) = 1$ as representative.

- Laguerre polynomials when $\deg \Phi = 1$;

  We will take $\Phi(x) = x$ as representative.

- Bessel polynomial when $\deg \Phi = 2$ and Φ has a single root;

  We will take $\Phi(x) = x^2$ as representative.

- Jacobi polynomials when $\deg \Phi = 2$ and Φ has two simple roots.

  We will take $\Phi(x) = (x-1)(x+1)$ as representative.

Between

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x)$$

and

$$Q_{n+2}(x) = (x - \widetilde{\beta}_{n+1})Q_{n+1}(x) - \widetilde{\gamma}_{n+1}Q_n(x),$$

we obtain

$$P_{n+1}(x) = Q_{n+1}(x) + (n+1)(\beta_{n+1} - \widetilde{\beta}_n)Q_n(x) + (n\gamma_{n+1} - (n+1)\widetilde{\gamma}_n)Q_{n-1}(x).$$

which leads to

$$\begin{aligned}
\widetilde{\gamma}_n &= \frac{n}{n+1}\vartheta_n\gamma_{n+1} \\
(n+2)\widetilde{\beta}_n - n\widetilde{\beta}_{n-1} &= (n+1)\beta_{n+1} - (n-1)\beta_n \\
\vartheta_{n+1}\widetilde{\beta}_{n+1} + (\vartheta_{n+1} - 2)\widetilde{\beta}_n &= (2\vartheta_{n+1} - 1)\beta_{n+2} - \beta_{n+1} \\
(n+1)\left(1 - \frac{n+3}{n+2}\vartheta_{n+1}\right)\gamma_{n+2} + \Big(1 + n(\vartheta_n - 1)\Big)\gamma_{n+1} + (n+1)(\beta_{n+1} - \widetilde{\beta}_n)^2 &= 0
\end{aligned}$$

where

$$\vartheta_n = \frac{(n+1)\frac{\Phi''(0)}{2} - \Psi'(0)}{(n)\frac{\Phi''(0)}{2} - \Psi'(0)}, \;\; n \geq 0.$$

This implies that $\vartheta_n = 1$ for any $n \geq 0$. so that

$$\begin{aligned}
\beta_n &= \beta_0 - (\beta_0 - \beta_1) n \\
\widetilde{\beta}_n &= \beta_0 - \frac{\beta_0 - \beta_1}{2}(2n+1) \\
\gamma_{n+1} &= (n+1)\left(\gamma_1 + \left(\tfrac{\beta_0 - \beta_1}{2}\right)^2 n\right) \\
\widetilde{\gamma}_n &= (n+1)\left(\gamma_1 + \left(\tfrac{\beta_0 - \beta_1}{2}\right)^2 (n+1)\right)
\end{aligned}$$

and, consequently,

$$\Phi(x) = k^{-1}\left(cx + c\beta_0 + \gamma_1\right) \quad \text{and} \quad \Psi(x) = k^{-1}(x - \beta_0).$$

There are two subcases to analyse depending on whether:

$\underbrace{c = 0}_{\text{Hermite polynomials}}$ or $\underbrace{c \neq 0}_{\text{Laguerre polynomials}}$

Set $\rho = -\Psi'(0)$ so that we have

$$\vartheta_n = \frac{n+\rho+1}{n+\rho} \text{ for all } n \geq 0$$

as well as

$$\begin{aligned}
\beta_n &= d + \frac{1}{2}\frac{c(\rho^2-1)(\rho+3)}{(2n+\rho+1)(2n+\rho-1)} \\
\widetilde{\beta}_n &= d + \frac{1}{2}\frac{c(\rho^2+1)(\rho+3)}{(2n+\rho+1)(2n+\rho+3)} \\
\gamma_{n+1} &= \frac{(n+1)(n+\rho)\Big(\mu n^2 + \mu(\rho+1)n + \gamma_1(\rho+1)^2(\rho+2)\Big)}{(2n+\rho)(2n+\rho+1)^2(2n+\rho+2)}
\end{aligned}$$

with

$$d = \frac{(\rho+1)}{2}\left(\beta_1 - \frac{\rho-1}{\rho+1}\widetilde{\beta}_0\right) \quad \text{and} \quad \mu = 4(\rho+2)\gamma_1 + c^2(\rho+3)^2$$

which imply

$$\Phi(x) = (x-d)^2 - \frac{\mu}{4} \quad \text{and} \quad \Psi(x) = k^{-1}(x-\beta_0). \tag{17}$$

We choose $\beta_0 = 0$ and $\gamma_1 = \frac{1}{2}$, so that

$$\Phi(x) = 1 \quad \text{and} \quad \Psi(x) = 2x, \tag{18}$$

and

$$\beta_n = 0 \quad \text{and} \quad \gamma_{n+1} = \frac{n+1}{2}, \;\; n \geq 0. \tag{19}$$

as well as

$$\widetilde{\beta}_n = 0 \quad \text{and} \quad \widetilde{\gamma}_{n+1} = \frac{n+1}{2}, \;\; n \geq 0. \tag{20}$$

Observe that this means that

$$P_n''(x) - 2xP_n'(x) = -2nP_n(x), \;\; n \geq 0.$$

In this case, the Hermite OPS is orthogonal for a linear functional $\mathscr{L}$ admitting the integral representation

$$\langle \mathscr{L}, f(x)\rangle = \int_{-\infty}^{+\infty} f(x)W(x)\mathrm{d}x, \quad \text{for all polynomials} \quad f(x),$$

where $W(x)$ is a solution of

$$W'(x) + 2xW(x) = 0,$$

subject to $f(x)W(x)\Big|_{-\infty}^{+\infty} = 0$ for any polynomial $f(x)$. Indeed, by solving the homogeneous differential equation, it follows that

$$W(x) = k\mathrm{e}^{-x^2}$$

for some integration constant k. Obviously k cannot be zero (otherwise $W(x) = 0$, identically), and we may choose it so that Ł$[1] = 1$, which means that

$$\int_{-\infty}^{+\infty} W(x)\mathrm{d}x = 1.$$

Hence we take $k = \frac{1}{\sqrt{\pi}}$ and we obtain

$$\langle \mathscr{L}, f(x)\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-x^2}\mathrm{d}x, \quad \text{for all polynomials} \quad f(x).$$

Rodrigues formula:

$$\exp(-x^2/2)P_n(x;\alpha,\beta) = \frac{(-1)^n}{2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\exp(-x^2/2)\right),\ n \geq 0.$$

Similar formulas can be obtained from

$$E(x)P_n(x) = 2^{-n}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + 2x - \frac{E'(x)}{E(x)}\right)^n E(x),\ n \geq 0,$$

for suitable choices of the analytic function $E(x)$.
Clearly, the Rodrigues formula can be obtained from the latter by setting $E(x) = \exp(-x^2/2)$. Another interesting example is when $E(x) = 1$, so that we obtain:

$$P_n(x) = 2^{-n}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + 2x\right)^n,\ n \geq 0.$$

Generating function. The Hermite polynomials can also be described via a generating function:

$$\exp\left(2xt - t^2\right) = \sum_{n\geq 0}\frac{2^n}{n!}P_n(x)t^n,$$

hence, $\displaystyle \frac{\partial^n}{\partial x^n}\left(\exp\left(2xt - t^2\right)\right)\Big|_{t=0} = 2^n P_n(x),\ n \geq 0.$

We choose β_0 and c such that $\beta_0 - \frac{\gamma_1}{c} = 0$ and $c = 1$ and we set $\gamma_1 = 1 + \alpha$ to obtain

$$\Phi(x) = x \quad \text{and} \quad \Psi(x) = x - (\alpha + 1), \tag{21}$$

and

$$\beta_n = 2n + \alpha + 1 \quad \text{and} \quad \gamma_{n+1} = (n+1)(n+\alpha+1), \;\; n \geq 0, \tag{22}$$
$$\widetilde{\beta}_n = 2n + \alpha + 2 \quad \text{and} \quad \widetilde{\gamma}_{n+1} = (n+1)(n+\alpha+2), \;\; n \geq 0, \tag{23}$$

provided that $\alpha \neq -n$ for any integer $n \geq 1$. So we write

$$P_n(x;\alpha) \quad \text{instead of} \quad P_n(x).$$

and, from the recurrence coefficients, we deduce that

$$P'_{n+1}(x;\alpha) = (n+1)P_n(x;\alpha+1).$$

and also

$$xP''_n(x;\alpha) - (x - \alpha - 1)P'_n(x;\alpha) = -nP_n(x;\alpha), \;\; n \geq 0. \tag{24}$$

We seek an integral representation for Ł

$$\langle \mathscr{L}, f(x) \rangle = \int_{-\infty}^{+\infty} f(x) W(x) \mathrm{d}x, \quad \text{for all polynomials} \quad f(x),$$

Hence $W(x)$ is a solution of

$$\Big(xW(x)\Big)' + (x - \alpha - 1) W(x) = c g(x),$$

subject to the conditions

$$\int_a^b W(x) \mathrm{d}x \neq 0 \quad \text{and} \quad p(x) W(x)|_a^b = 0, \text{ for any polynomial } p(x), \qquad (25)$$

With $c = 0$, the general solution of the latter differential equation is given by

$$W(x) = \begin{cases} k_1 \mathrm{e}^{-x} |x|^{\alpha} & \text{if} \quad x < 0 \\ k_2 \mathrm{e}^{-x} x^{\alpha} & \text{if} \quad x > 0. \end{cases}$$

So, $\alpha > -1$ and necessarily $k_1 = 0$ and $k_2 \neq 0$ s.t.

$$k_2 \int_0^{+\infty} \mathrm{e}^{-x} x^{\alpha} \mathrm{d}x = 1 \quad \Rightarrow \quad k_2 = \frac{1}{\Gamma(\alpha + 1)}.$$

Therefore, we conclude that the linear functional can be represented by

$$\langle \mathscr{L}, f(x) \rangle = \frac{1}{\Gamma(\alpha+1)} \int_0^{+\infty} f(x) \mathrm{e}^{-x} x^{\alpha} \mathrm{d}x, \quad \text{provided that} \quad \alpha > -1.$$

Rodrigues formula:

$$x^\alpha \exp(-x) P_n(x;\alpha,\beta) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{\alpha+n}\exp(-x)\right),\ n \geq 0.$$

Generating function: monic Laguerre polynomials can be described as follows

$$(1-x)^{-\alpha-1}\exp\left(\frac{xt}{t-1}\right) = \sum_{n\geq 0} P_n(x;\alpha)\frac{(-t)^n}{n!}$$

Explicit expression:

$$L_n(x;\alpha) = (-1)^n(\alpha+1)_n\, {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right)$$

We choose $\mu = 0$ and therefore $\Phi(x) = (x-d)^2$ and we can set $d = 0$ and $\gamma_1(\rho+2)(\rho+1)^2 = -4$. Hence $c^2(\rho+1)^2(\rho+3)^2 = 16$. We take $c = -4(\rho+1)^{-1}(\rho+3)^{-1}$ and set $\rho+1 = 2\alpha$ to obtain:

$$\Phi(x) = x^2 \quad \text{and} \quad \Psi(x) = -2(\alpha x + 1), \tag{26}$$

and

$$\beta_0 = -\frac{1}{\alpha}, \quad \beta_{n+1} = \frac{1-\alpha}{(n+\alpha)(n+\alpha+1)}, \tag{27}$$

$$\gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}, \; n \geq 0, \tag{28}$$

provided that $\alpha \neq -n$ for any integer $n \geq 0$. Denoting $\beta_n := \beta_n(\alpha)$, it follows that

$$\widetilde{\beta}_n = \beta_n(\alpha+1), \quad \widetilde{\gamma}_n = \gamma_n(\alpha+1).$$

Hence, Bessel polynomials depend on a parameter, so that we write

$$P_n(x;\alpha) \quad \text{instead of} \quad P_n(x).$$

The expressions of the recurrence coefficients also tells

$$P'_{n+1}(x;\alpha) = (n+1)P_n(x;\alpha+1), \; n \geq 0.$$

They satisfy

$$x^2 P''_n(x) + 2(\alpha x + 1)P'_n(x) = n(n+2\alpha-1)P_n(x), \; n \geq 0.$$

Rodrigues formula:

$$x^{2-2\alpha}\exp\left(\frac{2}{x}\right)P_n(x;\alpha)=\frac{(1)^n}{(-2n-2\alpha+2)_n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{-2+2\alpha+2n}\exp\left(-\frac{2}{x}\right)\right),\ n\geq 0.$$

Similar formulas may be obtained via the following:

$$E(x)P_n(x;\alpha)=\frac{1}{(2\alpha)_n}\left(-x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}-2\left(\alpha+\frac{n+1}{2}\right)x-2+x^2\frac{E'(x)}{E(x)}\right)^n E(x),\ n\geq 0,$$

for suitable choices of the analytic function $E(x)$.

Explicit expression.

$$P_n(x;\alpha)=\frac{2^n}{(n+2\alpha-1)_n}\,{}_2F_0\left(\begin{matrix}-n,\ n+2\alpha-1\\ -\end{matrix};-\frac{x}{2}\right)$$

or, equivalently,

$$P_n(x;\alpha)=x^n\,{}_1F_1\left(\begin{matrix}-n\\ -2n-2\alpha+2\end{matrix};\frac{2}{x}\right)$$

Here $\mu \neq 0$. A suitable linear transformation on the variable permits to place the two distinct roots at -1 and 1. For that, we take $\mu = 4$ and $d = 0$. The other two parameters ρ and c remain arbitrary, which we replace by other two parameters α and β, by setting

$$\rho = \alpha + \beta + 1 \quad \text{and} \quad c = \frac{2(\alpha - \beta)}{(\rho+1)(\rho+3)}.$$

With these conditions we obtain

$$\Phi(x) = x^2 - 1, \quad \text{and} \quad \Psi(x) = -(\alpha+\beta+2)x + \alpha - \beta,$$

and also

$$\beta_0 = \frac{\alpha-\beta}{\alpha+\beta+2}, \quad \beta_{n+1} = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}$$

$$\gamma_{n+1} = \frac{4(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}, \ n \geq 0.$$

Obviously, it is required that $\alpha+\beta \neq -(n+1)$, $\alpha \neq -(n+1)$ and $\beta \neq -(n+1)$ for all $n \geq 0$. Besides,

$$\widetilde{\beta}_n = \beta_n(\alpha+1, \beta+1), \quad \widetilde{\gamma}_n = \gamma_n(\alpha+1, \beta+1).$$

Hence $P_n(x;\alpha,\beta)$ satisfies

$$(x^2-1)P_n''(x;\alpha,\beta) + ((\alpha+\beta+2)x + \alpha - \beta)P_n'(x;\alpha,\beta) = n(n+\alpha+\beta+1)P_n(x;\alpha,\beta).$$

Since

$$((x^2-1)W(x))' + \Big(-(\alpha+\beta+2)x + \alpha - \beta\Big)W(x) = cg(x).$$

With $c = 0$, observe that the general solution is given by

$$W(x) = \begin{cases} k(1+x)^{\alpha}(1-x)^{\beta} & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1. \end{cases}$$

For $\alpha > -1$ and $\beta > -1$, then the conditions (25) are satisfied, so that we can represent the Jacobi linear functional as follows:

$$\langle \mathscr{L}, f(x)\rangle = \frac{1}{2^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha)\Gamma(\beta)} \int_{-1}^{1} f(x)(1+x)^{\alpha}(1-x)^{\beta}\mathrm{d}x, \quad \text{for any polynomial } f.$$

Rodrigues formula:

$$(1+x)^{\alpha}(1-x)^{\beta}P_n(x;\alpha,\beta)=\frac{(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1+x)^{\alpha+n}(1-x)^{\beta+n}\right),\ n\geq 0.$$

Generating function:

$$\frac{2^{\alpha+\beta}}{\sqrt{1-2xt+t^2}\,\left(1+t+\sqrt{1-2xt+t^2}\right)^{\alpha}\left(1-t+\sqrt{1-2xt+t^2}\right)^{\beta}}$$
$$=\sum_{n\geq 0}\frac{(n+\alpha+\beta+1)_n}{2^n n!}P_n(x;\alpha,\beta)t^n$$

Explicit expression:

$$P_n(x;\alpha,\beta)=\frac{2^n(\alpha+1)_n n!}{(n+\alpha+\beta+1)_n}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right)$$

and, additionally,

$$P_n(x;\alpha,\beta)=(-1)^n P_n(-x;\beta,\alpha)$$

Legendre Polynomials. With $\alpha = \beta = 0$, we obtain the Legendre polynomials. These are given by $P_n(x) = P_n(x;0,0)$ satisfying

$$\int_{-1}^{1} P_k(x)P_n(x)\mathrm{d}x = \frac{2^{2n+1}}{2n+1}\left(\binom{2n}{n}\right)^{-2}\delta_{n,k}, \ n,k \geq 0.$$

Chebyshev Polynomials of 1st kind. (when $\alpha = \beta = -\frac{1}{2}$):

$$\widehat{T}_1(x) = x \quad \text{and} \quad \widehat{T}_n(x) = 2^{-n}\cos(n\theta), \text{ for } n \neq 1 \quad \text{where} \quad x = \cos(\theta).$$

and can be expressed via the generating function

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n \geq 0} 2^{-n+\delta_{n,1}}\,\widehat{T}_n(x)t^n.$$

The recurrence relation becomes reduced to

$$\widehat{T}_{n+1}(x) = x\,\widehat{T}_n(x) - \frac{1}{4}\,\widehat{T}_{n-1}(x)$$

with $\widehat{T}_0(x) = 1$ and $\widehat{T}_1(x) = x$.

Chebyshev Polynomials of 2nd kind. (When $\alpha = \beta = \frac{1}{2}$) correspond to

$$\widehat{U}_n(x) = 2^{-n}\frac{\sin(n\theta)}{\sin(\theta)}, \quad \text{where} \quad x = \cos(\theta),$$

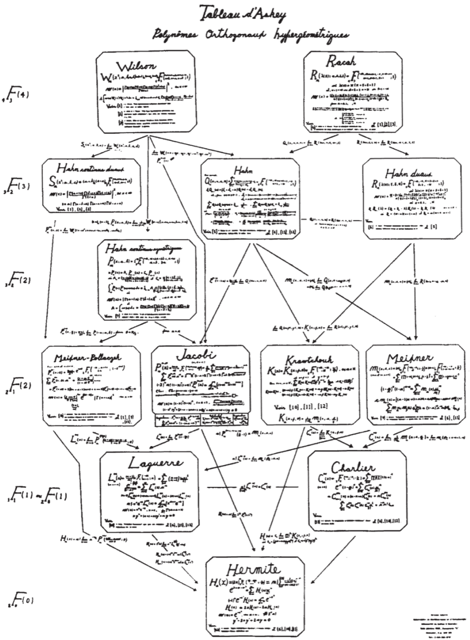
and can be expressed via a generating function

$$\frac{1}{1-2xt+t^2} = \sum_{n\geq 0} 2^n \widehat{U}_n(x) t^n.$$

Also, observe that

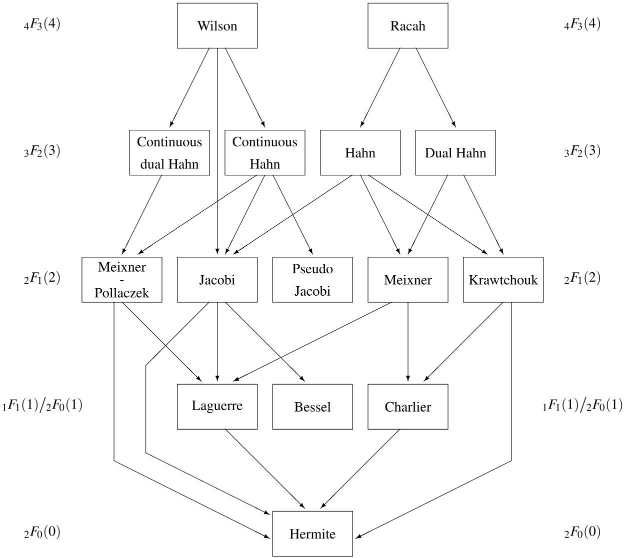
$$\frac{\mathrm{d}}{\mathrm{d}x}\widehat{T}_{n+1}(x) = (n+1)\widehat{U}_n(x), \; n \geq 0.$$

	Hermite	Laguerre $\alpha \neq -(n+1)$	Bessel $\alpha \neq -\frac{n}{2}$	Jacobi $\alpha,\beta \neq -(n+1)$ $\alpha+\beta \neq -(n+2)$
$\Phi(x)$	1	x	x^2	x^2-1
$\Psi(x)$	$2x$	$x-\alpha-1$	$-2(\alpha x+1)$	$-(\alpha+\beta+2)x+(\alpha-\beta)$
χ_n	$-2n$	$-n$	$n(n+2\alpha-1)$	$n(n+\alpha+\beta+1)$
ζ_n	$(-2)^{-n}$	$(-1)^n$	$\frac{\Gamma(n+2\alpha-1)}{\Gamma(2n+2\alpha-1)}$	$\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}$
β_n	0	$2n+\alpha+1$	$\dfrac{1-\alpha}{(n+\alpha-1)(n+\alpha)}$ $(\beta_0 = -\frac{1}{\alpha})$	$\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$
γ_{n+1}	$\frac{n+1}{2}$	$(n+1)(n+\alpha+1)$	$\frac{-(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}$	$\frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}$
	$\int_{-\infty}^{+\infty} f(x)\,\frac{e^{-x^2}}{\sqrt{\pi}}\,dx$	$\int_0^{+\infty} f(x)\,\frac{e^{-x}x^\alpha}{\Gamma(\alpha+1)}\,dx$ valid for $\alpha > -1$		$c_{\alpha,\beta}\int_{-1}^{1} f(x)(1+x)^\alpha(1-x)^\beta\,dx$ with $c_{\alpha,\beta} = \frac{2^{-(\alpha+\beta+1)}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}$ valid for $\alpha,\beta > -1$



Askey scheme as proposed by Jacques Labelle at the first OPSFA meeting in Bar-Le-Duc (France) in 1984

Askey Scheme



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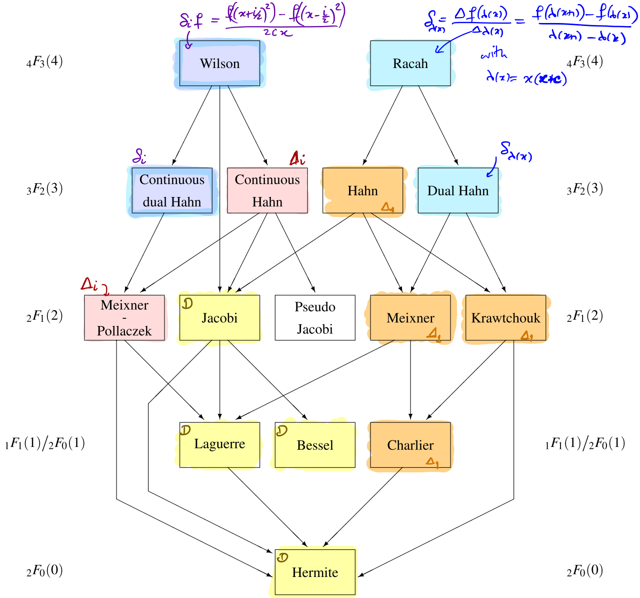

$\mathcal{D}f(x) = \frac{df}{dx}(x)$; $(\Delta_\omega f)(x) = \frac{f(x+\omega) - f(x)}{\omega}$

Consider the operator $\Delta_\omega : \mathscr{P} \longrightarrow \mathscr{P}$ s.t.

$$\Delta_\omega f(x) = \frac{f(x+\omega) - f(x)}{\omega}, \quad \omega \neq 0.$$

Definition. An orthogonal polynomial sequence $\{P_n\}_{n \geqslant 0}$ is Δ_ω-classical iff the polynomial sequence $\{Q_n\}_{n \geqslant 0}$ given by

$$Q_n(x) := \frac{1}{n+1} \Delta_\omega P_{n+1}(x)$$

is also orthogonal.

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In this case it makes all sense to analyse the polynomials on the modified Pochhammer basis

$$(x;-\omega)_n := \prod_{k=0}^{n-1} (x - \omega k)$$

so that

$$\Delta_\omega (x;-\omega)_{n+1} = \frac{(x;-\omega)_n}{-\omega} (x+\omega-(x-\omega n)) = (n+1)(x;-\omega)_n$$

Denoting by $\Delta_\omega^T : \mathscr{P}' \longrightarrow \mathscr{P}'$ the transposed of the operator $\Delta_\omega : \mathscr{P} \longrightarrow \mathscr{P}$, then we have

$$\Delta_\omega^T \mathscr{L} := -\Delta_{-\omega}\mathscr{L}$$

so, with some abuse of notation, we have

$$< \Delta_{-\omega}\mathscr{L}, f(x) >= - < \mathscr{L}, \Delta_{-\omega} f(x) >$$

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Theorem. For any OPS $\{P_n\}_{n\geqslant 0}$ for $\mathscr{L}$ the following are equivalent

(a) $\{P_n\}_{n\geqslant 0}$ is Δ_ω-classical.

(b) There exists Φ and Ψ with $\deg \Phi \leq 2$ and $\deg \Psi = 1$ s.t.

$$\Delta_{-\omega}\left(\Phi(x)\mathscr{L}\right) + \Psi(x)\mathscr{L} = 0$$

(c) There exists Φ and Ψ with $\deg \Phi \leq 2$ and $\deg \Psi = 1$ and coefficients $\lambda_n \neq 0$, for $n \geq 1$, s.t.

$$\Phi(x)\left(\Delta_\omega \circ \Delta_{-\omega} P_n\right)(x) - \Psi(x)\left(\Delta_{-\omega} P_n\right)(x) = \lambda_n P_n(x)$$

(d) There exists Φ with $\deg \Phi \leq 2$ and coefficients $\xi_n \neq 0$, for $n \geq 1$, s.t.

$$P_n(x)\mathscr{L} = \xi_n \Delta_{-\omega}^n \left(\left(\prod_{\sigma=0}^{n-1} \Phi(x+\omega\sigma) \right) \mathscr{L} \right)$$

Similar to the very classical polynomials, and under the same equivalence relation, one can define the corresponding equivalence classes for the Δ_ω-classical polynomials because....

If $\{P_n\}_{n\geqslant 0}$ is Δ_ω-classical w.r.t. $\mathscr{L}$, iff $\{\widetilde{P}_n := a^{-n}P_n(ax+b)\}_{n\geqslant 0}$ is also Δ_ω-classical w.r.t. $\widetilde{\mathscr{L}}$

so that, we have

$$\Delta_{-\omega}(\Phi(x)\mathscr{L}) + \Psi(x)\mathscr{L} = 0$$

and

$$\Delta_{-\omega a^{-1}}\left(\widetilde{\Phi}(x)\widetilde{\mathscr{L}}\right) + \widetilde{\Psi}(x)\widetilde{\mathscr{L}} = 0$$

where $\widetilde{\Phi}(x) = a^{-t}\,\Phi(ax+b)$, $\widetilde{\Psi}(x) = a^{1-t}\,\Psi(ax+b)$, where $t = \deg(\Phi) \leqslant 2$

(For more details see Abdelkarim& Maroni, 1997)

Consider the operator $D_q : \mathscr{P} \longrightarrow \mathscr{P}$ s.t.

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \in \mathbb{C}\backslash\{0\} \quad \text{and} \quad |q| \neq 1.$$

Definition. An orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ is D_q-classical iff the polynomial sequence $\{Q_n\}_{n\geqslant 0}$ given by

$$Q_n(x) := \frac{1}{[n+1]}\left(D_q P_{n+1}\right)(x)$$

is also orthogonal, where

$$[n] := \frac{q^n - 1}{q-1}$$

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Hahn-classical sequences with respect to $D_q f(x) := \frac{f(qx)-f(x)}{(q-1)x}$

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$$Q_n(x) := \frac{1}{[n+1]}(D_q P_{n+1})(x)$$

is also orthogonal, where

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Denoting by $D_q^T : \mathscr{P}' \longrightarrow \mathscr{P}'$ the transposed of the operator $D_q : \mathscr{P} \longrightarrow \mathscr{P}$, then we have

$$D_q^T \mathscr{L} := -D_q \mathscr{L}$$

so, with some abuse of notation, we have

$$< D_q \mathscr{L}, f(x) > = - < \mathscr{L}, D_q f(x) >$$

Theorem. For any OPS $\{P_n\}_{n\geqslant 0}$ for $\mathscr{L}$ the following are equivalent

(a) $\{P_n\}_{n\geqslant 0}$ is D_q-classical.

(b) There exists Φ and Ψ with $\deg\Phi \leq 2$ and $\deg\Psi = 1$ s.t.

$$D_q\left(\Phi(x)\mathscr{L}\right) + \Psi(x)\mathscr{L} = 0$$

(c) There exists Φ and Ψ with $\deg\Phi \leq 2$ and $\deg\Psi = 1$ and coefficients $\lambda_n \neq 0$, for $n \geq 1$, s.t.

$$\Phi(x)\left(D_q \circ D_{q^{-1}} P_n\right)(x) - \Psi(x)\left(D_{q^{-1}} P_n\right)(x) = \lambda_n P_n(x)$$

(d) There exists Φ with $\deg\Phi \leq 2$ and coefficients $\xi_n \neq 0$, for $n \geq 1$, s.t.

$$P_n(x)\mathscr{L} = \xi_n D_q^n\left(\left(\prod_{\sigma=0}^{n-1}\Phi(q^\sigma x)\right)\mathscr{L}\right)$$

Similar to the very classical polynomials, and under the equivalence relation

$$B_n(x) \sim P_n(x) \qquad \text{iff} \qquad \exists a \neq 0 \quad \text{s.t.} \quad B_n(x) = a^{-n} P_n(ax)$$

one can define the corresponding equivalence classes for the D_q-classical polynomials because....

$\{P_n\}_{n\geqslant 0}$ is D_q-classical w.r.t. $\mathscr{L}$, iff $\{\widetilde{P}_n := a^{-n} P_n(ax)\}_{n\geqslant 0}$ is also D_q-classical w.r.t. $\widetilde{\mathscr{L}} = h_{a^{-1}}\mathscr{L}$ since we have

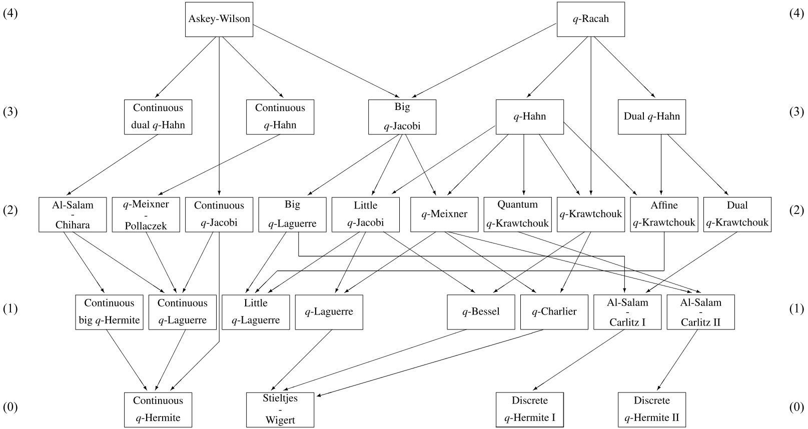
$$D_q\left(\Phi(x)\mathscr{L}\right) + \Psi(x)\mathscr{L} = 0$$

and

$$D_q\left(\widetilde{\Phi}(x)\widetilde{\mathscr{L}}\right) + \widetilde{\Psi}(x)\widetilde{\mathscr{L}} = 0$$

where $\widetilde{\Phi}(x) = a^{-t}\,\Phi(ax)$, $\widetilde{\Psi}(x) = a^{1-t}\,\Psi(ax)$, where $t = \deg(\Phi) \leqslant 2$

(For more details see Khériji & Maroni, 2002)



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Definition. An OPS $\{P_n\}_{n\geqslant 0}$ is semiclassical w.r.t. a linear functional $\mathscr{L}$ iff there exists a polynomial Φ and a polynomial Ψ with $\deg \Psi \geq 1$ s.t.

$$(\Phi(x)\mathscr{L})' + \Psi(x)\mathscr{L} = 0 \tag{29}$$

and the pair (Φ, Ψ) is such that $\max(\deg \Phi - 2, \deg \Psi - 1) \geq 1$ and needs to satisfy the so called *admissible conditions*.
Observe that the pair (Φ, Ψ) realising equation (29) is not unique and there is simplification criteria

- Simplification criteria: for

$$(\Phi(x)\mathscr{L})' + \Psi(x)\mathscr{L} = 0$$

$\exists c$ such that $\Phi(c) = 0$ and

$$\Big|\Phi'(c) + \Psi(c)\Big| + \Big| < u, \theta_c^2(\Phi) + \theta_c(\Psi) > \Big| = 0 \ , \qquad (30)$$

where $\theta_c(f)(x) = \frac{f(x)-f(c)}{x-c}$, for any $f \in \mathscr{P}$, and u would then fulfill

$$(\theta_c(\Phi)u)' + \Big(\theta_c^2(\Phi) + \theta_c(\Psi)\Big)\, u = 0.$$

- The class of $u = \mathbf{s}$ is given by $\min\limits_{(\Phi,\Psi)} [\max(\deg(\Phi) - 2, \deg(\Psi) - 1)]$

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- The class of $u = \mathbf{s}$ is given by $\min\limits_{(\Phi,\Psi)} \left[\max(\deg(\Phi) - 2, \deg(\Psi) - 1)\right]$

- Moreover, $\Phi(x)P'_{n+1}(x) = \sum\limits_{\nu=n-\mathbf{s}}^{n+\deg\Phi} \theta_{n,\nu} P_\nu(x)$ with $\theta_{n,n-\mathbf{s}}\theta_{n,n+t} \neq 0, \quad n \geqslant s.$

Theorem. For any monic polynomial Φ and any orthogonal sequence $\{P_n\}_{n\geqslant 0}$ for $\mathscr{L}$, the following are equivalent:

(a) $\exists\Psi$ with $\deg\Psi = p \geq 1$ s.t. $\quad (\Phi(x)\mathscr{L})' + \Psi(x)\mathscr{L} = 0$
where the pair (Φ,Ψ) is admissible and gives the class $s = \max(\deg\Phi - 2, \deg\Psi - 1)$ of the semiclassical linear functional $\mathscr{L}$.

(b) There exists an integer $s \geq 0$ s.t.

$$\Phi(x)P'_{n+1}(x) = \sum_{\nu=n-\mathbf{s}}^{n+\deg\Phi} \theta_{n,\nu}P_\nu(x)$$

with $\theta_{n,n-\mathbf{s}}\theta_{n,n+t} \neq 0, \quad n \geqslant s$.

(c) There exist an integer $s \geq 0$ and a polynomial Ψ with $\deg\Psi = p \geq 1$ s.t.

$$\Phi(x)P'_n(x) - \Psi(x)P_n(x) = \sum_{\nu=m-\deg\Phi}^{n+s_n} \widetilde{\lambda}_{n,\nu}P_{\nu+1}(x), \quad n \geq \deg\Phi$$

with $\widetilde{\lambda}_{n,n-\deg\Phi} \neq 0$ where

$$s_n = \begin{cases} p-1, & n = 0, \\ s = \max(\deg\Phi - 2, \deg\Psi - 1), & n \geq 1, \end{cases}$$

and we write

$$\widetilde{\lambda}_{n,\nu} = -(\nu+1)\frac{\langle\mathscr{L}, P_n^2(x)\rangle}{\langle\mathscr{L}, P_{\nu+1}^2(x)\rangle}\lambda_{\nu,n}, \ 0 \leq \nu \leq n+s.$$

Freud Weights (1976)

$$\langle \mathscr{L}, f(x)\rangle = \int_{\mathbb{R}} f(x)\mathrm{d}\mu(x) \quad \text{with} \quad \mathrm{d}\mu(x) = |x|^{\rho}\exp(-|x|^{m})$$

with $m = 2,4,6$ (Géza Freud, 1976) and earlier considered by Shohat in 1939.

Semiclassical extensions of modified Laguerre polynomials

$$\mathrm{d}\mu(x) = \underbrace{x^{\alpha}\exp(-x-s/x)}_{W(x;s,\alpha)}\mathrm{d}x, \quad x\in[0,+\infty), \quad \alpha>0, s\geq 0,$$

whose moments of order k are $m_k = 2(\sqrt{s})^{\alpha+k+1}K_{\alpha+k+1}(2\sqrt{s})$, and we have

$$(x^2W(x;s,\alpha))' + (x^2-(\alpha+2)x-s)W(x;s,\alpha) = 0$$

The recurrence coefficients are related to special solutions of *PIII* (but can be also seen as special solutions of the alternative discrete dPII) (Chen&Its, 2010)

many more examples can be found in the book (Van Assche, 2018).

Theorem. For any monic polynomial Φ and any orthogonal sequence $\{P_n\}_{n\geqslant 0}$ for $\mathscr{L}$, the following are equivalent:

(a) $\exists\Psi$ with $\deg\Psi = p \geq 1$ s.t.

$$\Delta_{-\omega}(\Phi(x)\mathscr{L}) + \Psi(x)\mathscr{L} = 0$$

where the pair (Φ,Ψ) is admissible and gives the class $s = \max(\deg\Phi - 2, \deg\Psi - 1)$ of the semiclassical linear functional $\mathscr{L}$.

(b) There exists an integer $s \geq 0$ s.t.

$$\Phi(x)\left(\Delta_\omega P_{n+1}\right)(x) = \sum_{\nu=n-\mathbf{s}}^{n+\deg\Phi} \theta_{n,\nu} P_\nu(x)$$

with $\theta_{n,n-\mathbf{s}}\theta_{n,n+t} \neq 0, \quad n \geqslant s.$

(c) There exist an integer $s \geq 0$ and a polynomial Ψ with $\deg\Psi = p \geq 1$ s.t.

$$\Phi(x)\left(\Delta_\omega P_n\right)(x) - \Psi(x)P_n(x) = \sum_{\nu=m-\deg\Phi}^{n+s_n} \widetilde{\lambda}_{n,\nu} P_{\nu+1}(x), \quad n \geq \deg\Phi$$

with $\widetilde{\lambda}_{n,n-\deg\Phi} \neq 0$ where $s_n = \begin{cases} p-1, & n=0, \\ s = \max(\deg\Phi - 2, \deg\Psi - 1), & n \geq 1, \end{cases}$

and we write $\widetilde{\lambda}_{n,\nu} = -(\nu+1)\frac{\langle\mathscr{L},P_n^2(x)\rangle}{\langle\mathscr{L},P_{\nu+1}^2(x)\rangle}\lambda_{\nu,n},\ 0 \leq \nu \leq n+s.$

Generalised Charlier polynomials

$$\langle \mathscr{L}, f(x)\rangle = \sum_{x\in\mathbb{N}} f(x)\,\underbrace{\frac{a^x}{x!(\beta)_x}}_{W(x;\beta,a)} \quad \beta, a > 0,$$

whose moments of order k are $m_k = 2(\sqrt{s})^{\alpha+k+1}K_{\alpha+k+1}(2\sqrt{s}),$ and we have

$$\Delta_{-1}(W(x;\beta,a)) + \frac{1}{a}(x^2 + (\beta-1)x - a)W(x;\beta,a) = 0$$

The recurrence coefficients of the corresponding OPS with recurrence relation

$$xp_n = a_{n+1}p_{n+1} + b_np_n + a_np_{n-1}$$

satisfy

$$\begin{cases} b_n + b_{n-1} - n + \beta = \frac{an}{a_n^2} \\ b_{n-1} - b_n + 1 = \frac{a_n^2}{an}(a_{n+1}^2 - a_{n-1}^2) \end{cases}$$

with initial conditions $b_0 = \frac{\sqrt{a}I_\beta(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a})}$ and $a_0^2 = 0$

Many more examples can be found in the book (Van Assche, 2018).

Theorem. For any monic polynomial Φ and any orthogonal sequence $\{P_n\}_{n\geqslant 0}$ for $\mathscr{L}$, the following are equivalent:

(a) $\exists\Psi$ with $\deg\Psi = p \geq 1$ s.t.

$$D_q(\Phi(x)\mathscr{L}) + \Psi(x)\mathscr{L} = 0$$

where the pair (Φ,Ψ) is admissible and gives the class $s = \max(\deg\Phi - 2, \deg\Psi - 1)$ of the semiclassical linear functional $\mathscr{L}$.

(b) There exists an integer $s \geq 0$ s.t.

$$\Phi(x)\,(D_q P_{n+1})\,(x) = \sum_{\nu=n-\mathbf{s}}^{n+\deg\Phi} \theta_{n,\nu} P_\nu(x)$$

with $\theta_{n,n-\mathbf{s}}\theta_{n,n+t} \neq 0, \quad n \geqslant s.$

(c) There exist an integer $s \geq 0$ and a polynomial Ψ with $\deg\Psi = p \geq 1$ s.t.

$$\Phi(x)\,(D_q P_n)\,(x) - \Psi(x) P_n(x) = \sum_{\nu=m-\deg\Phi}^{n+s_n} \widetilde{\lambda}_{n,\nu} P_{\nu+1}(x), \quad n \geq \deg\Phi$$

with $\widetilde{\lambda}_{n,n-\deg\Phi} \neq 0$ where $s_n = \begin{cases} p-1, & n = 0, \\ s = \max(\deg\Phi - 2, \deg\Psi - 1), & n \geq 1, \end{cases}$

and we write $\widetilde{\lambda}_{n,\nu} = -[\nu+1]\frac{\langle\mathscr{L},P_n^2(x)\rangle}{\langle\mathscr{L},P_{\nu+1}^2(x)\rangle}\lambda_{\nu,n},\ 0 \leq \nu \leq n+s.$

Semiclassical extensions of q-Laguerre polynomials (or the Stieltjes-Wigert) Starting with the indeterminate weight

$$\hat{W}(x) = \frac{x^\alpha}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \quad x \in [0,\infty)$$

where

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

then, the recurrence coefficients (a_n, b_n) of p_n defined by

$$xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}$$

are such that

$$\begin{cases} a_n^2 = q^{1-n}x_n + q^{-2n-\alpha+1} \\ b_n^2 q^{2n+2\alpha}x_n = x_{n+1} + q^{2n+2\alpha}x_{n-1}(x_n + q^{-n-\alpha})^2 + 2(x_n + q^{-\alpha}) \end{cases}$$

where

$$x_{n-1}x_{n+1} = \frac{(x_n + q^\alpha)^2}{(q^{n+\alpha}x_n + 1)^2}$$

with initial conditions $x_0 = -q^\alpha$ and $x_1 = b_0^2 = \left(\frac{m_1}{m_0}\right)^2$

are related to the q-discrete PIII.

Many more examples can be found in the book (Van Assche, 2018).

$\{P_n\}_{n\geqslant 0}$ is $\mathscr{O}$-semiclassical, whenever the corresponding regular form u_0 fulfils

$$ {}^t\mathscr{O}\,(\Phi u_0) + \Psi u_0 = 0 $$

with $\deg\Phi = t \geqslant 0$ and $\deg\Psi = p \geqslant 1$.

Semiclassical extensions of Hahn-classical polynomials

$\{P_n\}_{n\geqslant 0}$ is $\mathscr{O}$-semiclassical, whenever the corresponding regular form u_0 fulfils

$$ {}^t\mathscr{O}(\Phi u_0) + \Psi u_0 = 0 $$

with $\deg \Phi = t \geqslant 0$ and $\deg \Psi = p \geqslant 1$.

- ${}^t\mathscr{O} = D$:
  The recurrence coefficients of D-semiclassical polynomial sequences are often related to Painlevé type equations.

  Magnus (1995,1999), Clarkson (2008), Chen & Its (2010), Chen & Zhang (2010),
  Dai & Zhang (2010), Clarkson & Jordaan (2013), Clarkson, Jordaan & Kelil (2016), etc...

- ${}^t\mathscr{O} = \Delta_\omega$: (Maroni & Mejri, 2008), where the symmetric case is treated for the class $s = 1$.

  \- connections to discrete Painlevé type equations: (Boelen, Filipuk &Van Assche (2011,2012)), Clarkson & Jordaan (2013), etc.

- ${}^t\mathscr{O} = D_q$, we refer to (Khériji, 2003), (Ghressi & Khériji, 2009), (Mejri, 2009), (Ormerod, Witte & Forrester, 2011) , (Boelen, Smet & Van Assche, 2010)