ORTHOGONAL POLYNOMIALS AND SPECIAL FUNCTIONS

Part 3

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Outline

- Part 1. Special Functions
- Part 2. Orthogonal Polynomials: an introduction
  - Main properties
    Recurrence relations, zeros, distribution of the zeros and so on and on....
  - Classical Orthogonal Polynomials
    Hermite, Laguerre, Bessel and Jacobi!!
  - Other notions of "classical orthogonal polynomials"
    How to identify this on the Askey Scheme?
  - Semiclassical Orthogonal Polynomials
    How do these link to Random Matrix Theory, Painlevé equations and so on?
- **Part 3. Multiple Orthogonal Polynomials**
  When the orthogonality measure is spread across a vector of measures.

Let μ be a **positive Borel measure** with support S defined on $\mathbb{R}$ (represented by the linear functional $\mathscr{L}$) for which **moments** of all orders exist, *i.e.* ,

$$m_n = \int_S x^n \mathrm{d}\mu(x) \ < \ \infty, \quad n = 0, 1, 2, \ldots \ .$$

we have seen that...

A sequence of monic polynomials $\{P_n\}_{n\geq 0}$ with $\deg P_n = n$ is orthogonal w.r.t. the measure μ if

$$\langle \mathscr{L}, x^k P_n(x) \rangle := \int_S x^k P_n(x) \mathrm{d}\mu(x) = N_n \ \delta_{n,k} \quad k = 0, 1, 2, \ldots, n \ .$$

where S is the support of μ and N_n is the square of the weighted L^2-norm of P_n given by

$$N_n = \int_S x^n P_n(x) \mathrm{d}\mu(x) > 0.$$

The system

$$\langle \mathscr{L}, x^k P_n(x) \rangle := \int_S x^k P_n(x) \mathrm{d}\mu(x) = 0 \quad k = 0, 1, 2, \ldots, n-1 \ .$$

is a linear system of n equations for the n unknown coefficients $c_{n,k}$ of $P_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$ with $c_{n,n} = 1$.

The system has a unique solution because the matrix of the system is the Gram matrix

$$\begin{bmatrix} m_0 & m_1 & \ldots & m_{n-1} \\ m_1 & m_2 & \ldots & m_n \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \ldots & m_{2n-2} \end{bmatrix} \quad \text{where} \quad m_k = \int_S x^k \mathrm{d}\mu(x).$$

which is a positive definite matrix whenever the support of μ contains at least n points.

A sequence of Multiple Orthogonal Polynomials is a sequence of polynomials of **one variable** which is defined by orthogonality relations with respect to r different measures $\mu_1, \ldots, \mu_r$, where $r \geq 1$.
Some remarks:

- The case where $r = 1$ reduces to the standard notion of orthogonality;
- These polynomials should not be confused with *multivariate or multivariable orthogonal polynomials of several variables* nor with *matrix orthogonal polynomials*;
- Other terminology is also used such as:
  - *Hermite-Padé polynomials*, motivated by the link with Hermite-Padé approximation or simultaneous Padé approximation, following the works by (Nuttall, 84), (de Bruin, 85), (Sorokin, 84 & 90), (Bultheel *et al.*, 05);
  - *Polyorthogonal polynomials* after (Nikishin & Sorokin, 91);
  - *Vector orthogonal polynomials* following (Van Iseghem, 87), (Kaliaguine, 95), (Sorokin & Van Iseghem, 97)
  - The so-called *d-orthogonal polynomials* initiated by (Maroni, 89) and followed by Douak, Ben Cheikh and many others up to now: these are multiple orthogonal polynomials near the diagonal and d is the number of orthogonality measures.

Multiple orthogonal polynomials

There are two types of multiple orthogonal polynomials: type I and type II

In either cases, the polynomials will be depend on the **multi-index**

$$\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$$

with **length**

$$|\vec{n}| = n_1 + \ldots + n_r$$

Type I multiple orthogonal polynomials are collected in a vector of r polynomials

$$(A_{\vec{n},1}(x), \ldots A_{\vec{n},r}(x))$$

where $\deg A_{\vec{n},j}(x) \leq n_j - 1$ s.t.

$$\sum_{j=1}^{r} \int_S x^k A_{\vec{n},j}(x) \mathrm{d}\mu_j(x) = 0, \text{ for } \ k = 0, 1, \ldots |\vec{n}| - 2$$

$$\sum_{j=1}^{r} \int_S x^{|\vec{n}|-1} A_{\vec{n},j}(x) \mathrm{d}\mu_j(x) = 1 \quad \leftarrow \quad \text{(normalisation)}$$

which gives a linear system of $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of the polynomials $A_{\vec{n},j}(x)$ for $j = 1, \ldots r$.

Type I multiple orthogonal polynomials

$(A_{\vec{n},1}(x), \ldots A_{\vec{n},r}(x))$ where $\deg A_{\vec{n},j}(x) \leq n_j - 1$ s.t.

$$\sum_{j=1}^{r} \int_S x^k A_{\vec{n},j}(x) \mathrm{d}\mu_j(x) = 0, \text{ for } \ k = 0, 1, \ldots |\vec{n}| - 2 \tag{1}$$

$$\sum_{j=1}^{r} \int_S x^{|\vec{n}|-1} A_{\vec{n},j}(x) \mathrm{d}\mu_j(x) = 1 \tag{2}$$

This gives a linear system of $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of the polynomials $A_{\vec{n},j}(x)$ for $j = 1, \ldots r$.

The **index** $\vec{n}$ **is normal** if the relations (1) determine the polynomials uniquely, which corresponds to say that

$$\boxed{\det M_{\vec{n}} \neq 0} \quad \text{where} \quad M_{\vec{n}} = \left[\begin{array}{cccc} M_{n_1}^{(1)} & M_{n_2}^{(2)} & \cdots & M_{n_r}^{(r)} \end{array} \right]$$

with

$$M_{n_j}^{(j)} = \left[\begin{array}{cccc} m_0^{(j)} & m_1^{(j)} & \cdots & m_{n_j-1}^{(j)} \\ m_1^{(j)} & m_2^{(j)} & \cdots & m_{n_j}^{(j)} \\ \vdots & \vdots & \cdots & \vdots \\ m_{|\vec{n}|-1}^{(j)} & m_{|\vec{n}|}^{(j)} & \cdots & m_{|\vec{n}|+n_j-2}^{(j)} \end{array} \right] \quad \text{and} \quad m_k^{(j)} = \int_S x^k \mathrm{d}\mu_j(x)$$

The type II multiple orthogonal polynomialsfor $\vec{n}$ corresponds to the **monic** polynomials $P_{\vec{n}}(x)$ of degree $|\vec{n}|$ for which

$$\int_S x^k P_{\vec{n}}(x) \mathrm{d}\mu_j(x) = 0, \; k = 0, \ldots n_j - 1, \tag{3}$$

for $j = 1, \ldots, r$.

The conditions (3) give a system of $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of the monic $P_{\vec{n}}(x)$.

The matrix of this linear system is

$$M_{\vec{n}}^T = \begin{bmatrix} M_{n_1}^{(1)} & M_{n_2}^{(2)} & \cdots & M_{n_r}^{(r)} \end{bmatrix}^T$$

which is the transpose of $M_{\vec{n}}$.

Hence the system (3) has a unique solution if the **multi-index** $\vec{n}$ **is normal**, *i.e.* $\det M_{\vec{n}} \neq 0$.

A multi-index $\vec{n}$ is normal

$$\iff \det M_{\vec{n}} = \det \begin{bmatrix} M^{(1)}_{n_1} & M^{(2)}_{n_2} & \cdots & M^{(r)}_{n_r} \end{bmatrix} \neq 0$$

where $$M^{(j)}_{n_j} = \begin{bmatrix} m^{(j)}_0 & m^{(j)}_1 & \cdots & m^{(j)}_{n_j-1} \\ m^{(j)}_1 & m^{(j)}_2 & \cdots & m^{(j)}_{n_j} \\ \vdots & \vdots & \cdots & \vdots \\ m^{(j)}_{|\vec{n}|-1} & m^{(j)}_{|\vec{n}|} & \cdots & m^{(j)}_{|\vec{n}|+n_j-2} \end{bmatrix}$$ and $$m^{(j)}_k = \int_S x^k \mathrm{d}\mu_j(x)$$

$\iff$ the type I vector $(A_{\vec{n},1}(x), \ldots A_{\vec{n},r}(x))$ exists and is unique

$\iff$ the monic type II multiple orthogonal polynomials $P_{\vec{n}}$ exists and is unique

Definition. The vector measures $(\mu_1, \ldots, \mu_r)$ form an **Angelesco system** if the supports of the measures are subsets of disjoint intervals
i.e.,
$\mathrm{supp}(\mu_j) \subset S_j$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$.

Usually one allows that the intervals are touching, so that

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Theorem. (Angelesco, Nikishin)

The type II multiple orthogonal polynomials $P_{\vec{n}}$ has exactly n_j distinct zeros on $\overset{\circ}{S}_j$ for $j = 1, \dots, r$.

Definition. The system of linearly independent functions $\varphi_1, \ldots, \varphi_n$ form a **Chebyshev system** on $[a,b]$ if every linear combination $\sum_{i=1}^{n} a_i \varphi_i(x)$ with $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ has at most $n-1$ zeros on $[a,b]$.

Example. $e^{c_1 x}, xe^{c_1 x}, \ldots, x^{n_1-1}e^{c_1 x}, \ldots, e^{c_r x}, xe^{c_r x}, \ldots, x^{n_r-1}e^{c_r x}$, with $c_i \neq c_j$ whenever $i \neq j$, is a Chebyshev system of order $|\vec{n}|$ on $\mathbb{R}$.

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Definition. (AT-system) The measures $(\mu_1, \dots, \mu_r)$ form an **AT-system** on the interval $[a,b]$ if the measures are all absolutely continuous with respect to a positive measure μ on $[a,b]$, *i.e.*

$$\mathrm{d}\mu_j(x) = w_j(x)\mathrm{d}\mu(x), \; j = 1, \dots, r,$$

and, for every $\vec{n}$, the functions

$$w_1(x), \dots, x^{n_1-1} w_1(x), w_2(x), \dots, x^{n_2-1} w_2(x), \dots, w_r(x), \dots, x^{n_r-1} w_r(x)$$

are a Chebyshev system on $[a,b]$.

Theorem. If $(\mu_1, \ldots, \mu_r)$ is an **AT-system** on the interval $[a, b]$, then the type II multiple orthogonal polynomials $P_{\vec{n}}(x)$ has exactly $|\vec{n}|$ distinct zeros on (a, b) and hence $\vec{n}$ is a normal index.

Theorem. For an AT-system, the function

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x)$$

has exactly $|\vec{n}| - 1$ sign changes on (a, b).

In an AT-system every measure μ_k is absolutely continuous w.r.t. a given measure μ on $[a,b]$ and $\mathrm{d}\mu_k(x) = w_k(x)\mathrm{d}\mu(x)$.

In an Angelesco system we can define $\mu = \mu_1 + \ldots + \mu_r$. If all intervals $[a_j, b_j]$ are disjoint then $\mathrm{d}\mu_k(x) = w_k(x)\mathrm{d}\mu(x)$ where

$$w_k(x) = \chi_{[a_k,b_k]}(x) = \begin{cases} 1, & \text{if } x \in [a_k, b_k] \\ 0, & \text{if } x \notin [a_k, b_k] \end{cases}$$

If $b_j = a_{j+1}$ then we consider $\mu_j = \hat{\mu}_j + c_1\delta_{b_j}$ and $\mu_{j+1} = \hat{\mu}_{j+1} + c_2\delta_{a_{j+1}}$, so that $\hat{\mu}_j$ and $\hat{\mu}_{j+1}$ have no mass at $b_j = a_{j+1}$. Then the absolutely continuity w.r.t. to $\mu = \mu_1 + \ldots + \mu_r$ still holds, but with

$$w_j = \chi_{(a_j,b_j)}(x) + \frac{c_1}{c_1+c_2}\chi_{\{b_j\}}(x)$$
$$w_{j+1} = \chi_{(a_{j+1},b_{j+1})}(x) + \frac{c_2}{c_1+c_2}\chi_{\{a_{j+1}\}}(x)$$

Biorthogonality

For an AT-system and an Angelesco system we have

$$\mathrm{d}\mu_j(x) = w_j(x)\mathrm{d}\mu(x), \quad j = 1, 2, \ldots, r.$$

Based on the type I orthogonality relations, then for the **type I functions**

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x)$$

we have

$$\int_a^b Q_{\vec{n}}(x) x^k \mathrm{d}\mu(x) = 0, \quad k = 0, 1, \ldots, |\vec{n}| - 2,$$
$$\int_a^b Q_{\vec{n}}(x) x^{|\vec{n}|-1} \mathrm{d}\mu(x) = 1.$$

Biorthogonality

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$$\int_a^b Q_{\vec{n}}(x) x^{|\vec{n}|-1} \mathrm{d}\mu(x) = 1.$$

The **type II multiple orthogonal polynomials** $P_{\vec{n}}$ and these **type I functions** $Q_{\vec{n}}(x)$ satisfy biorthogonality:

$$\int_a^b P_{\vec{n}}(x) Q_{\vec{m}}(x) \mathrm{d}\mu(x) = \begin{cases} 0, & \text{if } \vec{m} \le \vec{n}, \\ 0, & \text{if } |\vec{n}| \le |\vec{m}| - 2, \\ 1, & \text{if } |\vec{n}| = |\vec{m}| - 1. \end{cases}$$

Nearest neighbour recurrence relations for **type II multiple orthogonal polynomials** (see Van Assche, 11)

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},1}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x)$$

$$\vdots$$

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_r}(x) + b_{\vec{n},r}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x)$$

where

$$\vec{e}_j = (0,\ldots,0,\underbrace{1}_{j\text{th entry}},0,\ldots,0)$$

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Nearest neighbour recurrence relations for **type I functions**

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_1}(x) + b_{\vec{n}-\vec{e}_1,1} Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} Q_{\vec{n}+\vec{e}_j}(x)$$

$$\vdots$$

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_r}(x) + b_{\vec{n}-\vec{e}_r,r} Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} Q_{\vec{n}+\vec{e}_j}(x)$$

where

$$\vec{e}_j = (0,\ldots,0, \underbrace{1}_{j\text{th entry}}, 0,\ldots,0)$$

Theorem. (Van Assche,11)
The recurrence coefficients $(a_{\vec{n},1},\ldots,a_{\vec{n},r})$ and $(b_{\vec{n},1},\ldots,b_{\vec{n},r})$ satisfy the partial difference equations

$$\begin{aligned} b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n},j} &= b_{\vec{n}+\vec{e}_j,i} - b_{\vec{n},i} \\ \sum_{k=1}^{r} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{r} a_{\vec{n}+\vec{e}_i,k} &= \det \begin{pmatrix} b_{\vec{n}+\vec{e}_j,i} & b_{\vec{n},i} \\ b_{\vec{n}+\vec{e}_i,j} & b_{\vec{n},j} \end{pmatrix} \\ \frac{a_{\vec{n},i}}{a_{\vec{n}+\vec{e}_j,i}} &= \frac{b_{\vec{n}+\vec{e}_j,j} - b_{\vec{n}+\vec{e}_i,i}}{b_{\vec{n},j} - b_{\vec{n},i}} \end{aligned}$$

for all $j = 1,\ldots,r$.

These are given by

$$\int_{-\infty}^{\infty} x^k H_{\vec{n}}(x) e^{-x^2+c_j x}\, dx = 0, \qquad k = 0, 1, \ldots, n_j - 1,$$

for $1 \leq j \leq r$, where $c_i \neq c_j$ whenever $i \neq j$. The recurrence relation is explicitly given as

$$xH_{\vec{n}}(x) = H_{\vec{n}+\vec{e}_k}(x) + \frac{c_k}{2} H_{\vec{n}}(x) + \frac{1}{2} \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x),$$

for $1 \leq k \leq r$, so that

$$b_{\vec{n},j} = c_j/2, \quad a_{\vec{n},j} = n_j/2, \qquad 1 \leq j \leq r.$$

see (Van Assche & Coussement, 01) and (Van Assche, 11)

Let M be a random Hermitian matrix of size $N \times N$, and consider the *ensemble* with probability distribution

$$\frac{1}{Z_N} \exp\left(-\mathrm{Tr}(M^2 - AM)\right) \mathrm{d}M, \quad \mathrm{d}M = \prod_{i=1}^{N} \mathrm{d}M_{i,i} \prod_{1 \le i \le j \le N} \mathrm{d}M_{i,j}$$

where A is a fixed Hermitian matrix (the *external source*).

Property. Suppose A has eigenvalues $c_1, \ldots, c_r$ with multiplicities $n_1, \ldots, n_r$, then

$$\mathbb{E}\left(\det\left(M - z\,\mathrm{I}_N\right)\right) = (-1)^{|\vec{n}|} H_{\vec{n}}(z).$$

For further information, I suggest to read (Martínez-Finkelshtein & Van Assche, 16) including the connection to links to non-intersecting Brownian motions.

These are given by the orthogonality relations

$$\int_0^\infty x^k L_{\vec{n}}(x) x^{\alpha_j} e^{-x}\, dx = 0, \qquad k = 0, 1, \ldots, n_j - 1,$$

for $1 \leq j \leq r$, where $\alpha_1, \ldots, \alpha_r > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$.

They can be obtained using the Rodrigues formula

$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}(x) = \prod_{j=1}^{r} \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j + \alpha_j} \right) e^{-x} \tag{4}$$

where the product of the differential operators can be taken in any order. This Rodrigues formula is useful for computing the recurrence coefficients.

Example. Multiple Laguerre polynomials of first kind. (cont.)

Indeed,

$$\int_0^\infty x^{n_j} L_{\vec{n}}(x) x^{\alpha_j} e^{-x} = (-1)^{|\vec{n}|} \int_0^\infty x^{n_j+\alpha_j-\alpha_1} \frac{d^{n_1}}{dx^{n_1}} x^{n_1+\alpha_1} \prod_{i=2}^{r} \left(x^{-\alpha_i} \frac{d^{n_i}}{dx^{n_i}} x^{n_i+\alpha_i} \right) e^{-x}\, dx$$

and integration by parts (n_1 times) gives

$$= \quad (-1)^{|\vec{n}|+n_1} \binom{n_j+\alpha_j-\alpha_1}{n_1} n_1! \int_0^\infty x^{n_j+\alpha_j} \prod_{i=2}^{r} \left(x^{-\alpha_i} \frac{d^{n_i}}{dx^{n_i}} x^{n_i+\alpha_i} \right) e^{-x}\, dx.$$

Repeating this r times gives

$$\int_0^\infty x^{n_j} L_{\vec{n}}(x) x^{\alpha_j} e^{-x} \quad = \quad \Gamma(n_j+\alpha_j+1) \prod_{i=1}^{r} \binom{n_j+\alpha_j-\alpha_i}{n_i} n_i!.$$

From the definition of multiple orthogonality, we have

$$a_{\vec{n},j} = \frac{\displaystyle\int x^{n_j} L_{\vec{n}}(x)\, d\mu_j(x)}{\displaystyle\int x^{n_j-1} L_{\vec{n}-\vec{e}_j}(x)\, d\mu_j(x)},$$

which implies

$$a_{\vec{n},j} = n_j(n_j+\alpha_j) \prod_{i=1, i\neq j}^{r} \frac{n_j+\alpha_j-\alpha_i}{n_j-n_i+\alpha_j-\alpha_i}, \qquad j=1,\ldots,r. \tag{5}$$

The recurrence coefficients $b_{\vec{n},k}$ can be obtained by comparing the coefficients of $x^{|\vec{n}|}$ in the recurrence relation

$$xL_{\vec{n}}(x) = L_{\vec{n}+\vec{e}_i}(x) + b_{\vec{n},i}L_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}L_{\vec{n}-\vec{e}_j}(x)\ ,\ i = 1, \ldots, r-1.$$

They are

$$b_{\vec{n},k} = |\vec{n}| + n_k + \alpha_k + 1, \qquad k = 1, \ldots, r. \tag{6}$$

These are given by the orthogonality relations

$$\int_0^\infty x^k L_{\vec{n}}(x) x^\alpha e^{-c_j x}\, dx = 0, \qquad k = 0, 1, \ldots, n_j - 1,$$

for $1 \leq j \leq r$, where $\alpha > -1$, $c_1, \ldots, c_r > 0$ and $c_i \neq c_j$ whenever $i \neq j$. They can be obtained using the Rodrigues formula

$$(-1)^{|\vec{n}|} \left(\prod_{j=1}^r c_j^{n_j} \right) x^\alpha L_{\vec{n}}(x) = \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right) x^{|\vec{n}|+\alpha} \tag{7}$$

where the differential operators in the product can be taken in any order. A useful integral is

$$\int_0^\infty e^{-\lambda x} \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right) x^{|\vec{n}|+\alpha}\, dx = (-1)^{|\vec{n}|} \frac{\Gamma(|\vec{n}|+\alpha+1)}{\lambda^{|\vec{n}|+\alpha+1}} \prod_{j=1}^r (c_j - \lambda)^{n_j}$$

which can be evaluated by using integration by parts in a similar way as in the previous example. Observe that the right hand side has a zero at $\lambda = c_j$ of multiplicity n_j. Using (7) we thus have for $\lambda > 0$

$$\int_0^\infty e^{-\lambda x} x^\alpha L_{\vec{n}}(x)\, dx = \frac{\Gamma(|\vec{n}|+\alpha+1)}{\lambda^{|\vec{n}|+\alpha+1}} \prod_{i=1}^r (1 - \lambda/c_i)^{n_i}.$$

Clearly

$$\frac{d^k}{d\lambda^k}\int_0^\infty e^{-\lambda x}x^\alpha L_{\vec{n}}(x)\,dx\Big|_{\lambda=c_j}=(-1)^k\int_0^\infty x^k e^{c_j x}x^\alpha L_{\vec{n}}(x)\,dx=0, \qquad 0\le k<n_j,$$

which confirms the orthogonality relations, and for $k=n_j$

$$\int_0^\infty x^{n_j}e^{c_j x}x^\alpha L_{\vec{n}}(x)\,dx=\frac{\Gamma(|\vec{n}|+\alpha+1)}{c_j^{|\vec{n}|+n_j+\alpha+1}}n_j!\prod_{i=1,i\neq j}^{r}\left(1-\frac{c_j}{c_i}\right).$$

The fact that

$$a_{\vec{n},j}=\frac{\int x^{n_j}L_{\vec{n}}(x)\,d\mu_j(x)}{\int x^{n_j-1}L_{\vec{n}-\vec{e}_j}(x)\,d\mu_j(x)},$$

implies

$$a_{\vec{n},j}=\frac{(|\vec{n}|+\alpha)n_j}{c_j^2}, \qquad 1\le j\le r. \tag{8}$$

For the coefficients $b_{\vec{n},k}$ a comparison of the coefficients of $x^{|\vec{n}|}$ on both sides of the recurrence relation satisfied by $L_{\vec{n}}(x)$, and Eq. (23.4.5) in (Ismail) leads to

$$b_{\vec{n},k}=\frac{|\vec{n}|+\alpha+1}{c_k}+\sum_{j=1}^{r}\frac{n_j}{c_j}. \tag{9}$$

John Wishart (1928) introduced the *Wishart distribution* for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \quad X \in \mathbb{C}^{N\times(N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ:

$$\frac{1}{Z_N} \exp\left(-\mathrm{Tr}(\Sigma^{-1}M)\right) (\det M)^p \, \mathrm{d}M.$$

If $\Sigma = I_N$, then Laguerre polynomials (with $\alpha = p$) play an important role.

If Σ^{-1} has eigenvalues $c_1, \ldots, c_r$ with multiplicities $n_1, \ldots, n_r$, then *multiple Laguerre polynomials of the second kind* are crucial:

$$\mathbb{E}\left(\det(M - z\ I_N)\right) = (-1)^{|\vec{n}|} L_{\vec{n}}^{p,\vec{c}}(z).$$

for further details see

Kuijlaars, A.B.J.: Multiple orthogonal polynomials in random matrix theory. In: Bhatia, R. (ed.) Proceedings of the International Congress of Mathematicians, vol. III, Hyderabad, India, pp. 1417-1432 (2010)

Example. Jacobi-Piñeiro multiple orthogonal polynomials

These are multiple orthogonal polynomials on $[0,1]$ for the Jacobi weights $x_i^{\alpha}(1-x)^{\beta}$, with $\alpha_1,\ldots,\alpha_r,\beta > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$. They satisfy

$$\int_0^1 x^k P_{\vec{n}}(x) x^{\alpha_j}(1-x)^{\beta}\, dx = 0, \qquad k = 0,1,\ldots,n_j-1,\ 1 \le j \le r.$$

They are given by the Rodrigues formula

$$(-1)^{|\vec{n}|}\prod_{j=1}^{r}(|\vec{n}|+\alpha_j+\beta+1)_{n_j}\ (1-x)^{\beta} P_{\vec{n}}(x) = \prod_{j=1}^{r}\left(x^{-\alpha_j}\frac{d^{n_j}}{dx^{n_j}}x^{n_j+\alpha_j}\right)(1-x)^{|\vec{n}|+\beta},$$

where the product of differential operators is the same as for the multiple Laguerre polynomials of the first kind.

One has

$$\int_0^1 x^\gamma P_{\vec{n}}(x)(1-x)^\beta \, dx = (-1)^{|\vec{n}|} \frac{\prod_{i=1}^r (\alpha_i - \gamma)_{n_i}}{\prod_{i=1}^r (|\vec{n}| + \alpha_i + \beta + 1)_{n_i}} \frac{\Gamma(\gamma+1)\Gamma(|\vec{n}|+\beta+1)}{\Gamma(|\vec{n}|+\beta+\gamma+2)}$$

so that

$$\int_0^1 x^{n_j+\alpha_j} P_{\vec{n}}(x)(1-x)^\beta \, dx = \frac{\prod_{i=1}^r \binom{n_j+\alpha_j-\alpha_i}{n_i} n_i!}{\prod_{i=1}^r (|\vec{n}| + \alpha_i + \beta + 1)_{n_i}} \frac{\Gamma(n_j+\alpha_j+1)\Gamma(|\vec{n}|+\beta+1)}{\Gamma(|\vec{n}|+n_j+\alpha_j+\beta+2)}.$$

Using the expression for $a_{\vec{n},j}$ (as in the previous examples) to then find for $1 \leq j \leq r$

$$\begin{aligned} a_{\vec{n},j} = {} & \prod_{i=1, i\neq j}^r \frac{n_j + \alpha_j - \alpha_i}{n_j - n_i + \alpha_j - \alpha_i} \prod_{i=1}^r \frac{|\vec{n}| + \alpha_i + \beta}{|\vec{n}| + n_i + \alpha_i + \beta} \\ & \times \frac{n_j(n_j+\alpha_j)(|\vec{n}|+\beta)}{(|\vec{n}|+n_j+\alpha_j+\beta+1)(|\vec{n}|+n_j+\alpha_j+\beta)(|\vec{n}|+n_j+\alpha_j+\beta-1)}. \end{aligned} \tag{10}$$

Example. Multiple Charlier polynomials

The orthogonality relations are

$$\sum_{k=0}^{\infty} C_{\vec{n}}(k) k^{\ell} \frac{a_j^k}{k!} = 0, \qquad \ell = 0, 1, \ldots, n_j - 1,$$

for $1 \leq j \leq r$, where $a_i > 0$ and $a_i \neq a_j$ whenever $i \neq j$. The recurrence relation is given by

$$xC_{\vec{n}}(x) = C_{\vec{n}+\vec{e}_k}(x) + (a_k + |\vec{n}|) C_{\vec{n}}(x) + \sum_{j=1}^{r} n_j a_j C_{\vec{n}-\vec{e}_j}(x),$$

so that

$$b_{\vec{n},j} = |\vec{n}| + a_j, \quad a_{\vec{n},j} = n_j a_j, \qquad 1 \leq j \leq r.$$

Let $(\vec{n}_k)_{k\geq 0}$ be a path in $\mathbb{N}^r$ starting from $\vec{n}_0 = 0$, such that $\vec{n}_{k+1} - \vec{n}_k = \vec{e}_i$ for some $i = 1, \ldots, r$. Then

$$xP_{\vec{n}_k}(x) = P_{\vec{n}_{k+1}}(x) + \sum_{j=0}^{r} \alpha_{\vec{n}_k,j} P_{\vec{n}_{k-j}}(x).$$

An important case is the **stepline**

$$\vec{n}_k = (\underbrace{i+1, \ldots, i+1}_{j}, \overbrace{i, \ldots, i}^{r-j}), \text{ with } k = ri + j, \text{ for } \quad j = 0, \ldots r-1.$$

where $|\vec{n}_k| = k = ri + j$.

For the the **stepline**

$$\vec{n}_k = (\underbrace{i+1,\ldots,i+1}_{j},\overbrace{i,\ldots,i}^{r-j}), \text{ with } k = ri+j, \text{ for } \quad j = 0,\ldots r-1.$$

we consider

$$B_k(x) = P_{\vec{n}_k}(x), \qquad \text{for} \qquad k \geq 0.$$

Hence $B_k(x)$ satisfies the following recurrence relation of order $r+1$, in the sense that there exist coefficients β_k and $\gamma_k^{(j)}$ for $j = 0,1,\ldots,r-1$ such that

$$xB_n = B_{n+1}(x) + \beta_n B_n(x) + \sum_{k=1}^{r} \gamma_{n-k}^{(r-k)} B_{n-1-k}$$

The polynomial sequence $\{B_n\}_{n\geq 0}$ is an r-**orthogonal polynomial sequence**... to be explained....

Definition. The monic r-orthogonal polynomial sequence $\{B_n\}_{n\geq 0}$ for $(\mu_1,\ldots,\mu_r)$ is such that

$$\int_S x^k B_n(x)\mathrm{d}\mu_j(x) = 0,\ n \geq rk+j,$$
$$\int_S x^n B_{rn+j-1}(x)\mathrm{d}\mu_j(x) = 0,\ n \geq 0,$$

for each $j = 1,\ldots,r$. (Maroni, 89)

Theorem. The monic polynomial sequence $\{B_n\}_{n\geq 0}$ is r-orthogonal iff

$$xB_n = B_{n+1}(x) + \beta_n B_n(x) + \sum_{k=1}^{r} \gamma_{n-k+1}^{(r-k)} B_{n-k}$$

with $\gamma_{n-r+1}^{(0)} \neq 0$ for $n \geq r$.

We can now introduce the banded Hessenberg matrix H_n such that

$$H_n \begin{pmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{n-1}(x) \end{pmatrix} = x \begin{pmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{n-1}(x) \end{pmatrix} - B_n(x) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Exercise. Find an explicit expression for H_n based on the recurrence relation

$$xB_n = B_{n+1}(x) + \beta_n B_n(x) + \sum_{k=1}^{r} \gamma_{n-k+1}^{(r-k)} B_{n-k}.$$

Hence, each zero of $B_n(x)$ is an eigenvalue for H_n.

The matrix H_n is not symmetric and there is no obvious way to do so for $r \geq 2$.

No reason, in general, for the eigenvalues (zeros) to be real. Nonetheless in several examples it is the case.

An important case is when the banded Hessenberg matrix has only non zero entries on the extremes of the band. This means that

$$xB_n = B_{n+1}(x) + \gamma_{n-r+1}^{(r-k)} B_{n-r}.$$

Comments on r-orhogonality and multiple orthogonality

- The examples seen are extensions of classical orthogonal polynomials, in the sense that the corresponding weight functions satisfy a *Pearson equation*
$$(\phi_k(x)w_k(x))' + \psi_k(x)w_k(x) = 0$$
with $\deg \phi_k \leq 2$ and $\deg \psi_k = 1$.

- There are also well studied cases where r-orthogonal polynomials arise from an extension of Hahn's classical character and this gives rise to weight functions that are solution to a second order differential equation (certainly not of Pearson type)! For such type of weights, the concept of multiple orthogonality is quite natural.

- The concept of "classical" in the context of multiple orthogonality is not unique. Depending on which property of the very classical polynomials one takes, this will give rise to completely independent multiple orthogonal polynomials.
- There are also examples of extensions of semiclassical polynomials into the context of multiple orthogonality. Typically, examples of Angelesco systems, such as the Jacobi-Angelesco polynomials or the Jacobi-Laguerre polynomials. The realm of extensions goes beyond these notions...

The case of 2-orthogonal polynomials

The monic 2-OPS $\{P_n\}_{n\geq 0}$ for $\mathbf{u} = (u_0, u_1)$ satisfies a third order recurrence relation (see Van Iseghem'88, Maroni'89)

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x) \tag{11}$$

with $P_0(x) = 1, \;\; P_1(x) = x - \beta_0$ and $P_2(x) = (x - \beta_1)P_1(x) - \alpha_1$.

The monic 2-OPS $\{P_n\}_{n\geq 0}$ for $\mathbf{u} = (u_0, u_1)$ satisfies a third order recurrence relation (see Van Iseghem'88, Maroni'89)

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x) \tag{11}$$

with $P_0(x) = 1$, $P_1(x) = x - \beta_0$ and $P_2(x) = (x - \beta_1)P_1(x) - \alpha_1$.

Expressions for the recurrence coefficients follow immediately from the definition. For instance,

$$\gamma_{2n+1} = \frac{< u_0, x^{n+1} P_{2n+2} >}{< u_0, x^n P_{2n} >}, \quad \gamma_{2n+2} = \frac{< u_1, x^{n+1} P_{2n+3} >}{< u_1, x^n P_{2n+1} >}, \quad n \geq 0.$$

Conversely, we also have

$$N_0(n) := < u_0, x^{n+1} P_{2n+2} > = \prod_{k=0}^{n} \gamma_{2k+1}$$

and

$$N_1(n) := < u_1, x^{n+1} P_{2n+3} > = \prod_{k=0}^{n} \gamma_{2k+2}, \quad \text{for} \quad n \geq 0.$$

The type II multiple Laguerre polynomials of second kind for $r = 2$ measures on the step-line are

$$B_{2n}(x) = L_{(n,n)(x)}, \quad B_{2n+1}(x) = L_{(n+1,n)(x)}$$

satisfy

$$xB_n(x) = B_{n+1}(x) + \beta_n B_n(x) + \gamma_n^{(1)} B_{n-1} + \gamma_{n-1}^{(0)} B_{n-2}$$

where

$$\begin{aligned}
&\beta_{2n} = 3n + \alpha_1 + 1, \quad \beta_{2n+1} = 3n + \alpha_2 + 2,\\
&\gamma_{2n}^{(1)} = n(3n + \alpha_1 + \alpha_2), \quad \gamma_{2n+1}^{(1)} = 3n^2 + n(3 + \alpha_1 + \alpha_2) + \alpha_1 + 1,\\
&\gamma_{2n}^{(0)} = n(n + \alpha_1)(n + \alpha_1 - \alpha_2), \; \gamma_{2n+1}^{(0)} = n(n + \alpha_2)(n + \alpha_2 - \alpha_1).
\end{aligned}$$

Example 1 – The 2-orthogonal polynomials with constant rec coef

The sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ satisfying the recurrence relation

$$P_{n+1}(x) = xP_n(x) - 3^{\delta_{n,0}}\frac{4}{27}P_{n-2}(x)$$

is 2-orthogonal with respect to $U = (u_0, u_1)$ such that

$$\begin{cases} (x^3-1)u_0'' + \frac{3}{2}x^2u_0' - \frac{1}{2}xu_0 = 0 \\ u_1 = 3(x^3-1)u_0' - \frac{3}{2}x^2u_0 \end{cases}$$

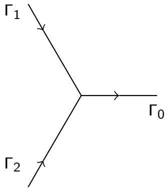
Such vector functional admits an integral representation on the real line as follows

$$\begin{aligned} < u_0, f(x) > &= \int_0^1 f(x)\frac{9\sqrt{3}}{4\pi}\left[(1+\sqrt{1-x^3})^{1/3} - (1-\sqrt{1-x^3})^{1/3}\right]\mathrm{d}x \\ &\quad + \int_0^{+\infty} f(x)3\mathrm{e}^{-x}\left[\lambda_1\sqrt{x}\cos(\sqrt{3}x) + \lambda_2 x^2\sin(\sqrt{3}x)\right]\mathrm{d}x, \\ < u_1, f(x) > &= \int f(x)U_1(x)\mathrm{d}x, \end{aligned}$$

(See Douak&Maroni'97 for further details.)

Consider the monic polynomials $P_{n,m}$ of degree $n+m$ for which

$$\int_{\Gamma_0\cup\Gamma_1} x^j P_{n,m}(x)\exp(-x^3+tx)dx = 0,\ j=0,\dots,n-1,$$
$$\int_{\Gamma_0\cup\Gamma_2} x^j P_{n,m}(x)\exp(-x^3+tx)dx = 0,\ j=0,\dots,m-1,$$



with $\Gamma_k = \{z\in\mathbb{C} : \arg z = e^{2k\pi i/3}\},\ k=0,1,2.$
(see Van Assche & Filipuk & Zhang (2015))

Rodrigues' formula:

$$\mathrm{e}^{-x^3+tx}P_{n,n+m}(x) = \frac{(-1)^n}{3^n}\frac{d^n}{dx^n}\left(\mathrm{e}^{-x^3+tx}P_{0,m}(x)\right)$$
$$\mathrm{e}^{-x^3+tx}P_{n+m,n}(x) = \frac{(-1)^n}{3^n}\frac{d^n}{dx^n}\left(\mathrm{e}^{-x^3+tx}P_{m,0}(x)\right)$$

where $P_{m,0}$ and $P_{0,m}$ are orthogonal polynomials...

and $\{P_{k,k}\}_k$ is 2-OPS.

(Case $t=0$ already in Pólya and Szegő (1925).
Special case of Gould-Hopper polynomials (1962).)

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Definition

A monic 2-OPS $\{P_n\}_{n\geq 0}$ is "**classical**" in Hahn's sense when the sequence of its derivatives $\{Q_n\}_{n\geq 0}$, with

$$Q_n(x) = \frac{1}{n+1} P'_{n+1}(x)$$

is also a 2-OPS.

Hence, as a monic 2-OPS, the sequence $\{Q_n\}_{n\geq 0}$ satisfies a third order recurrence relation:

$$Q_{n+1}(x) = (x - \widetilde{\beta}_n) Q_n(x) - \widetilde{\alpha}_n Q_{n-1}(x) - \widetilde{\gamma}_{n-1} Q_{n-2}(x), \ n \geq 2, \qquad (12)$$

with $Q_0 = 1$, $Q_1(x) = x - \widetilde{\beta}_0$ and $Q_2(x) = (x - \widetilde{\beta}_1) Q_1(x) - \widetilde{\alpha}_1$.

On the other hand, the 2-orthogonality of $\{P_n\}_{n\geq 0}$ for $U = (u_0, u_1)$ and the 2-orthogonality of $\{Q_n\}_{n\geq 0}$ for $V = (v_0, v_1)$ implies

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \mathbf{\Phi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \tag{13}$$

and also that

$$\begin{bmatrix} v_0' \\ v_1' \end{bmatrix} = -\mathbf{\Psi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}. \tag{14}$$

with

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{0,0} & \phi_{0,1} \\ \phi_{1,0} & \phi_{1,1} \end{bmatrix} \quad \text{and} \quad \mathbf{\Psi} = \begin{bmatrix} 0 & 1 \\ \psi(x) & \zeta \end{bmatrix}$$

where $\psi(x) = \frac{2}{\gamma_1} P_1(x)$ and $\zeta = -\frac{2\alpha_1}{\gamma_1}$,

whilst $\deg\{\phi_{0,0}, \phi_{0,1}, \phi_{1,1}\} \leq 1$ and $\deg \phi_{1,0} \leq 2$.

Theorem
The monic 2-OPS $\{P_n\}_{n\geq 0}$ for $U = (u_0, u_1)$ is "classical" iff there are polynomials ψ and $\phi_{i,j}$, with $i,j \in \{0,1\}$, and a constant ζ such that

$$\left(\begin{bmatrix} \phi_{0,0} & \phi_{0,1} \\ \phi_{1,0} & \phi_{1,1} \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right)' + \begin{bmatrix} 0 & 1 \\ \psi(x) & \zeta \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (15)$$

where $\deg\{\phi_{0,0}, \phi_{0,1}, \phi_{1,1}\} \leq 1$, $\deg \phi_{1,0} \leq 2$ *and* $\deg \psi = 1$.

Relation (15) reads as follows

$$\left(\mathbf{\Phi}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right)' + \mathbf{\Psi}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(Maroni& Douak'92, Maroni'99)

Corollary

If the monic 2-OPS $\{P_n\}_{n\geq 0}$ for $U=(u_0,u_1)$ is "classical", then

$$\begin{cases} c(x)\left((\phi\, u_0)'' - \Big((\phi' + d(x) - a(x))u_0\Big)' + (b(x) - a'(x))u_0\right) \\ = c'(x)\left(\phi\, u_0' - d(x)u_0\right), \\ \\ c(x)u_1 = \phi\, u_0' - d(x)u_0\ . \end{cases}$$

where

$$\begin{cases} a(x) = \phi_{0,0}(x)\left(\phi_{1,1}'(x)+\zeta\right) - \phi_{1,0}(x)\left(\phi_{0,1}'(x)+1\right) \\ b(x) = \phi_{0,0}'(x)\left(\phi_{1,1}'(x)+\zeta\right) - \left(\phi_{0,1}'(x)+1\right)\left(\phi_{1,0}'(x)+\psi(x)\right) \\ c(x) = \phi_{0,1}(x)\left(\phi_{1,1}'(x)+\zeta\right) - \phi_{1,1}(x)\left(\phi_{0,1}'(x)+1\right) \\ d(x) = \phi_{0,1}(x)\left(\phi_{1,0}'(x)+\psi(x)\right) - \phi_{1,1}(x)\phi_{0,0}'(x) \end{cases} \tag{16}$$

and

$$\phi(x) = \det \mathbf{\Phi} = \phi_{0,0}(x)\phi_{1,1}(x) - \phi_{0,1}(x)\phi_{1,0}(x)$$

Consequently, we have

$$\widetilde{\gamma}_{2n+1} = \frac{2n+1}{2n+3}\left(\frac{1-(n+1)\phi'_{0,1}(0)}{1-n\phi'_{0,1}(0)}\right)\gamma_{2n+2}$$

and

$$\widetilde{\gamma}_{2n+2} = \frac{n+1}{n+2}\left(\frac{1-(n+1)\frac{\phi''_{1,0}(0)}{2\psi'(0)}}{1-n\frac{\phi''_{1,0}(0)}{2\psi'(0)}}\right)\gamma_{2n+3}$$

Lengthier expressions can be obtained relating the recurrence coefficients β_n, $\widetilde{\beta}_n$, α_n, and $\widetilde{\alpha}_n$.

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x)$$

with

$$\begin{aligned}
\beta_n &= 3n^2 + (2\alpha + 2\beta + 3)n + (1+\alpha)(1+\beta) \\
\alpha_n &= n(3n + \alpha + \beta)(n+\alpha)(n+\beta),\ \ n \geq 1, \\
\gamma_n &= n(n+1)(n+\alpha+1)(n+\alpha)(n+\beta+1)(n+\beta),\ \ n \geq 2,
\end{aligned}$$

They satisfy the 3rd order recurrence relation

$$x^2 P_n''' + (3+\alpha+\beta)xP_n'' + ((\alpha+1)(\beta+1) - x)P_n' = -nP_n$$

and are 2-OPS for $U = (u_0, u_1)$ satisfying

$$x^2 u_0'' - (\alpha+\beta-1)xu_0' - (x - \alpha\beta)u_0 = 0 \quad , \quad (\alpha+1)(\beta+1)u_1 = -(xu_0)'$$

Such vector functional $U = (u_0, u_1)$ admits the following integral representation

$$\begin{aligned}
< u_0, f(x) > &= \tfrac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{+\infty} f(x) x^{(\alpha+\beta)/2} K_{\alpha-\beta}(2\sqrt{x}) \mathrm{d}x, \\
< u_1, f(x) > &= \tfrac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{+\infty} f(x) \left(x^{(\alpha+\beta)/2} K_{\alpha-\beta}(2\sqrt{x}) \right)' \mathrm{d}x,
\end{aligned}$$

(See Ben Cheikh&Douak'00 and Van Assche&Yakubovich'00.)

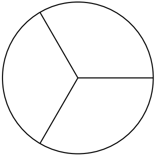
Definition

A monic polynomial sequence $\{B_n\}_{n\geq 0}$ is 3-fold symmetric if and only if

$$B_n(\mathrm{e}^{\frac{2i\pi}{3}}x) = \mathrm{e}^{\frac{2in\pi}{3}} B_n(x)$$

and

$$B_n(\mathrm{e}^{\frac{4i\pi}{3}}x) = \mathrm{e}^{\frac{4in\pi}{3}} B_n(x), \ n \geq 0.$$



In other words, this is to say that there exist three sequences $\{B_n^{[j]}\}_{n\geq 0}$ with $j \in \{0,1,2\}$ such that

$$\begin{aligned} B_{3n}(x) &= B_n^{[0]}(x^3), \\ B_{3n+1}(x) &= xB_n^{[1]}(x^3), \\ B_{3n+2}(x) &= x^2 B_n^{[2]}(x^3), \end{aligned}$$

(The sequences $\{B_n^{[j]}\}_{n\geq 0}$ are the components of the cubic decomposition of the 3-fold symmetric sequence $\{B_n\}_{n\geq 0}$.)

(see Barrucand&Dickinson'66)

Whilst we are dealing with 3-fold symmetric and 2-orthogonal sequences, we recall the following result.

Theorem (Douak & Maroni'92)

Let $\{P_n\}_{n\geq 0}$ be a 2-orthogonal polynomial sequence for $U=(u_0,u_1)$. Then, $\{P_n\}_{n\geq 0}$ is 3-fold symmetric iff if satisfies the third order recurrence relation

$$P_{n+1}(x)=xP_n(x)-\gamma_{n-1}P_{n-2}(x),\ n\geq 2,$$

with $P_0(x)=1,\ P_1(x)=x$ and $P_2(x)=x^2$.

(Observe that this is a three-term recurrence relation!)

Moreover, we have

Lemma (Douak & Maroni'92)

If the a 3-fold symmetric sequence $\{P_n\}_{n\geq 0}$ *is 2-orthogonal, then the three components in the cubic decomposition of* $\{P_n\}_{n\geq 0}$ *are also 2-orthogonal fulfilling the recurrence relations:*

$$P_{n+1}^{[k]}(x) = (x-\beta_n^{[k]})P_n^{[k]}(x) - \alpha_n^{[k]}P_{n-1}^{[k]}(x) - \gamma_{n-1}^{[k]}P_{n-2}^{[k]}(x),$$

where

$$\begin{aligned}
\beta_n^{[k]} &= \gamma_{3n-1+k}+\gamma_{3n+k}+\gamma_{3n+1+k},\ n\geq 0,\\
\alpha_n^{[k]} &= \gamma_{3n-2+k}\gamma_{3n+k}+\gamma_{3n-1+k}\gamma_{3n-3+k}+\gamma_{3n-2+k}\gamma_{3n-1+k},\ n\geq 1,\\
\gamma_n^{[k]} &= \gamma_{3n-2+k}\gamma_{3n+k}\gamma_{3n+2+k}\neq 0,\ n\geq 2,
\end{aligned}$$

for each $k=0,1,2$.

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Theorem. (Aptekarev *et al.*'00)
If $\gamma_n > 0$ for $n \geq 1$ in

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x),$$

then $\{P_n\}_{n\geq 0}$ is a 2-OPS w.r.t. the vector of linear functionals (u_0, u_1) and

$$< u_0, f(x) >= \int_S f(x)\mathrm{d}\mu_0(x) \tag{17}$$

$$< u_1, f(x) >= \int_S f(x)\mathrm{d}\mu_1(x) \tag{18}$$

where S represents the starlike set

$$S := \bigcup_{k=0}^{2} \Gamma_k \qquad \text{with} \qquad \Gamma_k = [0, \mathrm{e}^{2\pi ik/3}\infty),$$

and the measures have a common support which is a subset of S and are invariant under rotations of $2\pi/3$.

Theorem. (Ben Romdhane'08)
Let $\{P_n\}_{n\geq 0}$ be a 2-OPS satisfying

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x).$$

If $\gamma_n > 0$, then the following statements hold

(a) If x is a zero of P_{3n+j}, then $\omega^k x$ are also zeros of P_{3n+j} with $\omega = \mathrm{e}^{2\pi i/3}$

(b) 0 is a zero of P_{3n+j} of multiplicity j when $j = 1,2$

(c) P_{3n+j} has n distinct positive real zeros

(d) Between two real zeros of P_{3n+j+3} there exist only one zero of P_{3n+j+2} and only one zero of P_{3n+j+1}

Theorem. (AL & Van Assche'18)
Let $\{P_n\}_{n\geq 0}$ be a 2-OPS satisfying $P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x)$.
If

$$0 < \gamma_n \leq cn^{\alpha} + o(n^{\alpha})$$

(with $c,\ \alpha > 0$), then the **largest zero** $x_{n,n}$ is s.t.

$$x_{n,n} \leq \frac{3}{2^{2/3}} c^{1/3} n^{\alpha/3} + o(n^{\alpha/3}).$$

These satisfy a third order differential equation

Lemma (Douak&Maroni'97)

If a 2-symmetric 2-OPS $\{P_n\}_{n\geq 0}$ is "classical", then each polynomial is a solution of the third order differential equation

$$(a_n x^3 - b_n)P'''_{n+1} + c_n x^2 P''_{n+1} + d_n x P'_{n+1} = e_n P_{n+1}$$

where

$$\begin{aligned}
a_n &= (\vartheta_n - 1)(\vartheta_{n+1} - 1) \\
b_n &= \tfrac{\gamma_{n+3}((n+3)\vartheta_{n+2}-(n+2))((n+4)\vartheta_{n+1}-(n+3))((n+5)\vartheta_{n+2}-(n+4))}{(n+3)(n+4)} \\
c_n &= \vartheta_n\vartheta_{n+1} - 1 - (n-3)(\vartheta_n - 1)(\vartheta_{n+1} - 1) \\
d_n &= n\vartheta_{n+1} - (n-1)\vartheta_n(2\vartheta_{n+1} - 1) \\
e_n &= n\vartheta_{n+1}, \quad \textit{for any} \quad n \geq 1,
\end{aligned}$$

with $a_0 = b_0 = c_0 = d_0 = e_0 = 0$.

Here $Q_n(x) := \frac{1}{n+1} P'_{n+1}(x) = P_n(x)$. Additionally

$$\gamma_{n+1} = (n+1)(n+2), \qquad \text{and} \qquad \begin{cases} u_0'' - x\ u_0 = 0 \\ \\ u_1 = -u_0' \end{cases}$$

and

$$-P'''_{n+1}(x) + xP'_{n+1}(x) = nP_{n+1}(x),\ n \geq 0.$$

Remark. The polynomials appear in the Vorob'ev-Yablonski polynomials associated with rational solutions of Painlevé II equations (Clarkson & Mansfield'03)

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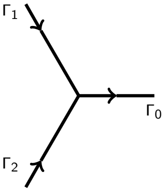
Integral representation (AL&VA)

$$
\begin{aligned}
< u_0, f > &= \int_\Gamma f(x) W_0(x) dx, \text{ for all } f \in \mathscr{P}, \\
< u_1, f > &= \int_\Gamma f(x) W_1(x) dx, \text{ for all } f \in \mathscr{P},
\end{aligned}
$$

where $W_0 : \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \longrightarrow \mathbb{R}$ defined by

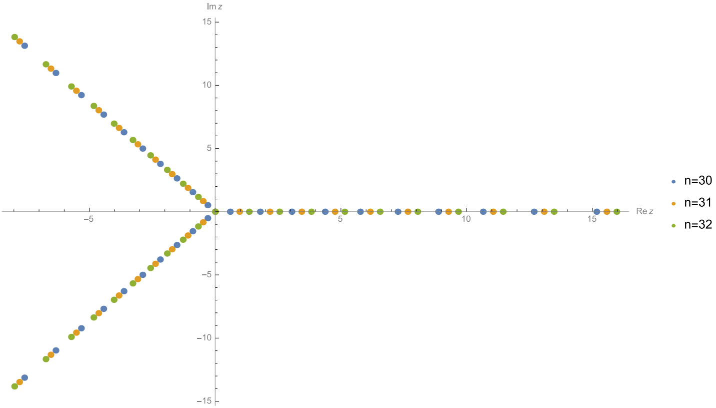
$$W_0(x) = \mathrm{Ai}(x)\mathbb{I}_{\Gamma_0} - \mathrm{e}^{-2\pi i/3}\mathrm{Ai}(\mathrm{e}^{-2\pi i/3}x)\mathbb{I}_{\Gamma_1} - \mathrm{e}^{2\pi i/3}\mathrm{Ai}(\mathrm{e}^{2\pi i/3}x)\mathbb{I}_{\Gamma_2}$$

with $\Gamma_k = \left\{ w : \arg(w) = \frac{2k\pi}{3} \right\}$, with $k = 0, 1, 2,$



where the orientations of Γ_k are all taken from left to right

Plot of the zeros of P_{30}, P_{31} and P_{32}



Remarks.

- All the zeros of $P_n(x)$ are located on $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$
- In each Γ_k, between two zeros of P_{n+2} there is one zero of P_n and P_{n+1}.

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