

“Symmetry Methods for Differential Equations”

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There is much current interest in the determination of symmetry reductions (sometimes known as similarity reductions) of physically and mathematically interesting partial differential equations. Symmetry reductions are usually obtained either by seeking a solution in a special form or, more generally, by exploiting symmetries of the equation. Such symmetries provide a technique for obtaining exact and special solutions, typically in terms of solutions of ordinary differential equations. Recently there have been considerable developments in the application of symmetry analysis (or group methods), which are highly algorithmic, to partial differential equations.

In this series of lectures I shall discuss various techniques for deriving symmetry reductions, in particular

- (i) the classical Lie group method of infinitesimal transformations which dates back to the work of Sophus Lie in the last century,
- (ii) the “direct method” due to Clarkson & Kruskal [*J. Math. Phys.*, **30** (1989) 2201–2213], and
- (iii) the “nonclassical method” of group-invariant solutions due to Bluman & Cole [*J. Math. Mech.*, **18** (1969) 1025–1042], which is also known as the method of conditional symmetries.

Emphasis will be placed on explicit computational algorithms to discover symmetries admitted by differential equations and the construction of reductions and exact solutions arising from these symmetries. I shall give a comparison of these three methods discussing their relative strengths and weaknesses. The use of symbolic manipulation packages, e.g. in MAPLE and MATHEMATICA, which considerably facilitate the calculations will also be discussed.

All the aforementioned symmetry methods involve the solution of an overdetermined system of partial differential equations. An introduction will also be given to the theory of Differential Gröbner Bases which provide a “triangularization” of the system and have made the analysis of such systems more tractable.

To conclude, I shall describe some of the many important applications of symmetry analysis, e.g., to study properties such as asymptotics and “blow-up”, to the design of numerical algorithms and testing computer coding.

Tentative outline of lectures — Peter Clarkson

1 Classical Lie Group Method

In this lecture I shall give an introduction to the classical Lie group method of infinitesimal transformations for deriving symmetry reductions of partial differential equations which dates back to the work of Sophus Lie in the last century.

2 Direct Method of Clarkson and Kruskal

In this lecture I shall describe the “direct method” developed by Clarkson and Kruskal [*J. Math. Phys.*, **30** (1989) 2201–2213] in their study of symmetry reductions of the Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx}.$$

Other applications of the direct method include the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + u(1 - u)(u - \alpha)$$

where α is an arbitrary constant and the Zabalotskaya-Khoklov equation

$$(u_t + uu_x + u_{xx})_x + u_{yy} = 0.$$

The direct method involves making an ansatz and as such involves no use of group theory. I shall also discuss some other ansatz-based methods for deriving exact solutions of partial differential equations.

3 Nonclassical Method of Bluman and Cole

In this lecture I shall describe the “nonclassical method” of group-invariant solutions developed by Bluman and Cole [*J. Math. Mech.*, **18** (1969) 1025–1042] in their study of symmetry reductions of the linear heat equation. This technique, also known as the “method of conditional symmetries”, is a generalization of the classical Lie group method and will be applied to various equations including the Boussinesq equation, the nonlinear heat equation

$$u_t = u_{xx} + f(u)$$

where $f(u)$ is an arbitrary function, and a shallow water wave equation

$$u_{xxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0$$

where α and β are arbitrary constants. I shall give a comparison of the three methods (classical Lie, direct and nonclassical methods) discussing their relative strengths and weaknesses.

4 Differential Gröbner Bases

Symmetry methods usually involve the solution of an overdetermined system of partial differential equations. In this lecture I shall give an introduction will also be given to the theory of Differential Gröbner Bases which provide a “triangularization” of the system and have made the analysis of such overdetermined systems more tractable.