ltccsoln4.tex

LTCC SOLUTIONS 4. 28.10.2013

Q1 *Symmetric stable processes.* $X_c(t) = c^{-1}X(c^\alpha t)$ has CF

$$\begin{aligned} E\exp\{isX_c(t)\} &= E\exp\{i(s/c)X(c^\alpha t)\} \\ &= \exp\{-(c^\alpha t)(s/c)^\alpha\} \\ &= \exp\{-ts^\alpha\}. \end{aligned}$$

So X_c has the same CF as X. Like X, it is a Lévy process. So it has the same Lévy measure as X. As X is symmetric stable of index α, so too is X_c. The above scaling property says that X is *self-similar* with *index* α.

As with Brownian motion (or any other self-similar process), the path of X is a fractal, and so is its zero-set.

Q2. For $t \neq 0$, X is Gaussian with zero mean (as B is), and continuous (again, as B is). The covariance of B is $\min(s, t)$. The covariance of X is

$$\begin{aligned} cov(X_s, X_t) &= cov(sB(1/s), tB(1/t)) \\ &= E[sB(1/s).tB(1/t)] \\ &= st.E[B(1/s)B(1/t)] \\ &= st.cov(B(1/s), B(1/t)) \\ &= st.\min(1/s, 1/t) = \min(t, s) = \min(s, t). \end{aligned}$$

This is the same covariance as Brownian motion. So, away from the origin, X *is* Brownian motion, as a Gaussian process is uniquely characterized by its mean and covariance (from the properties of the multivariate normal distribution). So X is continuous. So we can define it at the origin by continuity. So X is Brownian motion everywhere – X is BM.

Q3. We are given that BM is zero at (infinitely many) times increasing to infinity (this can be proved in various ways – e.g., from the law of the iterated logarithm – LIL – for BM). Hence by Q1, BM is zero at (infinitely many) random times, t_n say, decreasing to zero.

We can start BM afresh at each t_n (this uses the *strong Markov property*, which we quote). So in the same way: for each t_n, there are infinitely many random times $t_{nm} \downarrow t_n$ at which BM is zero, etc.

Continuing in the same way: it is (at first sight) difficult to see how BM can ever get away from zero – but it does! The key to all this is *excursion* *theory* (due to K. Itô, in 1972). We can think of the Brownian path as its infinitely many zeros, separated from each other by *excursions* away from 0. These are positive or negative according to 'independent coin tosses', and are independently drawn from the same (infinite) excursion measure. It would take us too far afield to discuss this further; for details, see e.g. Itô's paper in Vol. III of the *Proceedings of the Sixth Berkeley Symposium in Probability and Statistics*, U. California Press, 1972, or look up 'excursions', or 'excursion theory', in the index of a good book on Stochastic Processes.

NHB