Set Theory
Descriptive Set Theory

Mirna Džamonja, University of East Anglia

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1 General Information

Descriptive set theory is the part of set theory where one studies 'nice' subsets of 'nice' topological spaces. What do we mean by nice? Early considerations in set theory have shown that by transfinite induction (so using the axiom of choice) one can construct sets with very unusual properties. For example one can show that there is a nonmeasurable subset of the reals, or that there is a subset of the plane which intersects every line in exactly two points. Equally, there are old and simple questions about sets which are now known to be independent of the basic axioms of mathematics, namely ZFC. One such question is if the Continuum Hypothesis CH, which postulates that every subset of the set $\mathbf{R}$ of the reals is either countable or has the same size as $\mathbf{R}$, is true. However, it was shown already by Cantor that when one restricts oneself to say closed subsets of the reals, CH is true: every closed set is either countable or has the size of $\mathbf{R}$. Rather more obviously, every closed subset of $\mathbf{R}$ is measurable.

This may lead one to argue that 'pathological' examples obtained by transfinite induction or other means would not appear in the course of a proof in ordinary mathematics. Therefore it seems reasonable to seek a class of sets in which CH and other similar questions are decidably true or false, but which still contains enough sets to allow us to do most constructions that appear in ordinary mathematics.

Descriptive set theory is the study of various such classes of sets. One restrics oneself to studying 'nice' topological spaces: for example $\mathbf{R}$, and their 'nice' subsets, for example closed and open sets or sets obtained from these by simple operations. Such sets include the class of Borel sets, forming the smallest σ-algebra that contains the open sets, or projective sets, which form the smallest class of sets containing Borel sets and closed under projections from higher dimensional spaces. The course will present a number of results of classical descriptive set theory studying concepts naturally arising in ordinary mathematics.

Amazingly enough, it does turn out that descriptive set theory is intimately connected with the study of independence results in set theory, as well as the notion of large cardinals. These connections are achieved through a study of certain games on the reals. The course will mention some of them.

Finally, the newest developments in descriptive set theory develop connections with dynamical systems, ergodic theory, etc. through the study of orbit equivalence relations and induced quotient spaces. These spaces are typically nonstandard, i.e. the induced Borel structure is degenerate, nevertheless one can study their properties by lifting them to the original space and using advanced methods of effective decriptive set theory and other techniques. This leads to deep considerations about classification problems in various areas of mathematics. Descriptive set theory provides a general framework and tools for studying these problems and it allows us to properly formulate and give precise answers to some previously vague classification problems in various areas of mathematics.

2 Course outline

- Basic notions of set theory, such as ordinals. Axiom of Choice.
- Constructions using the Axiom of Choice.
- Borel sets. Definable CH. Projective sets.
- Determinacy.
- Effective descriptive set theory.
- Borel equivalence relations.
- Connections to other parts of mathematics.

3 Bibliography

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3. R. Mansfield and G. Weitkamp: "Recursive Aspects of Descriptive Set Theory", Oxford, 1985.

4. A.W. Miller, "Descriptive Set theory and Forcing, Springer", 1995.