Methods of Industrial Mathematics – Problem Sheet on Session 2.

Issue No: 1.0

Prepared By: Dr John Curtis

Date: 26/02/2018

1. (i) Consider a one-dimensional steady heat flow in the region $0 \le x \le a$. Suppose

$\theta(0) = \theta_0$, $\theta(a) = \theta_A$, and $\frac{\partial \theta}{\partial x} = g = const$ at $x = 0$, and that a is unknown. Find a.

This is an example of using an extra boundary condition to determine an unknown boundary.

(ii) Two different materials occupying the one-dimensional regions 1: $a \le x \le c$ and

2: $c \le x \le b$ are in contact at $x = c$. The conductivities of the materials are k_1 and k_2 and the temperatures in each region are $\theta_1(x)$ and $\theta_2(x)$ respectively. Heat is flowing steadily. The temperature at $x = a$ is $\theta(a) = \theta_A$ and the temperature at $x = b$ is $\theta(b) = \theta_B$. Two Robin conditions hold at $x = c$, namely

$$-k_1 \frac{\partial \theta_1}{\partial x} = -k_2 \frac{\partial \theta_2}{\partial x} = h(\theta_1 - \theta_2).$$

The first equation states that the heat flux leaving material 1 equals that entering material 2. The second gives the magnitude of the heat flux. Find the temperature fields $\theta_1(x)$ and $\theta_2(x)$.

2. See Session 2 slides 17 and 18. For the 1-D steady heat flow in the fin, solve case 3 with convection at the end as well as at sides

$$\theta = T_0 - T_\infty \; at \; x = 0; k\frac{\partial \theta}{\partial x} + h\theta = 0 \; at \; x = L.$$

3. Consider a hot axisymmetric highly conductive pipe of outer radius a containing hot liquid at the steady temperature θ_P, surrounded by poorly conductive insulation of conductivity k of thickness $b - a$. So the outside of the insulation is in contact with the surrounding air at temperature θ_0 at radius $r = b$. Suppose we have heat loss at the outer surface of the insulation given by the convection condition:

$$-k\frac{\partial \theta}{\partial r} = h(\theta - \theta_0).$$

Show that the heat flux is given by:

$$\frac{(\theta_P - \theta_0)}{\left[\frac{1}{bh} + \frac{1}{k} ln\left(\frac{b}{a}\right)\right]}$$

4. Find the discretisation of the flux at the node 1 that is accurate to second order, *i.e.* error $O(\Delta x^3)$ in terms of the node temperatures at $x_1, x_2 = x_1 + \Delta x, x_3 = x_1 + 2\Delta x$,

   $x_4 = x_1 + 3\Delta x$. See slide 40.

5. **Challenge Question**. Show that making the functional $I[\theta] = k\int_a^b \theta'^2 dx + 2C\theta(b)$ stationary subject to the boundary condition $\theta(a) = \theta_A$ by considering small variations $\theta = \theta_0 + \varepsilon\theta_1 + o(\varepsilon)$ results in the Euler equation $\theta_0'' = 0$ and the natural flux boundary condition:$-k\theta_0'(b) = C$. [Hint: $\theta_1(b)$ may take arbitrary values.]

   Show that $I[\theta] \geq I[\theta_0]$. Slightly harder than the example in the slides.

   You may then find it interesting to go through the development of the finite element scheme for this case in a manner analogous to the case of two Dirichlet conditions in the slides. The text by Davies shows this case but you will see I have changed notation slightly in my slides in the interests (I hope!) of clarity.