London Taught Course on Spectral Theory
Problems for weeks 1 – 3.

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The following problems are similar to those that might be asked in the question that I am expected to produce for a final test. You may send me written work, but please only do so when you do not feel confident about your solution for a problem. Also make sure that what you send me is easily readable.

1. Let $\mathcal{B} = C[0, a]$ and define the Volterra operator V by
$$(Vf)(x) = \int_0^x f(s)\,\mathrm{d}s.$$
Prove that V is a bounded operator and that it has no eigenvalues. Prove also that $0 \in \mathrm{Spec}(V)$.

2. Prove that if A is a bounded operator on $\mathcal{B}$ and $\lambda, \mu \notin \mathrm{Spec}(A)$ then
$$\begin{aligned} R(\lambda, A)R(\mu, A) &= R(\mu, A)R(\lambda, A), \\ R(\lambda, A) - R(\mu, A) &= (\mu - \lambda)R(\mu, A)R(\lambda, A). \end{aligned}$$

3. Let $\theta \in \mathbf{R}$ and let $\mathcal{H} = L^2(0, 2\pi)$. Define the operator A by $(Af)(x) = if'(x)$ on the domain
$$\mathcal{D} = \{f \in C^1[0, 2\pi] : f(2\pi) = \mathrm{e}^{i\theta} f(0)\}.$$
Find all of the eigenvalues of A and use Lemma 1.2.2 to prove that A is essentially self-adjoint on $\mathcal{D}$.

4. Let $\mathcal{H} = L^2(0, 1)$ and define A by
$$(Af)(x) = a(x)f''(x) + b(x)f'(x) + c(x)f(x)$$

on the domain

$$\mathcal{D} = f \in C^2[0,1] : f(0) = f(1) = 0\}.$$

Assuming that a, b, c are all sufficiently differentiable, real-valued functions on [0,1], use integration by parts to find the precise conditions that ensure that A is a symmetric operator.

5. Let $\mathcal{H}$ be a Hilbert space and let $P : \mathcal{H} \to \mathcal{H}$ be a bounded operator satisfying $P = P^* = p^2$. Prove that $\mathcal{L} = \mathrm{Ker}(P)$ and $\mathcal{M} = \mathrm{Ran}(P)$ are orthogonal linear subspaces and that $\mathcal{H} = \mathcal{L} + \mathcal{M}$ as an algebraic direct sum.

6. Let $\mathcal{H} = L^2(0,1)$ and define A by $(Af)(x) = a(x)f(x)$ where $a : [0,1] \to \mathbf{C}$ is a bounded continuous function. Use the definition of spectrum to determine $\mathrm{Spec}(A)$. Can A have any eigenvalues?

7. Let $\mathcal{H} = \ell^2(\mathbf{N})$ where $\mathbf{N}$ is the set of all natural numbers and define the bounded operator A by

$$(Af)(n) = \cos(n)f(n)$$

Find the set of all eigenvalues of A and prove that $\mathrm{Spec}(A) = [-1, 1]$.

8. Let $\mathcal{H} = \ell^2(\mathbf{N})$ and define the bounded operators A_m on $\mathcal{H}$ by

$$(A_m f)(n) = \begin{cases} f(n) & \text{if } n \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\|A_m\| = 1$ for all $m \in \mathbf{N}$ but that $\lim_{m\to\infty} \|A_m f\| = 0$ for all $f \in \mathcal{H}$. In other words A_m converge to 0 strongly but not in norm.

9. Let $\mathcal{H} = \ell^2(\mathbf{N})$ and define the bounded 'left shift' operator L by

$$(Lf)(n) = f(n+1).$$

Prove that $\|L\| = 1$ and that every $\lambda \in \mathbf{C}$ such that $|\lambda| < 1$ is an eigenvalue of L. Find $\mathrm{Spec}(L)$.