London Taught Course on Spectral Theory
Problems for week 4.

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1. Let H be the operator defined on the domain $C^2[-1,1]$ in $\mathcal{H} = L^2((-1,1), \mathrm{d}x)$ by
$$(Hf)(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{(1-x^2)^2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right\}.$$
Prove that H is symmetric and find its associated quadratic form. Prove that if $\mathcal{P}_n$ is the space of all polynomials of degree at most n then $H(\mathcal{P}_n) \subseteq \mathcal{P}_n$ for all n. Use the Gram-Schmidt method of the last lecture to find a series of eigenvalues of H.

2. Let $\mathcal{H} = \ell^2(\mathbf{N})$ and let
$$(Af)_n = a_n f_n$$
for all f on the usual maximal domain of A, where a_n is a complex-valued sequence. Prove that A is unitary if and only if $|a_n| = 1$ for all n and find the spectrum of A.

3. Prove that the map $U : L^2((-\pi,\pi), \mathrm{d}x) \to \ell^2(\mathbf{Z})$ defined by $(Uf)_n = a_n$ where a_n is the Fourier coefficient
$$a_n = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-inx} \, \mathrm{d}x$$
is unitary. What is the inverse of U?

4. Use Fourier series methods as in the last problem to find the spectrum of the differential operator H on $L^2((-\pi,\pi), \mathrm{d}x)$ defined by $Hf = af'' + bf' + cf$, on the domain $\mathcal{D}$ of C^2 functions f such that $f(-\pi) = f(\pi)$ and $f'(-\pi) = f'(\pi)$, where a, b, c are complex constants. Find the precise conditions that ensure that H is self-adjoint on an appropriate domain.