

LTCC course on ‘Spectral Theory’

E B Davies

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Core Audience: First year postgraduates: pure (also of interest to applied)

Course Format: extended (10 hours at 2 hours per week)

Location: De Morgan House in Russell Square,

Timing: Mondays 10:50 - 12:50 from 14 January 2013 to 11 February 2013 inclusive.

Lecturer: Prof. E. B. Davies

Lecturer home institution: King’s College London

Lecturer e-mail: E.Brian.Davies@kcl.ac.uk

Keywords: linear operator, spectrum, self-adjoint

Summary: This course is an introduction to the many aspects of spectral theory for both self-adjoint and non-self-adjoint linear operators and in particular for differential operators. It is hoped but not required that students will be familiar with the spectral theorem for self-adjoint operators. The course will focus on theorems and how they are used in some very simple examples, and will not involve any numerical computation.

Supporting material:

E B Davies, *Spectral Theory and Differential Operators*, CUP, 1995 covers much of the material in the course and provides proofs of some matters that the course will describe without proof.

E B Davies, *Linear Operators and their Spectra*, CUP, 2007 is a more comprehensive account, concentrating particularly, but not exclusively, on non-self-adjoint operators.

The importance of spectral theory

Applications of spectral theory arise in a wide range of areas in applied mathematics, physics and engineering, but there are also applications to geometry, the analysis of finite graphs, probability and many other fields. The origins of spectral theory can be traced to Laplace at the start of the nineteenth century and, if one includes its connections with musical harmonies, back to Pythagoras 2500 years ago. Its present day manifestations cover everything from the design and testing of mechanical structures to the algorithms that are used in the Google search engine. There is an extensive mathematical theory behind the subject and there are computer packages that enable scientists and engineers to determine the spectral features of the problems that they are studying. Spectral theory is of great interest for its own sake as a branch of pure mathematics, but it can also be used as a way of writing the solution of an applied problem as a linear combination of the solutions of a sequence of eigenvalue problems.

The word *eigenvalue* is a combination of the German root ‘eigen’, meaning characteristic or distinctive, and the English ‘value’, which suggests that one is seeking a numerical quantity λ that is naturally associated with the application being considered. This number is often interpreted as the energy or frequency of some non-zero solution f of an equation of the form

$$Lf = \lambda f. \tag{1}$$

The symbol L refers to the linear operator that describes the particular problem being considered. The solution f of the equation is called the *eigenfunction* (or *eigenvector*, depending on the context) corresponding to the eigenvalue λ , and it may be a function of one or more variables. Although much of the literature concentrates on determining the eigenvalues of some model, the eigenfunctions carry much more information.

Depending on the particular evolution equation involved, if an eigenvalue $\lambda = u + iv$ associated with some problem is complex, then u is interpreted as the *frequency of oscillation* or vibration of the system being studied. If $v < 0$ then the vibration being considered is stable and v is interpreted as its *rate of decay*. If $v > 0$ then the vibration is unstable and the size of v determines how unstable; unstable vibrations in engineering structures can lead to catastrophic failure.

We conclude by mentioning inverse problems, in which one seeks to determine important features of some problem from measurements of its associated spectrum. This field is rapidly developing at the present time and has a wide variety of important applications, ranging from engineering to seismology.

Summary of material to be covered

1. Bounded and unbounded operators. Closed operators. The definition of the spectrum in the two cases and its basic properties.
2. The spectral theorem for self-adjoint operators. Some different statements of the theorem, including an account of the functional calculus.
3. Quadratic forms for non-negative self-adjoint operators. Variational methods for calculating eigenvalues.
4. Classification of the spectrum. Essential spectrum, discrete spectrum, Fredholm operators.
5. Fourier transforms and the Laplacian. Other constant coefficient operators.
6. Perturbation theory. Schrödinger operators.
7. Unitary groups. One-parameter semigroups, their generators and their resolvents.