The Riemann-Hilbert method and the Painlevé equations

General Information

The classical Painlevé equations are getting increasingly involved in many areas of modern mathematical physics and analysis. Indeed, it is now clear that the Painlevé transcendents play the same role in nonlinear problems that the "linear" special functions, such as Airy functions, Bessel functions etc., play in linear science. During the last twenty to twenty five years, great progress in the theory of Painlevé equations themselves has been achieved. This progress has been based on the so-called Riemann-Hilbert, or Isomonodromy, Method.

The Riemann-Hilbert method reduces a particular problem at hand to the *Riemann-Hilbert problem* of analytic factorization of a given matrix - valued function defined on an oriented contour in the complex plane. The main benefit of this reduction arises in asymptotic analysis.

The principal goal of this course is to demonstrate the power of the Riemann-Hilbert approach to the asymptotic analysis of nonlinear problems considering the Painlevé equations as *a case study*. This would allow us to present the Riemann-Hilbert scheme in a rather elementary, although sufficiently general, manner. Simultaneously, the Painlevé equations will be introduced as an intrinsic part of the general Fuchsian monodromy theory. We are also planning to outline some of the other applications of the Riemann-Hilbert technique which range from integrable PDEs of KdV type (the area where the Riemann-Hilbert method was in fact originated) to exactly solvable quantum field and statistical mechanics models and (most recently) to the theory of orthogonal polynomials, matrix models, and random permutations.

The background required is the standard basic complex analysis and linear algebra.

Course Outline:

1. Riemann-Hilbert problems, the Fuchsian systems, and Hilbert's twenty-first problem.

2. Isomonodromy deformations and the Painlevé equations.

3. Asymptotic solutions of the Painlevé equations; connection formulae, asymptotics on the complex plane, distributions of poles, quasi-linear and nonlinear Stokes phenomena:

    - The inverse monodromy problem approach - the Deift-Zhou nonlinear steepest descent method
    - The direct monodromy problem approach - the complex WKB method

4. The Riemann-Hilbert method for Soliton equations.

5. The Riemann-Hilbert method and the Painlevé transcendents in the theory of orthogonal polynomials, Toeplitz and Hankel determinants, and matrix models.

Basic Text:

- A. Fokas, A. Its, A. Kapaev, V. Novokshenov, Painlevé Transcendents: The Riemann-Hilbert Approach, AMS Mathematical Surveys and Monographs, vol. 128, 2006

Supporting Texts:

- P. A. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Courant Lecture Notes in Mathematics, **3**, CIMS, New York (1999).

- P. A. Deift, A. R. Its and X. Zhou, " Long-Time Asymptotics for Integrable Nonlinear Wave Equations", in Important Developments in Soliton Theory, ed. A. S. Fokas and V. E. Zakharov, *Springer-Verlag*, 181-204 (1993).

- S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons (the Inverse Scattering Method)* Consultants Bureau, New York (1984).

- E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

- M. Jimbo, T. Miwa, K. Ueno Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, *Physica D* **2** (1980) 306–352; M. Jimbo, T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, *Physica D* **2** (1981) 407–448; M. Jimbo, T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III, *Physica D* **4** (1981) 26–46.

- E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge, University Press, 1927