C*-ALGEBRAS

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These notes are written for an LTCC[2] postgraduate course on C*-algebras first given in 2008. They are intended as a *brief* introduction to the basic theory of C*-algebras and their representations on Hilbert spaces, as well as the Murray-von Neumann classification of von Neumann algebras. The only prerequisite for what follows is a basic knowledge of topology, algebra and analysis, at a level comparable to that of, say, G.F. Simmon's book "Introduction to topology and modern analysis". Needless to say, these notes (for 10-hour lectures only!) are not a complete text for the theory of C*-algebras. A selection of books are listed at the end of the notes, of which [11] can be used as a main textbook and [5] a short but leisurely one.

Chapter 1. What is a C*-algebra

All vector spaces $\mathcal{A}$ in these notes are over the complex number field $\mathbb{C}$ so that a *conjugate* linear map $x \in \mathcal{A} \mapsto x^* \in \mathcal{A}$ is called an *involution* if it has period 2. The identity of an algebra will always be denoted by $\mathbf{1}$. We recall that a Banach space is a complete normed vector space.

Definition 1.1. A Banach space $\mathcal{A}$ is called a *Banach algebra* if it is an algebra in which the multiplication satisfies

$$\|xy\| \leq \|x\|\|y\| \qquad (x, y \in \mathcal{A}).$$

We note that multiplication in a Banach algebra is continuous. A Banach algebra $\mathcal{A}$ is called *unital* if it contains an identity, in which case there is an equivalent norm $|\cdot|$ on A such that $(\mathcal{A}, |\cdot|)$ is a Banach algebra and $|\mathbf{1}| = 1$, where

$$|x| := \sup\{\|xy\| : \|y\| \leq 1\}.$$

Therefore, there is no loss of generality to assume, in the sequel, that in a unital Banach algebra we always have $\|\mathbf{1}\| = 1$. A Banach *-algebra is a Banach algebra $\mathcal{A}$ which admits an involution $* : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying $(xy)^* = y^*x^*$ and $\|x^*\| = \|x\|$ for all $x, y \in \mathcal{A}$.

Definition 1.2. A Banach *-algebra $\mathcal{A}$ is called a C*-*algebra* if its norm and involution satisfy

$$\|x^*x\| = \|x\|^2 \qquad (x \in \mathcal{A}).$$

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In the sequel, $\mathcal{A}$ always denotes a C*-algebra unless otherwise stated!

Remark 1.3. A Banach algebra $\mathcal{A}$ which admits an involution $*$ satisfying $(xy)^* = y^*x^*$ and $\|x^*x\| = \|x\|^2$ is a C*-algebra since $\|x\| = \|x^*\|$ follows from $\|x\|^2 = \|x^*x\| \leq \|x^*\|\|x\|$ whence $\|x\| \leq \|x^*\|$ and the inequality can be reversed for x^*. Historically C*-algebras were first defined with the extra condition that $\mathbf{1} + x^*x$ is invertible, but this was later found to be superfluous.

Example 1.4. Let Ω be a locally compact Hausdorff space and let $C_0(\Omega)$ be the Banach algebra of complex continuous functions on Ω vanishing at infinity, with pointwise multiplication and the supremum norm

$$\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\} \qquad (f \in C_0(\Omega)).$$

It is a Banach *-algebra with the complex conjugation as involution

$$f^*(\omega) = \overline{f(\omega)} \qquad (\omega \in \Omega).$$

If Ω is compact, then $C_0(\Omega)$ coincides with the algebra $C(\Omega)$ of all complex continuous functions on Ω.

Further, $C_0(\Omega)$ is a C*-algebra which is also abelian. In fact, all abelian C*-algebras are of this form.

A map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ between two C*-algebras is called a **-map* if it preserves the involution: $\varphi(a^*) = \varphi(a)^*$. As usual, we denote by $\mathcal{A}^*$ the *dual space* of a Banach space $\mathcal{A}$, consisting of all continuous linear functionals on $\mathcal{A}$.

Theorem 1.5. *Let $\mathcal{A}$ be an abelian C*-algebra. Then it is isometrically *-isomorphic to the C*-algebra $C_0(\Omega_{\mathcal{A}})$ of complex continuous functions on a locally compact Hausdorff space $\Omega_{\mathcal{A}}$, vanishing at infinity.*

Proof. We equip the dual space $\mathcal{A}^*$ of $\mathcal{A}$ with the weak* topology in which a net (f_α) in $\mathcal{A}^*$ weak* converges to $f \in \mathcal{A}^*$ if, and only if, the net $(f_\alpha(a))$ in $\mathbb{C}$ converges to the number $f(a)$ for all $a \in \mathcal{A}$.

Each algebra homomorphism $\omega : \mathcal{A} \longrightarrow \mathbb{C}$, that is, a multiplicative linear functional ω, satisfies $\|\omega\| \leq 1$. We sometimes call ω a *character* if $\omega \neq 0$. Let

$$\Omega_{\mathcal{A}} = \{\omega \in \mathcal{A}^* : \omega \text{ is a nonzero homomorphism from } \mathcal{A} \text{ to } \mathbb{C}\}$$

and let

$$\Omega = \Omega_{\mathcal{A}} \cup \{0\}.$$

Then Ω is weak* closed in the closed unit ball $\mathcal{A}_1^* = \{f \in \mathcal{A}^* : \|f\| \leq 1\}$. By Tychonoff's theorem, $\mathcal{A}_1^*$ is compact in the weak* topology of $\mathcal{A}^*$. Hence Ω is weak* compact. Since $\{0\}$ is closed, $\Omega_{\mathcal{A}}$ is open in Ω and is therefore locally compact.

We define a map $\widehat{\ } : \mathcal{A} \longrightarrow C_0(\Omega_{\mathcal{A}})$, called the *Gelfand transform*, by

$$\widehat{a}(\omega) = \omega(a) \qquad (\omega \in \Omega).$$

This map is well-defined since $\widehat{a}$ is clearly continuous on $\Omega_{\mathcal{A}}$ and given $\varepsilon > 0$, the set

$$\{\omega \in \Omega_{\mathcal{A}} : |\widehat{a}(\omega)| \geq \varepsilon\}$$

is closed in Ω and hence weak* compact.

It follows that the Gelfand transform $\widehat{\ }$ is an isometric algebra isomorphism preserving the involution:

$$\widehat{a^*} = \overline{\widehat{a}}.$$

□

Remark 1.6. If $\mathcal{A}$ has an identity $\mathbf{1}$, then each character ω of $\mathcal{A}$ satisfies $\omega(\mathbf{1}) = 1$ and hence $\Omega_{\mathcal{A}}$ is weak* closed in $\mathcal{A}_1^*$, and is therefore weak* compact.

Definition 1.7. We call $\Omega_{\mathcal{A}}$ above the *spectrum* of $\mathcal{A}$

Example 1.8. Let $\{\mathcal{A}_\alpha : \alpha \in J\}$ be a family of C*-algebras. The Cartesian product

$$\times_{\alpha\in J}\mathcal{A}_\alpha$$

is clearly a *-algebra in the pointwise product and involution. We define the C*-*direct sum* of $\{\mathcal{A}_\alpha\}_{\alpha\in J}$ to be the following *-subalgebra of $\times_{\alpha\in J}\mathcal{A}_\alpha$:

$$\bigoplus_{\alpha\in J}\mathcal{A}_\alpha = \{(a_\alpha) \in \times_{\alpha\in J}\mathcal{A}_\alpha : \sup_\alpha \|a_\alpha\| < \infty\}.$$

Then $\bigoplus_{\alpha\in J}\mathcal{A}_\alpha$ is a C*-algebra with the norm

$$\|(a_\alpha)\| = \sup_\alpha \|a_\alpha\|.$$

Example 1.9. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Then the Banach algebra $B(H)$ of all bounded linear operators from H to itself is a C*-algebra in which T^* is the adjoint of T:

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad (x, y \in H)$$

and the identity $\|T^*T\| = \|T\|^2$ is well-known. If $\dim H = n$, then $B(H)$ is just the algebra M_n of $n \times n$ complex matrices. The identity $\mathbf{1}$ in $B(H)$ is the identity operator on H.

Evidently, a closed *-*subalgebra* $\mathcal{A}$ of $B(H)$, that is, a subalgebra $\mathcal{A}$ of $B(H)$, closed in the norm topology of $B(H)$ and satisfying $x \in A \Rightarrow x^* \in A$, is a C*-algebra. In fact, every C*-algebra is (isomorphic to) a closed *-subalgebra of some $B(H)$.

Theorem 1.10. *Let $\mathcal{A}$ be a C*-algebra. Then there exists an isometric *-monomorphism $\pi : \mathcal{A} \longrightarrow B(H)$, for some Hilbert space H.*

Proof. Call a linear functional $f : \mathcal{A} \longrightarrow \mathbb{C}$ *positive*, in symbols $f \geq 0$, if $f(a^*a) \geq 0$ for all $a \in \mathcal{A}$. Let

$$N_f = \{a \in \mathcal{A} : f(a^*a) = 0\}.$$

The quotient $\mathcal{A}/N_f$ is an inner product space with the inner product

$$\langle a + N_f, b + N_f \rangle := f(b^*a).$$

Let H_f be the completion of A/N_f and define a map $\pi_f : \mathcal{A} \longrightarrow B(H_f)$ by

$$\pi_f(a)(b + N_f) = ab + N_f \qquad (b + N_f \in H_f).$$

Take the direct sum of all the maps π_f induced by $f \geq 0$:

$$\pi = \bigoplus_{f\geq 0} \pi_f : \mathcal{A} \longrightarrow B\left(\bigoplus_{f\geq 0} H_f\right)$$

where $\bigoplus_{f\geq 0} H_f$ is the usual direct sum of the Hilbert spaces H_f:

$$\bigoplus_{f\geq 0} H_f = \left\{ (x_f)_{f\geq 0} : x_f \in H_f \text{ and } \sum_{f\geq 0} \|x_f\|^2 < \infty \right\}$$

and π is defined by

$$\left(\bigoplus_{f\geq 0} \pi_f\right)(a)((x_f)_{f\geq 0}) = (\pi_f(a)(x_f))_{f\geq 0} \qquad (a \in \mathcal{A}).$$

The map π is called the universal representation of $\mathcal{A}$ and is the required isometric *-monomorphism. □

The construction of the map π_f above is often called the *GNS-construction*, named after Gelfand, Naimark and Segal. We left out some details in the proofs of the two theorems above. To complete these details, we need to use some results developed in the following chapters.

Chapter 2. SPECTRAL THEORY

Let $\mathcal{A}$ be a C*-algebra. We define the *unit extension* of $\mathcal{A}$ to be the vector space direct sum $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$, equipped with the following product and involution:

$$(a \oplus \lambda)(b \oplus \mu) = (ab + \lambda b + \mu a) \oplus (\lambda\mu), \quad (a \oplus \lambda)^* = a^* \oplus \overline{\lambda}.$$

Then $\mathcal{A}_1$ is a C*-algebra in the following norm

$$\|x\| = \sup\{\|xa\| : a \in \mathcal{A}, \|a\| \leq 1\} \qquad (x \in \mathcal{A}_1).$$

$\mathcal{A}_1$ is unital with identity $0 \oplus 1$.

Given a unital Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$, we define the *spectrum* of a to be the following subset of $\mathbb{C}$:

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \text{ is not invertible in } \mathcal{A}\}.$$

We will write λ for $\lambda\mathbf{1}$ if there is no confusion. The complement $\mathbb{C}\backslash\sigma(a)$ is called the *resolvent set* of a. If a C*-algebra $\mathcal{A}$ is not unital, then we define the *quasi-spectrum* of an element $a \in \mathcal{A}$ to be the following set:

$$\sigma'(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \text{ is not invertible in } \mathcal{A}_1\}.$$

We always have $0 \in \sigma'(a)$! If $\mathcal{A}$ is unital, we have

$$\sigma'(a) = \sigma(a) \cup \{0\}.$$

Lemma 2.1. *Let $\mathcal{A}$ be unital Banach algebra and let $a \in \mathcal{A}$ satisfy $\|a\| < 1$. Then $\mathbf{1} - a$ is invertible in $\mathcal{A}$.*

Proof.

$$(\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

□

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra and let $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ be an algebra homomorphism. Then ω is continuous and $\|\omega\| \leq 1$.*

Proof. There is nothing to prove if $\omega = 0$. Otherwise, extend ω to an algebra homomorphism $\widetilde{\omega}$ on the unit extension $\mathcal{A}_1$ of $\mathcal{A}$ by defining

$$\widetilde{\omega}(a + \alpha) = \omega(a) + \alpha \qquad (a + \alpha \in \mathcal{A}_1).$$

Then $\omega(x) - x$ is not invertible for each $x \in \mathcal{A}$ since $\omega(x) - x$ is in the kernel of $\widetilde{\omega}$. Hence we have $|\omega(x)| \leq \|x\|$ by Lemma **2.1**. □

Lemma 2.3. *Let $\mathcal{A}$ be unital Banach algebra. The set $G(\mathcal{A})$ of invertible elements in $\mathcal{A}$ is open.*

Proof. Let $a \in G(\mathcal{A})$. Then the open ball

$$B\left(a, \frac{1}{\|a^{-1}\|}\right) = \left\{x \in \mathcal{A} : \|x - a\| < \frac{1}{\|a^{-1}\|}\right\},$$

centred at a with radius $\frac{1}{\|a^{-1}\|}$, is contained in $G(\mathcal{A})$ since $x \in B(a, \frac{1}{\|a^{-1}\|})$ implies $\|\mathbf{1} - a^{-1}x\| = \|a^{-1}(a - x)\| < 1$ and hence $a^{-1}x$ is invertible, so is x. □

Proposition 2.4. *Let $\mathcal{A}$ be unital Banach algebra and let $a \in \mathcal{A}$. Then the spectrum $\sigma(a)$ is a compact set in $\mathbb{C}$.*

Proof. The function $f : \lambda \in \mathbb{C} \mapsto (\lambda - a) \in \mathcal{A}$ is clearly continuous. Hence $\mathbb{C}\backslash\sigma(a) = f^{-1}(G(\mathcal{A}))$ is open.

If $|\lambda| > \|a\|$, then $\left\|\dfrac{a}{\lambda}\right\| < 1$ implies $\mathbf{1} - \dfrac{a}{\lambda}$, and hence $\lambda - a$, is invertible. Therefore $\sigma(a) \subset \{z \in \mathbb{C} : |z| \leq \|a\|\}$, that is, $\sigma(a)$ is bounded and compactness follows from the Heine-Borel Theorem. □

Proposition 2.5. *Let $\mathcal{A}$ be unital Banach algebra and let $a \in \mathcal{A}$. Then $\sigma(a) \neq \emptyset$*

Proof. Define the resolvent map $R : \mathbb{C}\backslash\sigma(a) \longrightarrow \mathcal{A}$ by

$$R(\lambda) = (\lambda - a)^{-1} \qquad (\lambda \in \mathbb{C}).$$

Then we have

$$\frac{R(\lambda) - R(\mu)}{\lambda - \mu} = -R(\mu)R(\lambda) \qquad (\lambda, \mu \in \mathbb{C})$$

and R is an $\mathcal{A}$-valued analytic function.

For each $\varphi \in \mathcal{A}^*$, the function $\varphi \circ R : \mathbb{C}\backslash\sigma(a) \longrightarrow \mathbb{C}$ is analytic. If $\sigma(a) = \emptyset$, then $\varphi \circ R$ is an entire function. Since

$$|\varphi \circ R(\lambda)| \leq \|\varphi\|\|R(\lambda)\| = \|\varphi\||\lambda|^{-1}\|(\mathbf{1} - a/\lambda)^{-1}\| \longrightarrow 0$$

as $|\lambda| \to \infty$, we must have $\varphi \circ R$ identically 0, by Liouville Theorem.

Since φ was arbitrary, we have $R = 0$ which is impossible. Hence $\sigma(a) \neq \emptyset$. □

An algebra with identity is called a *division algebra* if every nonzero element in it is invertible.

Theorem 2.6. (Gelfand-Mazur Theorem) *Let $\mathcal{A}$ be a unital Banach algebra. If $\mathcal{A}$ is a division algebra, then $\mathcal{A} = \mathbb{C}\mathbf{1}$.*

Proof. If there exists $x \in \mathcal{A}\backslash\mathbb{C}\mathbf{1}$. Then, for all $\lambda \in \mathbb{C}$, we have $\lambda\mathbf{1} - x \neq 0$ and is invertible, that is, $\lambda \notin \sigma(a)$. Thus $\sigma(a) = \emptyset$ which is impossible. □

Definition 2.7. Let $\mathcal{A}$ be a C*-algebra and let $a \in \mathcal{A}$. The supremum

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma'(a)\}$$

is called the *spectral radius* of a.

By the proof of Proposition **2.4**, we have $r(a) \leq \|a\|$. In fact, we have the following useful formula for the spectral radius.

Theorem 2.8. *Let $a \in \mathcal{A}$. Then we have*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. We may assume that $\mathcal{A}$ has an identity. We first note that $\lambda \in \sigma(a) \implies \lambda^n \in \sigma(a^n)$ since

$$\lambda^n - a^n = (\lambda - a)\sum_{k=0}^{n-1} \lambda^k a^{n-1-k}.$$

It follows that $|\lambda|^n \le \|a^n\|$ for all n and

$$r(a) \le \lim_{n\to\infty} \inf \|a^n\|^{1/n}.$$

For each $\varphi \in \mathcal{A}^*$, the complex function $\varphi \circ R$ is analytic in the domain

$$D = \{\lambda \in \mathbb{C} : |\lambda| > r(a)\} \subset \mathbb{C}\backslash\sigma(a).$$

By analyticity of $R(\lambda)$, it has the Laurent series

$$R(\lambda) = (\lambda\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$$

for $\lambda \in D$ and hence

$$\varphi \circ R(\lambda) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{\lambda^{n+1}}$$

which is the Laurent series of $\varphi \circ R$ in D. By convergence of the series, we have

$$\left|\frac{\varphi(a^n)}{\lambda^{n+1}}\right| \le C_\varphi \qquad \text{for all } n,$$

for some constant C_φ depending on φ.

By the Uniform Boundedness Principle, we must have, for $\lambda \in D$,

$$\frac{\|a^n\|}{|\lambda|^{n+1}} \le C \qquad \text{for all } n.$$

Hence

$$\|a^n\|^{1/n} \le C^{1/n}|\lambda|^{1+\frac{1}{n}}$$

for all $|\lambda| > r(a)$ which yields

$$\lim_{n\to\infty} \sup \|a^n\|^{1/n} \le r(a)$$

and the proof is complete. □

Definition 2.9. Let a be an element in a C*-algebra $\mathcal{A}$. It is called *self-adjoint* or *hermitian* if $a^* = a$. It is called *normal* if $a^*a = aa^*$. It is called a *projection* if $a = a^* = a^2$. It is called *unitary* if $a^*a = aa^* = \mathbf{1}$, given that $\mathcal{A}$ is unital.

Every element $a \in \mathcal{A}$ can be written in the form

$$a = a_1 + ia_2$$

where a_1 and a_2 are self-adjoint. In fact,

$$a_1 = \frac{1}{2}(a + a^*) \quad \text{and} \quad a_2 = \frac{1}{2i}(a - a^*).$$

Corollary 2.10. *Let a be a normal element in a C*-algebra $\mathcal{A}$. Then we have $\|a\| = r(a)$.*

Proof. We first show this for a self-adjoint element a. We have $\|a^2\| = \|a^*a\| = \|a\|^2$ and hence, by iteration,

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

For a normal element a, we have

$$r(a)^2 \leq \|a\|^2 = \|a^*a\| = \lim_{n\to\infty} \|(a^*a)^n\|^{1/n} \leq \lim_{n\to\infty} \|(a^*)^n\|^{1/n}\|(a)^n\|^{1/n} = r(a)^2.$$

$\square$

Proposition 2.11. *Let $\mathcal{A}$ be a unital C*-algebra. If $u \in \mathcal{A}$ is unitary, then $\sigma(u) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If $a \in \mathcal{A}$ is self-adjoint, then $\sigma(a) \subset \mathbb{R}$.*

Proof. We have $\|u\|^2 = \|u^*u\| = \|\mathbf{1}\| = 1$ and since $u^* = u^{-1}$, the set

$$\sigma(u^*) = \{\overline{\lambda} : \lambda \in \sigma(u)\} = \sigma(u^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(u)\}$$

is contained in the unit *disc* in $\mathbb{C}$, and hence $\sigma(u)$ is contained in the unit *circle* in $\mathbb{C}$.

Given a self-adjoint element $a \in \mathcal{A}$, the element

$$u = \exp(ia) = \sum_{n=0}^{\infty} \frac{(ia)^n}{n!}$$

is unitary since $u^* = \exp(-ia)$. We have

$$\{\exp i\lambda : \lambda \in \sigma(a)\} = \sigma(u) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

which implies $\sigma(a) \subset \mathbb{R}$. $\square$

Let $\mathcal{A}$ be a C*-algebra. A subalgebra $I \subset \mathcal{A}$ is called a *left ideal* if

$$a \in \mathcal{A} \quad \text{and} \quad x \in I \Longrightarrow ax \in I.$$

A *right ideal* of $\mathcal{A}$ is a subalgebra I satisfying

$$a \in \mathcal{A} \quad \text{and} \quad x \in I \Longrightarrow xa \in I.$$

We call I *proper* if $I \neq \mathcal{A}$. Plainly, if $\mathcal{A}$ is unital, then I is proper if, and only if, $\mathbf{1} \notin I$. A *two-sided ideal* of $\mathcal{A}$ is a subalgebra which is both a left and a right ideal. If $\mathcal{A}$ is abelian, we speak simply of ideals rather than left or right ideals, in which case a *maximal ideal* is a proper ideal not properly contained in any proper ideal.

If I is a proper ideal in a unital $\mathcal{A}$, then I cannot contain any invertible element, that is $I \subset \mathcal{A}\backslash G(\mathcal{A})$. Since $G(\mathcal{A})$ is open by Lemma **2.3**, the closure $\overline{I}$

is also contained in $\mathcal{A}\backslash G(\mathcal{A})$ and is therefore a proper ideal in $\mathcal{A}$. It follows that maximal ideals in $\mathcal{A}$ are closed.

There is a one-one correspondence between maximal ideals in an abelian C*-algebra $\mathcal{A}$ and the characters of $\mathcal{A}$.

Proposition 2.12. *Let $\mathcal{A}$ be a unital abelian C*-algebra with spectrum $\Omega_{\mathcal{A}}$. Then the mapping*

$$\omega \in \Omega_{\mathcal{A}} \mapsto \omega^{-1}(0) \subset \mathcal{A}$$

is a bijection onto the set of all maximal ideals of $\mathcal{A}$.

Proof. Let $\omega \in \Omega_{\mathcal{A}}$. Then $\omega^{-1}(0)$ is a proper ideal in $\mathcal{A}$ since $\mathbf{1} \notin \omega^{-1}(0)$. It is maximal because it has codimension 1.

Given a maximal ideal $M \subset \mathcal{A}$, the quotient Banach space $\mathcal{A}/M$ is a unital commutative Banach algebra in the product

$$(a+M)(b+M) := ab + M$$

and the quotient map

$$q : \mathcal{A} \longrightarrow \mathcal{A}/M$$

is an algebra homomorphism. By maximality of M, the Banach algebra $\mathcal{A}/M$ has no nontrivial ideal. Hence every nonzero element in $[a] := a + M$ in $\mathcal{A}/M$ is invertible for otherwise, $[a]\,(\mathcal{A}/M)$ would be a nontrivial ideal. It follows from Mazur's theorem that there is an isomorphism $\varphi : \mathcal{A}/M \longrightarrow \mathbb{C}$ and hence $M = (\varphi \circ q)^{-1}(0)$ with $\varphi \circ q \in \Omega_{\mathcal{A}}$. □

We can now give complete details of the proof of Theorem **1.5**.

Proposition 2.13. *Let $\mathcal{A}$ be an abelian C*-algebra. Then the Gelfand map $\widehat{\ }$: $\mathcal{A} \longrightarrow C_0(\Omega_{\mathcal{A}})$ defined in Theorem* **1.5** *is an isometry and a *-map.*

Proof. If $\mathcal{A}$ is unital, then each non-invertible element $b \in \mathcal{A}$ is contained is some maximal ideal since $b\mathcal{A}$ is a proper ideal containing b and Zorn's lemma applies. It follows that

$$\begin{aligned} \alpha \in \sigma(a) \quad &\Leftrightarrow \quad \alpha - a \in \omega^{-1}(0) \quad \text{for some} \quad \omega \in \Omega_{\mathcal{A}} \\ &\Leftrightarrow \quad \alpha = \omega(a) = \widehat{a}(\omega) \quad \text{for some} \quad \omega \in \Omega_{\mathcal{A}} \end{aligned}$$

and we have

$$\textbf{(2.1)} \qquad \sigma(a) = \widehat{a}(\Omega_{\mathcal{A}}).$$

If $\mathcal{A}$ is non-unital, then the quasi-spectrum $\sigma'(a)$ is the spectrum $\sigma_{\mathcal{A}_1}(a)$ of a in the unit extension $\mathcal{A}_1$ of $\mathcal{A}$ and we have

$$\textbf{(2.2)} \qquad \sigma'(a) = \sigma_{\mathcal{A}_1}(a) = \widehat{a}(\Omega_{\mathcal{A}_1}) = \widehat{a}(\Omega_{\mathcal{A}})$$

where $\Omega_{\mathcal{A}_1} = \Omega_{\mathcal{A}} \cup \{\omega_0\}$ and $\omega_0(a \oplus \beta) = \beta$ for $a \oplus \beta \in \mathcal{A}_1$.

In both cases, we have $\|a\| = r(a) = \|\widehat{a}\|$ since a is normal in the abelian algebra $\mathcal{A}$.

To see that the Gelfand map is a *-map, let $a \in \mathcal{A}$ and $a = a_1 + ia_2$ where a_1 and a_2 are self-adjoint. For each $\omega \in \Omega$, we have

$$\begin{aligned} \widehat{a^*}(\omega) &= \omega(a^*) = \omega(a_1 - ia_2) \\ &= \omega(a_1) - i\omega(a_2) = \overline{\omega(a_1) + i\omega(a_2)} \\ &= \overline{\omega(a)} = \overline{\widehat{a}}(\omega) \end{aligned}$$

where $\omega(a_i) = \widehat{a_i}(\omega) \in \sigma(a_i) \subset \mathbb{R}$ for $i = 1, 2$. This proves $\widehat{a^*} = \overline{\widehat{a}}$. □

Remark 2.14. The above proposition shows that

$$\|a\| = \sup\{|\omega(a)| : \omega \in \Omega_{\mathcal{A}}\}$$

in an abelian C*-algebra $\mathcal{A}$.

To conclude the proof of Theorem **1.5**, we observe that the Gelfand map $\widehat{\ }: \mathcal{A} \longrightarrow C(\Omega_{\mathcal{A}})$ is surjective if $\mathcal{A}$ is unital since $\widehat{\mathcal{A}}$ is a closed *-subalgebra of $C(\Omega_{\mathcal{A}})$ containing constant functions on $\Omega_{\mathcal{A}}$ and separating points of $\Omega_{\mathcal{A}}$ which imply $\widehat{\mathcal{A}} = C(\Omega)$ by the Stone-Weierstrass Theorem.

If $\mathcal{A}$ is non-unital, the Gelfand map $\widehat{\ }: \mathcal{A} \longrightarrow C_0(\Omega_{\mathcal{A}})$ is also surjective. Indeed, each $f \in C_0(\Omega_{\mathcal{A}})$ can be extended to a continuous function $\overline{f}$ on

$$\Omega_{\mathcal{A}_1} = \Omega_{\mathcal{A}} \cup \{\omega_0\}$$

by defining $\overline{f}(\omega_0) = 0$, and therefore, the surjectivity of the Gelfand map on $\mathcal{A}_1$ implies that $\overline{f} = \widehat{b}$ for some $b = a \oplus \beta \in \mathcal{A}_1$; but $\beta = \omega_0(a \oplus \beta) = \omega_0(b) = \overline{f}(\omega_0) = 0$. Hence $f = \widehat{a} \in \widehat{\mathcal{A}}$.

Chapter 3. Functional Calculus

Given a non-empty subset S of a C*-algebra $\mathcal{A}$, the smallest C*-subalgebra of $\mathcal{A}$ containing S clearly exists, by taking the intersection of all C*-subalgebras containing S. We call it the C*-*algebra generated by* S in $\mathcal{A}$. We begin with the following two fundamental results which enable us to derive properties of C*-algebras by reduction to the commutative case.

Theorem 3.1. *Let $\mathcal{A}$ be a unital C*-algebra and let $a \in \mathcal{A}$ be normal. Then the C*-subalgebra $C(a, \mathbf{1})$ generated by a and $\mathbf{1}$ is isometrically *-isomorphic to the abelian C*-algebra $C(\sigma(a))$ of complex continuous functions on the spectrum $\sigma(a)$ of a in $\mathcal{A}$.*

Proof. Since a is normal, the C*-algebra $C(a, \mathbf{1})$, consisting of polynomials in a and a^*, as well as their limits, is abelian and can be identified with the C*-algebra $C(\Omega)$ of continuous functions on the spectrum Ω of $C(a, \mathbf{1})$, via the Gelfand transform $\widehat{\ }$. Let $\sigma_C(a)$ be the spectrum of a in $C(a, \mathbf{1})$. By (**2.1**) in the proof of Proposition **2.13**, we have $\sigma_C(a) = \widehat{a}(\Omega)$ and the map

$$\omega \in \Omega \mapsto \widehat{a}(\omega) \in \sigma_C(a)$$

is a homeomorphism since $C(a, \mathbf{1})$ is generated by a. It follows that

$$f \in C(\sigma_C(a)) \mapsto f \circ \widehat{a} \in C(\Omega)$$

is an isometric *-isomorphism.

It remains to show that $\sigma(a) = \sigma_C(a)$. Evidently, we have $\sigma(a) \subset \sigma_C(a)$. Let $\alpha \in \sigma_C(a)$. Regard $f = a - \alpha\mathbf{1}$ as a function in $C(\sigma_C(a))$. For any $\varepsilon > 0$, the set $K = \{x \in \sigma_C(a) : |f(x)| \geq \varepsilon\}$ is compact in $\sigma_C(a)$ and we can find a function $g \in C(\sigma_C(a))$ satisfying $0 \leq g \leq 1$, $g(\alpha) = 1$ and $g(K) = \{0\}$, by Urysohn Lemma, so that $\|(a - \alpha)g\| \leq \varepsilon$ and hence $\alpha \in \sigma(a)$, for if $b(a - \alpha) = \mathbf{1}$ for some $b \in \mathcal{A}$, we can choose $g \in C(\sigma_C(a))$ with $\|g\| = 1$ and $\|(a - \alpha)g\| < \|b\|^{-1}$, giving a contradiction. This proves $\sigma(a) = \sigma_C(a)$. □

Theorem 3.2. *Let $\mathcal{A}$ be a C*-algebra and let $a \in \mathcal{A}$ be normal. Then the C*-subalgebra $C(a)$ generated by a is isometrically *-isomorphic to the abelian C*-algebra $C_0(\sigma'(a)\backslash\{0\})$ of complex continuous functions on $\sigma'(a)\backslash\{0\}$, vanishing at infinity, where $\sigma'(a)$ is the quasi-spectrum of a in $\mathcal{A}$.*

Proof. Normality of a implies that $C(a)$ is an abelian C*-algebra and we have

$$C(a) \simeq C_0(\Omega)$$

via the Gelfand transform, where Ω is the spectrum of $C(a)$. Since a generates $C(a)$, we have $\omega(a) \neq 0$ for all $\omega \in \Omega$ and by (**2.2**), the map

$$\omega \in \Omega \mapsto \widehat{a}(\omega) \in \sigma'_{C(a)}(a)\backslash\{0\}$$

is a homeomorphism and we have $C(a) \simeq C_0(\sigma'_{C(a)}(a)\backslash\{0\})$. As in the proof of the above theorem, the non-zero quasi-spectrum $\sigma'(a)\backslash\{0\}$ coincides with the non-zero quasi-spectrum $\sigma'_{C(a)}(a)\backslash\{0\}$ of a in $C(a)$. Hence we have $C(a) \simeq C_0(\sigma'(a)\backslash\{0\})$. □

Let $a \in \mathcal{A}$ be a self-adjoint element and let $C(a) \subset \mathcal{A}$ be the C*-algebra generated by a. Denote by $\varphi : C_0(\sigma'(a)\backslash\{0\}) \longrightarrow C(a)$ be the *-isomorphism in Theorem **4.4**. For each $f \in C_0(\sigma'(a)\backslash\{0\})$, we write $f(a) = \varphi(f)$. Evidently $a = \varphi(\iota) = \iota(a)$ where $\iota \in C_0(\sigma'(a)\backslash\{0\})$ is the identity map $\iota : \sigma'(a)\backslash\{0\} \longrightarrow \sigma'(a)\backslash\{0\}$. Since $\sigma(a) \subset \mathbb{R}$, we can define $f_1, f_2, f_3, f_4 \in C_0(\sigma'(a)\backslash\{0\})$ to be the following real-valued functions

$$\begin{aligned} f_1(\lambda) &= \lambda \vee 0 \\ f_2(\lambda) &= \lambda \wedge 0 \\ f_3(\lambda) &= |\lambda| \\ f_4(\lambda) &= \sqrt{\lambda}. \end{aligned}$$

Write $a_+ = f_1(a)$, $a_- = f_2(a)$, $|a| = f_3(a)$ and $a^{1/2} = f_4(a)$. Then we have $a = a_+ - a_-$, $|a| = a_+ + a_-$ and $a_+a_- = 0$.

Lemma 3.3. *If a functional $\varphi \in \mathcal{A}^*$ satisfies $f(a^*a) = 0$ for all $a \in \mathcal{A}$, then $\varphi = 0$.*

Proof. By a remark after Definition **2.9**, it suffices to show that $\varphi(a) = 0$ for each self-adjoint element $a \in \mathcal{A}$. By functional calculus, we have $\varphi(a) = \varphi(a_+ - a_-) = \varphi(a_+^{1/2}a_+^{1/2}) - \varphi(a_-^{1/2}a_-^{1/2}) = 0$. □

Chapter 4. Homomorphisms between C*-algebras

A simple spectral analysis shows that a *-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is automatically continuous. In fact, it is contractive.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a *-homomorphism. Then we have*

$$\|\varphi(a)\| \le \|a\| \qquad (a \in \mathcal{A}).$$

Proof. Since φ is a homomorphism, we have

$$\sigma'(a) \supset \sigma'(\varphi(a))$$

and hence the following inequalities for the spectral radii:

$$r(\varphi(a)) \le r(a) \le \|a\|$$

for all $a \in \mathcal{A}$. Noting that the spectral radius of a self-adjoint element coincides with its norm, we obtain, for each $a \in \mathcal{A}$,

$$\|\varphi(a)\|^2 = \|\varphi(a)\varphi(a)^*\| = \|\varphi(aa^*)\| = r(\varphi(aa^*)) \le \|aa^*\| = \|a\|^2$$

which completes the proof. □

Corollary 4.2. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a *-isomorphism from a C*-algebra $\mathcal{A}$ onto another one $\mathcal{B}$. Then φ is an isometry, that is, $\|\varphi(a)\| = \|a\|$ for all $a \in \mathcal{A}$.*

Proof. Apply the above theorem to the *-isomorphisms φ and its inverse φ^{-1}. □

Corollary 4.3. (Uniqueness of C*-norm) *Let $(\mathcal{A}, \|\cdot\|)$ be a C*-algebra. If $\|\cdot\|_1$ is a norm on $\mathcal{A}$ such that $(\mathcal{A}, \|\cdot\|_1)$ is a C*-algebra. Then $\|\cdot\| = \|\cdot\|_1$.*

One can show further that a *-isomorphism from a C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$ is an isometry. We recall that the *dual linear map* $\varphi^* : F^* \longrightarrow E^*$ of a continuous linear map $\varphi : E \longrightarrow F$ between Banach spaces is defined by

$$\varphi^*(\omega)(x) = \omega(\varphi(x)) \qquad (\omega \in F^*, x \in E).$$

Theorem 4.4. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a *-isomorphism from a C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$. Then φ is an isometry.*

Proof. We do not know, *a priori*, that the image $\varphi(\mathcal{A})$ is a C*-algebra since in general, a continuous image of a Banach space need not be a Banach space. Therefore the above arguments cannot be applied directly to the inverse of φ.

Let $a \in \mathcal{A}$. By considering the C*-subalgebras generated by a and $\varphi(a)$, we may assume both $\mathcal{A}$ and $\mathcal{B}$ are abelian. By adding the identity, we may assume further that $\mathcal{A}$ and $\mathcal{B}$ are unital, and that $\varphi(\mathbf{1}) = \mathbf{1}$.

Identify $\mathcal{A}$ with $C(\Omega_{\mathcal{A}})$ and $\mathcal{B}$ with $C(\Omega_{\mathcal{B}})$ via the Gelfand transform. As usual, we can identify $\Omega_{\mathcal{A}}$ as a weak* compact subspace of the dual $C(\Omega_{\mathcal{A}})^*$ by the evaluation map:

$$\omega \in \Omega_{\mathcal{A}} \mapsto \varepsilon_\omega \in C(\Omega_{\mathcal{A}})^*, \quad \text{where} \quad \varepsilon_\omega(a) = a(\omega) \quad \text{for} \quad a \in C(\Omega_{\mathcal{A}}) \text{ and } \omega \in \Omega_{\mathcal{A}}.$$

Likewise $\Omega_{\mathcal{B}} \subset C(\Omega_{\mathcal{B}})^*$.

Since $\Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{B}}$ are spectra of $\mathcal{A}$ and $\mathcal{B}$ respectively, and since φ is an algebra isomorphism, the dual map $\varphi^* : C(\Omega_{\mathcal{B}})^* \longrightarrow C(\Omega_{\mathcal{A}})^*$ carries $\Omega_{\mathcal{B}}$ into $\Omega_{\mathcal{A}}$, and $\varphi^*(\Omega_{\mathcal{B}})$ is weak* compact, by continuity of φ^*, and hence weak* closed.

We claim that $\varphi^*(\Omega_{\mathcal{B}}) = \Omega_{\mathcal{A}}$. Indeed, if $\varphi^*(\Omega_{\mathcal{B}}) \neq \Omega_{\mathcal{A}}$, then we can find two nonzero functions $f, g \in C(\Omega_{\mathcal{A}})$ such that $fg = 0$ and $g(\chi) = 1$ for all $\chi \in \varphi^*(\Omega_{\mathcal{B}})$. By the Gelfand representation, we can then find nonzero $a, b \in \mathcal{A}$ such that $ab = 0$ and $\chi(b) = 1$ for all $\chi \in \varphi^*(\Omega_{\mathcal{B}})$. Since $\varphi(a) \neq 0$, there exists $\omega \in \Omega_{\mathcal{B}}$, such that $\omega(\varphi(a)) \neq 0$. It follows from $\varphi(a)\varphi(b) = 0$ that

$$0 = \omega(\varphi(a)\varphi(b)) = \omega(\varphi(a))\omega(\varphi(b)) = \omega(\varphi(a))\varphi^*(\omega)(b) = \omega(\varphi(a)) \neq 0$$

which is a contradiction. Therefore $\varphi^*(\Omega_{\mathcal{B}}) = \Omega_{\mathcal{A}}$ and

$$\begin{aligned} \|\varphi(a)\| &= \sup\{|\varphi(a)(\omega)| : \omega \in \Omega_{\mathcal{B}}\} \\ &= \sup\{|\varphi^*(\omega)(a)| : \omega \in \Omega_{\mathcal{B}}\} \\ &= \sup\{|\chi(a)| : \chi \in \Omega_{\mathcal{A}}\} \\ &= \|a\|. \end{aligned}$$

□

A natural question arises: *is the converse of Theorem* **4.4** *true*? Is an isometry $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ from $\mathcal{A}$ <u>into</u> $\mathcal{B}$ a *-isomorphism? The answer is negative, but one can show that it is *"almost"* such *locally*: for each $a \in \mathcal{A}$, we have

$$\varphi(xx^*x) = \varphi(x)\varphi(x)^*\varphi(x) \qquad (x \in C(a))$$

modulo a projection $p \in \mathcal{B}^{**}$, meaning that both sides of the above equality should be multiplied on the right by p. This result and more details are given in [2, 3].

However, if φ is *surjective*, then it is a well-known result of Kadison [6] that

$$\varphi(xx^*x) = \varphi(x)\varphi(x)^*\varphi(x) \quad \text{for all} \quad x \in \mathcal{A}.$$

In particular, if $\mathcal{A}$ and $\mathcal{B}$ are unital and $\varphi(\mathbf{1}) = \mathbf{1}$, then φ is a *-isomorphism.

The above results can be viewed as non-commutative generalisations of a well-know fact in topology that two compact Hausdorff spaces X and Y are homeomorphic if, and only if, the corresponding continuous functions spaces $C(X)$ and $C(Y)$ are linearly isometric to each other.

Chapter 5. STATES AND REPRESENTATIONS

Let $\mathcal{A}$ be a C*-algebra. A linear functional $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is called *positive* if $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$.

Lemma 5.1. *A positive linear functional φ on a unital C*-algebra satisfies*

(i) $\varphi(x^*) = \overline{\varphi(x)} \quad (x \in \mathcal{A})$;
(ii) $|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y) \quad (x, y \in \mathcal{A})$;
(iii) φ *is continuous and* $\|\varphi\| = \varphi(\mathbf{1})$;
(iv) $|\varphi(y^*xy)| \leq \|x\|\varphi(y^*y) \quad (x, y \in \mathcal{A})$;
(v) $\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$ *if* ψ *is also a positive linear functional on* $\mathcal{A}$.

Proof. By positivity, φ induces a positive semidefinite sesquilinear form, i.e. a semi inner product, $\langle \cdot, \cdot \rangle : (x, y) \in \mathcal{A} \times \mathcal{A} \mapsto \varphi(y^*x)$. Hence (i) follows from Hermitian symmetry $\langle x, y \rangle = \overline{\langle y, x \rangle}$, and (ii) is the Schwarz inequality. To see (iii), we first note that (ii) implies

$$|\varphi(x)|^2 \leq \varphi(x^*x)\varphi(\mathbf{1}) \qquad (x \in \mathcal{A}).$$

The element $a = x^*x$ is self-adjoint and the C*-subalgebra $C(a, \mathbf{1})$ of $\mathcal{A}$, generated by a and $\mathbf{1}$, is identified with complex continuous functions on a compact Hausdorff space $\sigma(a)$, by Theorem **3.1**.

If $\|x\| \leq 1$, then $\|a\| = \|x\|^2 \leq 1$ and we must have, as functions on $\sigma(a)$, that $-\mathbf{1} \leq a \leq \mathbf{1}$ and hence $\mathbf{1} - a = b^*b$ for some $b \in C(a, \mathbf{1})$. Positivity of φ gives $\varphi(\mathbf{1} - x^*x) = \varphi(b^*b) \geq 0$ and $\varphi(x^*x) \leq \varphi(\mathbf{1})$. It follows that

$$\sup\{|\varphi(x)| : \|x\| \leq 1\} \leq \varphi(\mathbf{1})$$

and φ is continuous with $\|\varphi\| \leq \varphi(\mathbf{1})$. But $\|\mathbf{1}\| = 1$, we conclude $\|\varphi\| = \varphi(\mathbf{1})$.

For (iv), we observe that the linear functional $\varphi_y : x \in \mathcal{A} \mapsto \varphi(y^*xy) \in \mathbb{C}$ is positive and hence $\|\varphi_y\| = \varphi_y(\mathbf{1})$, by (iii), which gives $|\varphi_y(x)| \leq \|x\|\varphi(y^*y)$.

Finally, given another positive linear functional ψ on $\mathcal{A}$, the sum $\varphi + \psi$ is also positive and we have

$$\|\varphi + \psi\| = (\varphi + \psi)(\mathbf{1}) = \varphi(\mathbf{1}) + \psi(\mathbf{1}) = \|\varphi\| + \|\psi\|.$$

□

Remark 5.2. The assertions (i) and (ii) above do not require the identity of $\mathcal{A}$. Also (i) implies $\varphi(x) \in \mathbb{R}$ if x is self-adjoint.

Definition 5.3. A positive linear functional φ on a C*-algebra $\mathcal{A}$ is called a *state* if $\|\varphi\| = 1$.

Lemma 5.4. *Let $\mathcal{A}$ be a unital C*-algebra and let $\mathcal{B}$ be a C*-subalgebra containing the identity $\mathbf{1}$ of $\mathcal{A}$. Then every state φ on $\mathcal{B}$ extends to a state on $\mathcal{A}$, that is, there is a state $\widetilde{\varphi}$ on $\mathcal{A}$ such that $\widetilde{\varphi}|_{\mathcal{B}} = \varphi$.*

Proof. By Hahn-Banach theorem, φ extends to a linear functional $\widetilde{\varphi}$ on $\mathcal{A}$ with $\|\widetilde{\varphi}\| = \|\varphi\|$=1. We have $\widetilde{\varphi}(\mathbf{1}) = 1$. We need to show $\widetilde{\varphi}(x^*x) \geq 0$ for all $x \in \mathcal{A}$.

We first show that $\widetilde{\varphi}(a) \in \mathbb{R}$ if $a = a^*$. Indeed, if $\widetilde{\varphi}(a) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, then the element

$$y = \beta^{-1}(a - \alpha\mathbf{1})$$

is self-adjoint and $\widetilde{\varphi}(y) = i$ and hence for all $r \in \mathbb{R}$, we have

$$\begin{aligned} (r+1)^2 &= |i + ri|^2 = |\widetilde{\varphi}(y + ri)|^2 \\ &\leq \|y + ri\|^2 = \|(y + ri)^*(y + ri)\| \\ &= \|y^2 + r^2\| \leq \|y\|^2 + r^2 \end{aligned}$$

which is impossible.

To show $\widetilde{\varphi}(x^*x) \geq 0$, we may assume $\|x\| \leq 1$, by linearity of $\widetilde{\varphi}$. The C*-subalgebra $C(x^*x, \mathbf{1})$ generated by x^*x and $\mathbf{1}$ identifies with the algebra $C(\Omega)$ of complex continuous functions on a compact Hausdorff space Ω, and $\mathbf{1}$ with the constant function on Ω taking value 1. As a function in $C(\Omega)$, the element x^*x has supremum norm at most 1 and hence $\|\mathbf{1} - x^*x\| \leq 1$. It follows that

$$1 \geq \widetilde{\varphi}(\mathbf{1} - x^*x) = \widetilde{\varphi}(\mathbf{1}) - \widetilde{\varphi}(x^*x) = 1 - \widetilde{\varphi}(x^*x)$$

and $\widetilde{\varphi}(x^*x) \geq 0$, where $\widetilde{\varphi}(\mathbf{1} - x^*x) \in \mathbb{R}$ because $\mathbf{1} - x^*x$ is self-adjoint. □

Lemma 5.5. *Let $\mathcal{A}$ be a C*-algebra and $a \in \mathcal{A}$. If $\varphi(a^*a) = 0$ for every state φ of $\mathcal{A}$, then $a = 0$.*

Proof. First, assume $\mathcal{A}$ has an identity $\mathbf{1}$. Let

$$\widehat{\ } : C(a^*a, \mathbf{1}) \longrightarrow C(\Omega)$$

be the Gelfand transform identifying the C*-subalgebra generated by a^*a and $\mathbf{1}$ with the algebra $C(\Omega)$ of complex continuous functions on the spectrum Ω of a^*a, shown in Theorem **3.1**.

Each $\omega \in \Omega$ induces a state $\psi = \varepsilon_\omega \circ \widehat{\ }$ on $C(a^*a, \mathbf{1})$, where

$$\varepsilon_\omega(f) = f(\omega) \qquad (f \in C(\Omega)).$$

By Lemma **5.4**, ψ extends to a state φ of $\mathcal{A}$ and hence $\psi(a^*a) = \varphi(a^*a) = 0$ which gives

$$\widehat{a^*a}(\omega) = 0.$$

Since $\omega \in \Omega$ was arbitrary, we have $\widehat{a^*a} = 0$ and therefore $a^*a = 0$ as well as $a = 0$.

Now, if $\mathcal{A}$ lacks identity, we consider its unit extension $\mathcal{A}_1$. Each state ψ of $\mathcal{A}_1$ restricts to a state φ of $\mathcal{A}$ and by hypothesis, we have $\psi(a^*a) = \varphi(a^*a) = 0$. It follows that $a = 0$ by the above conclusion for the unital case.

□

Definition 5.6. Let $\mathcal{A}$ be a C*-algebra and let H be a Hilbert space. A *representation* π of $\mathcal{A}$ on H is a *-algebra homomorphism $\pi : \mathcal{A} \longrightarrow B(H)$, in other words, π is a linear map from $\mathcal{A}$ into $B(H)$ satisfying

$$\pi(ab) = \pi(a)\pi(b) \quad \text{and} \quad \pi(a^*) = \pi(a)^*$$

for all $a, b \in \mathcal{A}$. A representation π is called *faithful* if it is injective.

Two representations $\pi : \mathcal{A} \longrightarrow B(H)$ and $\tau : \mathcal{A} \longrightarrow B(K)$ are said to be (*unitarily*) *equivalent*, in symbols: $\pi \simeq \tau$, if there is a surjective linear isometry $u : H \longrightarrow K$ such that

$$u\pi(a) = \tau(a)u \qquad (a \in \mathcal{A}).$$

We call u an *intertwining operator* between π and τ.

Let $\pi : \mathcal{A} \longrightarrow B(H)$ be a representation of $\mathcal{A}$. Given a closed subspace $K \subset H$ *invariant* under $\pi(\mathcal{A})$, that is, $\pi(\mathcal{A})(K) \subset K$, we can define a representation $\pi_K : \mathcal{A} \longrightarrow B(K)$ by restriction:

$$\pi_K(a) = \pi(a)|_K \qquad (a \in \mathcal{A}).$$

The representation π_K is called a *sub-representation* of π.

Definition 5.7. A representation $\pi : \mathcal{A} \longrightarrow B(H)$ is called *irreducible* if $\pi(\mathcal{A})$ has no invariant subspace other than $\{0\}$ and H.

Let $\pi : \mathcal{A} \longrightarrow B(H)$ be a representation and let $p \in B(H)$ be a projection. Then the range space $p(H)$ of p is an invariant subspace of $\pi(\mathcal{A})$ if, and only if, p commutes with every element in $\pi(\mathcal{A})$. Indeed, if p commutes with $\pi(\mathcal{A})$, then

$$\pi(\mathcal{A})p(H) = p\pi(\mathcal{A})(H) \subset p(H).$$

Conversely, the invariance of $p(H)$ implies that $p\pi(a)p\eta = \pi(a)p\eta$ for every $\eta \in H$. On the other hand, we have

$$p\pi(a)\eta = p\pi(a)(p\eta + (\mathbf{1} - p)\eta) = p\pi(a)p\eta + p\pi(a)(\mathbf{1} - p)\eta = p\pi(a)p\eta$$

since $H = p(H) \oplus (\mathbf{1} - p)(H)$ and $\pi(\mathcal{A})(\mathbf{1} - p)H \subset (\mathbf{1} - p)H$.

Given a representation $\pi : \mathcal{A} \longrightarrow B(H)$ and a vector $\xi \in H$, it is easily verified that the function $f : \mathcal{A} \longrightarrow \mathbb{C}$ defined by

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle \qquad (a \in \mathcal{A})$$

is a positive linear functional. Conversely, every positive functional φ of $\mathcal{A}$ induces a representation π_φ of $\mathcal{A}$ by the Gelfand-Naimark-Segal construction described below.

Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a positive linear functional. Let

$$N_\varphi = \{a \in \mathcal{A} : \varphi(a^*a) = 0\}.$$

By the Schwarz inequality $|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y)$, we see that N_φ is a closed subspace of $\mathcal{A}$. In fact, N_φ is a left ideal of $\mathcal{A}$. This is not needed below, but one can see it immediately from Lemma **5.1** (iv) which is also valid if $\mathcal{A}$ is not unital.

The quotient $\mathcal{A}/N_\varphi$ is an inner product space with the inner product

$$\langle a + N_\varphi, b + N_\varphi \rangle := \varphi(b^*a)$$

since $\langle a + N_\varphi, a + N_\varphi \rangle = 0$ if, and only if, $a \in N_\varphi$. Denote the inner product norm in $\mathcal{A}/N_\varphi$ by

$$\|a + N_\varphi\|_\varphi = \varphi(a^*a)^{1/2}.$$

Let H_φ be the completion of A/N_φ. We can define a map $\pi_\varphi : \mathcal{A} \longrightarrow B(H_\varphi)$ satisfying

$$\pi_\varphi(a)(b + N_\varphi) = ab + N_\varphi \qquad (a \in \mathcal{A}, b + N_\varphi \in \mathcal{A}/N_\varphi).$$

Indeed, for $a \in \mathcal{A}$, the above formula defines a bounded linear operator $\pi_\varphi(a) : \mathcal{A}/N_\varphi \longrightarrow \mathcal{A}/N_\varphi$ since

$$\|\pi_\varphi(a)(b + N_\varphi)\|_\varphi^2 = \|ab + N_\varphi\|_\varphi^2 = \varphi(b^*a^*ab) \leq \|a\|^2\varphi(b^*b) = \|a\|^2\|b + N_\varphi\|_\varphi^2.$$

Hence $\pi_\varphi(a)$ can be extended to a bounded linear operator on the completion H_φ, and is still denoted by $\pi_\varphi(a)$.

We see that π_φ is a *-homomorphism since

$$\begin{aligned} & \langle \pi_\varphi(a)^*(b + N_\varphi), c + N_\varphi \rangle = \langle b + N_\varphi, \pi_\varphi(a)(c + N_\varphi) \rangle \\ = \;& \varphi(c^*a^*b) = \langle \pi_\varphi(a^*)(b + N_\varphi), c + N_\varphi \rangle. \end{aligned}$$

In particular, π_φ is continuous and $\|\pi_\varphi\| \leq 1$ by Theorem **4.1**.

If $\mathcal{A}$ has identity $\mathbf{1}$, then we have

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle \qquad (a \in \mathcal{A})$$

where $\xi_\varphi = \mathbf{1} + N_\varphi$. In this case, $\pi_\varphi(\mathcal{A})\xi_\varphi = \mathcal{A}/N_\varphi$ and is dense in H_φ. If $\mathcal{A}$ has no identity, one can still show that there is a vector $\xi_\varphi \in H_\varphi$ such that

$$\varphi(x) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle \qquad (x \in \mathcal{A})$$

and $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in H_φ. We refer to [11, p.39] for a proof.

We call the representation π_φ constructed above the GNS-*representation* of φ, and ξ_φ a *cyclic vector* for π_φ.

Definition 5.8. Let $\mathcal{A}$ be a C*-algebra. The set $S(\mathcal{A})$ of all states of $\mathcal{A}$ is called the *state space* of $\mathcal{A}$.

If $\mathcal{A}$ is unital, Lemma **5.1** implies that its state space $S(\mathcal{A})$ is a weak* closed convex subset of the dual ball $\{\varphi \in \mathcal{A}^* : \|\varphi\| \leq 1\}$ and is therefore weak* compact.

We are now ready to show that the direct sum $\bigoplus_\varphi \pi_\varphi$ of all the representations π_φ induced by the states φ of $\mathcal{A}$ is a *-isomorphism from $\mathcal{A}$ into $B(\bigoplus_\varphi H_\varphi)$. We first show that the mapping

$$\pi = \bigoplus_{\varphi \in S(\mathcal{A})} \pi_\varphi : \mathcal{A} \longrightarrow B\left(\bigoplus_{\varphi \in S(A)} H_\varphi\right)$$

defined by

$$\pi(a)(\oplus_\varphi x_\varphi) = \left(\bigoplus_{\varphi \in S(\mathcal{A})} \pi_\varphi\right)(a)(\oplus x_\varphi) = \bigoplus_{\varphi \in S(\mathcal{A})} \pi_\varphi(a)(x_\varphi) \qquad (a \in \mathcal{A})$$

is indeed well-defined. This follows from

$$\begin{aligned}
\left\| \bigoplus_{\varphi \in S(\mathcal{A})} \pi_\varphi(a)(x_\varphi) \right\|^2 &= \sum_{\varphi \in S(\mathcal{A})} \|\pi_\varphi(a)(x_\varphi)\|^2 \\
&\leq \sum_{\varphi \in S(\mathcal{A})} \|\pi_\varphi(a)\|^2 (x_\varphi)\|^2 \\
&\leq \|a\|^2 \sum_{\varphi \in S(\mathcal{A})} \|(x_\varphi)\|^2 \\
&= \|a\|^2 \| \oplus x_\varphi\|^2
\end{aligned}$$

which also implies $\|\pi(a)\| \leq \|a\|$. In fact, π is an isometry since it is a *-isomorphism and Theorem **4.4** applies. It is clear that π is a *-homomorphism. If $\pi(a) = 0$, then $\pi_\varphi(a) = 0$ for all $\varphi \in S(\mathcal{A})$. Hence

$$\varphi(aa^*aa^*) = \|aa^* + N_\varphi\|^2 = \|\pi_\varphi(a)(a^* + N_\varphi)\|^2 = 0$$

for all $\varphi \in S(\mathcal{A})$. By Lemma **5.5**, we have $aa^* = 0$ and $a = 0$. This shows that π is a *-monomorphism and completes the proof of Theorem **1.10**.

Definition 5.9. Given two linear functionals ψ and φ of a C*-algebra $\mathcal{A}$. We write $\psi \leq \varphi$ if $\psi(a^*a) \leq \varphi(a^*a)$ for all $a \in \mathcal{A}$.

Definition 5.10. A positive linear functional φ of $\mathcal{A}$ is called *pure* if for any positive linear functional ψ of $\mathcal{A}$ satisfying $\psi \leq \varphi$, we have $\psi = \alpha\varphi$ for some $\alpha \geq 0$. It follows that $\alpha \leq 1$

Theorem 5.11. *Let $\mathcal{A}$ be a C*-algebra and let φ be a state of $\mathcal{A}$. The following conditions are equivalent.*

(i) *φ is a pure state.*
(ii) *The GNS-representation π_φ is irreducible.*

Proof. (i) $\Longrightarrow$ (ii). Let φ be a pure state and let $\pi_\varphi : \mathcal{A} \longrightarrow B(H_\varphi)$ be the GNS-representation such that

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle \qquad (a \in \mathcal{A})$$

for some cyclic vector $\xi_\varphi \in H_\varphi$ and $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in H_φ.

Let $K \subset H_\varphi$ be a closed subspace satisfying $\pi_\varphi(\mathcal{A})K \subset K$. We show that $K = \{0\}$ or H_φ. This amounts to showing that the projection $P : H_\varphi \longrightarrow K$ is either 0 or the identity operator I on H_φ.

By a remark after Definition **5.7**, P commutes with $\pi_\varphi(a)$ for all $a \in \mathcal{A}$. Define a positive linear functional ψ on $\mathcal{A}$ by

$$\psi(x) = \langle \pi_\varphi(x)P\xi_\varphi, P\xi_\varphi\rangle \qquad (x \in \mathcal{A}).$$

We have $\psi \leq \varphi$ since

$$\begin{aligned}\psi(a^*a) &= \|\pi_\varphi(a)P\xi_\varphi\|_\varphi^2 = \|P\pi_\varphi(a)\xi_\varphi\|_\varphi^2 \\ &\leq \|\pi_\varphi(a)\xi_\varphi\|_\varphi^2 = \varphi(a^*a).\end{aligned}$$

By purity of φ, we have $\psi = \alpha\varphi$ for some $0 \leq \alpha \leq 1$.

For any $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\langle \alpha\pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi\rangle &= \alpha\varphi(b^*a) = \psi(b^*a) \\ &= \langle \pi_\varphi(a)P\xi_\varphi, \pi_\varphi(b)P\xi_\varphi\rangle \\ &= \langle P\pi_\varphi(a)\xi_\varphi, P\pi_\varphi(b)\xi_\varphi\rangle \\ &= \langle P\pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi\rangle.\end{aligned}$$

Since $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in H_φ, we conclude that $P = \alpha I$ and hence $\alpha = 0$ or 1 since $P^2 = P$. This proves irreducibility of π_φ.

(ii) $\Longrightarrow$ (i). Given that the above GNS-representation π_φ is irreducible, we show that φ is pure. Let $0 \leq \psi \leq \varphi$.

For $a, b \in \mathcal{A}$, the Schwarz inequality implies that

$$|\psi(b^*a)|^2 \leq \|\pi_\varphi(a)\xi_\varphi\|^2 \|\pi_\varphi(b)\xi_\varphi\|^2$$

and therefore

$$\ll \pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi \gg \, = \, \psi(b^*a)$$

defines a positive semidefinite sesquilinear form on the dense subspace $\pi_\varphi(\mathcal{A})\xi_\varphi$ of H_φ. It follows that there is a bounded operator $T \in B(H_\varphi)$ such that

$$\ll \pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi \gg \, = \, \langle T\pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi\rangle \qquad (a, b \in \mathcal{A}).$$

Since

$$\langle T\pi_\varphi(a)\xi_\varphi, \pi_\varphi(a)\xi_\varphi\rangle = \psi(a^*a) \geq 0 \qquad (a \in \mathcal{A})$$

and since $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in H_φ, the operator T is a positive operator on H_φ which means

$$\langle T\eta, \eta\rangle \geq 0$$

for all $\eta \in H_\varphi$.

We next show that T commutes with each element in $\pi_\varphi(\mathcal{A})$. For $x, y, z \in \mathcal{A}$, we have

$$\begin{aligned}\langle T\pi_\varphi(x)\pi_\varphi(y)\xi_\varphi, \, \pi_\varphi(z)\xi_\varphi\rangle &= \psi(z^*xy) = \psi((x^*z)^*y) \\ &= \langle T\pi_\varphi(y)\xi_\varphi, \, \pi_\varphi(x^*z)\xi_\varphi\rangle \\ &= \langle \pi_\varphi(x)T\pi_\varphi(y)\xi_\varphi, \, \pi_\varphi(z)\xi_\varphi\rangle\end{aligned}$$

which implies T commutes with $\pi_\varphi(\mathcal{A})$ by density of $\pi_\varphi(\mathcal{A})$ in H_φ. It follows that all spectral projections of T commutes with each element of $\pi_\varphi(\mathcal{A})$ and they are therefore either 0 or I by irreducibility of π_φ. Hence the spectrum of T reduces to a singleton and $T = \alpha I$ for some $\alpha \geq 0$.

Now we have

$$\psi(a^*a) = \langle T\pi_\varphi(a)\xi_\varphi, \, \pi_\varphi(a)\xi_\varphi\rangle = \alpha\langle \pi_\varphi(a)\xi_\varphi, \, \pi_\varphi(a)\xi_\varphi\rangle = \alpha\varphi(a^*a).$$

By Lemma **3.3**, we obtain $\psi = \alpha\varphi$. Hence φ is pure. □

Chapter 6. Locally convex topologies for $B(H)$

Let H be a complex Hilbert space. The C*-algebra $B(H)$ of bounded operators on H can be equipped with several locally convex topologies. The frequently used topologies are the *operator norm topology*, the *strong operator topology* and the *weak operator topology*, where the word "*operator*" is often omitted.

The operator norm topology is also called the *uniform topology*. It is induced by the operator norm

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| \leq 1\}.$$

This topology is metrizable and a sequence (T_n) converges to T in this topology if, and only if, $\lim_{n\to\infty} \|T_n - T\| = 0$.

Both the strong operator topology and the weak operator topology are not metrizable in general. If H is separable, then both topologies are metrizable on the closed unit ball of $B(H)$ (see [11, p. 71] for a proof).

The strong operator topology is the weakest topology for which the mappings

$$T \in B(H) \mapsto Tx \in H \qquad (x \in H)$$

are continuous. It is a locally convex topology and is defined by the seminorms

$$\{p_x : x \in H\}$$

where

$$p_x(T) = \|Tx\|$$

for $x \in H$. A net (T_α) converges to T in the strong operator topology if, and only if,

$$\|(T_\alpha - T)x\| \longrightarrow 0 \quad \text{as} \quad \alpha \to \infty$$

for each $x \in H$.

The weak operator topology is the weakest topology for which the mappings

$$T \in B(H) \mapsto \langle Tx, y\rangle \in H \qquad (x, y \in H)$$

are continuous. It is locally convex and is defined by the seminorms

$$\{p_{x,y} : x, y \in H\}$$

where

$$p_{x,y}(T) = |\langle Tx, y\rangle|$$

for $x, y \in H$. A net (T_α) converges to T in the strong operator topology if, and only if,

$$\langle (T_\alpha - T)x, y\rangle \longrightarrow 0 \quad \text{as} \quad \alpha \to \infty$$

for every $x, y \in H$.

We have the following proper inclusions for the above three topologies:

weak operator topology $\subset$ *strong operator topology* $\subset$ *operator norm topology.*

An operator $T \in B(H)$ is called *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. For any $T \in B(H)$, the operator T^*T and TT^* are positive since

$$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle \geq 0 \quad \text{and} \quad \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle \geq 0.$$

Also, every projection $E \in B(H)$ is positive. A positive operator T is clearly self-adjoint since $\langle Tx, x\rangle$ is real for all $x \in H$. Given self-adjoint operators $T, S \in B(H)$, we write $T \geq S$ or $S \leq T$ if $T - S$ is positive. It is readily verified that $\leq$ is a partially ordering on the norm closed subspace of self-adjoint operators in $B(H)$. Hence one can define the notion of a *least upper bound* (sup) and a *greatest lower bound* (inf) of a set in the usual way.

Lemma 6.1. *Let $T \in B(H)$ be self-adjoint. Then $T \leq \|T\|\mathbf{1}$.*

Proof. For each $x \in H$, we have

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \langle \|T\|x, x\rangle.$$

□

Lemma 6.2. *Let $T \geq 0$ in $B(H)$ and let $S \in B(H)$. Then $S^*TS \geq 0$.*

Proof. For each $x \in H$, we have

$$\langle S^*TSx, x\rangle = \langle TSx, Sx\rangle \geq 0.$$

□

Lemma 6.3. *Let $S, T \geq 0$ in $B(H)$. If $ST = TS$, then $ST \geq 0$.*

Proof. By spectral theory, there is a positive operator $S^{1/2} \in B(H)$ such that $S = (S^{1/2})^2$ and $S^{1/2}$ is in the C*-subalgebra generated by S and $\mathbf{1}$ (cf. [9, p.314]). Hence

$$ST = S^{1/2}S^{1/2}T = S^{1/2}TS^{1/2} \geq 0.$$

□

Lemma 6.4. *Let $T \in B(H)$ and $0 \leq T \leq P$ for some projection $P \in B(H)$. Then $T = TP = PT$.*

Proof. By Lemma **6.2**, we have

$$((\mathbf{1} - P)T^{1/2})((\mathbf{1} - P)T^{1/2})^* = (\mathbf{1} - P)T(\mathbf{1} - P) = 0$$

which gives $(\mathbf{1} - P)T^{1/2} = 0$. Hence $(\mathbf{1} - P)T = 0$ and $T = PT = PT$. □

Proposition 6.5. *Let (T_n) be an increasing sequence of self-adjoint operators in $B(H)$ with $0 \leq T_n \leq \mathbf{1}$. Then the least upper bound $\sup_n T_n$ exists and is the limit of the sequence (T_n) in the strong operator topology.*

Proof. For each $x \in H$, the sequence $(\langle T_n x, x\rangle)$ of real numbers is increasing and bounded above by $\|x\|^2$ and hence

$$Q(x,x) = \sup\langle T_n x, x\rangle = \lim_{n\to\infty} \langle T_n x, x\rangle$$

exists and defines a bounded real quadratic form on H. By polarization, the limit

$$Q(x,y) = \lim_{n\to\infty} \langle T_n x, y\rangle \qquad (x, y \in H)$$

exists and defines a bounded conjugate bilinear Hermitian form on H. Therefore there is a self-adjoint operator $T \in B(H)$ such that

$$\langle Tx, y\rangle = Q(x,y).$$

Plainly, (T_n) converges to T in the weak operator topology and $T_n \le T$. If $S \ge T_n$ for all n, then

$$\langle Sx, x\rangle \ge \langle T_n x, x\rangle \qquad (x \in H)$$

and hence

$$\langle Sx, x\rangle \ge \sup\langle T_n x, x\rangle = \langle Tx, x\rangle \qquad (x \in H)$$

and $S \ge T$. Therefore $T = \sup_n T_n$.

Finally, we have, for each $x \in H$,

$$\begin{aligned} \|(T - T_n)x\|^2 &\le \|(T-T_n)^{1/2}\|^2 \|(T-T_n)^{1/2}x\|^2 \\ &\le \langle (T-T_n)x, x\rangle \longrightarrow 0 \end{aligned}$$

since $0 \le T - T_n \le T \le \mathbf{1}$ implies $\|(T-T_n)^{1/2}\| \le 1$. This proves (T_n) converges to T in the strong operator topology. □

Corollary 6.6. *Let $T \in B(H)$ and let $E : H \longrightarrow \overline{T(H)}$ be the natural orthogonal projection. Then E is the smallest projection satisfying*

$$TE = T = ET.$$

If $0 \le T \le \mathbf{1}$, then $T \le E$ and E is the limit of a sequence of polynomials in T without constant terms, in the strong operator topology.

Proof. Let $P \in B(H)$ be a projection. Then $PT = 0$ if, and only if, $PE = 0$ since $E(H) = \overline{T(H)}$. In particular, for $P = \mathbf{1} - E$, we have $(\mathbf{1} - E)T = 0 = T(\mathbf{1} - E)$ which gives $TE = T = ET$.

If $P \in B(H)$ is a projection satisfying $PT = T$, then $(\mathbf{1} - P)T = 0$ and therefore $(\mathbf{1} - P)E = 0$ and

$$\langle Px, x\rangle = \langle PEx, x\rangle + \langle P(\mathbf{1} - E)x, x\rangle \ge \langle Ex, x\rangle$$

where $P(\mathbf{1} - E) \ge 0$ as $PE = E = EP$.

Let $0 \le T \le \mathbf{1}$. Then $E - T = E(\mathbf{1} - T) \ge 0$ by Lemma **6.3** and hence $E \ge T$.

We note that $\|T\| \le 1$ and also $T \le T^{1/2}$ since $T^{1/2} - T = T^{1/4}(\mathbf{1} - T^{1/2})T^{1/4} \ge 0$ where $T^{1/2} \le \|T^{1/2}\|\mathbf{1} \le \mathbf{1}$.

Since $T^{1/2}$ is a norm limit of polynomials in T without constant terms, we have $ET^{1/2} = T^{1/2} = T^{1/2}E$ and hence $T^{1/2} = ET^{1/2}E \leq E\mathbf{1}E = E$. It follows that

$$T \leq T^{1/2} \leq T^{1/4} \leq \cdots \leq T^{1/2^n} \leq \cdots \leq E.$$

Let $Q = LUB\{T^{1/2^n} : n = 1, 2, \ldots\}$. Then $Q \leq E$. By Proposition **6.5**, we have

$$Q = \lim_{n\to\infty} T^{1/2^n}$$

where the limit is taken in the strong operator topology and

$$Q^2 = \lim_{n\to\infty} (T^{1/2^n})^2 = \lim_{n\to\infty} T^{1/2^{n-1}} = Q.$$

Now $T \leq Q$ implies $QT = T = TQ$ and hence $E \leq Q$. It follows that $E = Q$ and E is the strong limit of a sequence of polynomials in T without constant terms. $\square$

Definition 6.7. Let $T \in B(H)$. The projection $E : H \longrightarrow \overline{T(H)}$ above is called the *range projection* of T and is denoted by $[T]$.

Plainly, the range projection $[P]$ of a projection P is itself. Given $T \in B(H)$, we have

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = \|(T^*T)^{1/2}x\|^2 \qquad (x \in H)$$

and hence T and $(T^*T)^{1/2}$ have the same kernel. Since the orthogonal complement $\overline{T^*(H)}^{\perp}$ of $\overline{T^*(H)}$ is the kernel $T^{-1}(0)$, it follows that $[T^*] = \big[(T^*T)^{1/2}\big]$.

Chapter 7. von Neumann algeras

In this final chapter, we discuss a very important class of C*-algebras, namely, the von Neumann algebras. We first recall that the algebra $B(H)$ of bounded operators on a Hilbert space H has a predual which is the Banach space $Tr(H)$ of trace-class operators.

A positive operator $T \in B(H)$ is of *trace class* if

$$\text{trace}\,(T) = \sum_\alpha \langle T\xi_\alpha, \xi_\alpha\rangle < \infty$$

for some orthonormal basis (ξ_α) of H, where the sum does not depend on the choice of (ξ_α). In fact,

$$\sum_\alpha \langle T\xi_\alpha, \xi_\alpha\rangle = \sum_\alpha \|T^{1/2}\xi_\alpha\|^2 = \sum_\beta \sum_\alpha |\langle T^{1/2}\xi_\alpha, \eta_\beta\rangle|^2 = \sum_\beta \langle T\eta_\beta, \eta_\beta\rangle$$

for any orthonormal basis (η_β) of H. An operator $T \in B(H)$ is of *trace class* if $|T|$ is of trace class as defined above. It is well-known that $Tr(H)$ is a norm-closed ideal in $B(H)$ although it is not closed in the weak operator topology unless $\dim H < \infty$.

We can identify $B(H)$ with the dual $Tr(H)^*$ via the linear isometry

$$T \in B(H) \mapsto \psi_T \in Tr(H)^*, \quad \psi_T(S) = \text{trace}\,(TS) \quad (S \in Tr(H)).$$

In this duality, the weak* topology of $Tr(H)^* = B(H)$ coincides with the weak operator topology of $B(H)$, and a subspace X of $B(H) = Tr(H)^*$ is weak* closed if, and only if, it is the dual of a quotient space of $Tr(H)$, namely, $Tr(H)/X^0$ where $X^0 = \{S \in Tr(H) : \psi(S) = 0, \forall\, \psi \in Tr(H)^*\}$.

Definition 7.1. Let H be a Hilbert space. A unital C*-subalgebra $\mathcal{A}$ of the algebra $B(H)$ of bounded operators on H is called a *von Neumann algebra* (or *W^*-algebra*) (*acting on* H) if it is closed in the weak operator topology of $B(H)$.

We often omit mentioning the underlying Hilbert space H for a von Neumann algebra $\mathcal{A} \subset B(H)$ if it is understood. A unital C*-algebra $\mathcal{A}$ is also called a von Neumann algebra if it admits a faithful representation $\pi : \mathcal{A} \longrightarrow B(H)$ such that $\pi(\mathcal{A})$ is a von Neumann algebra.

By Corollary **6.6**, a von Neumann algebra contains the range projections of its positive elements. Although von Neumann algebras are C*-algebras, they have a distinctive feature characterised by the existence of sufficiently many projections determining their intrinsic structures. For this reason, the study of projections is central in the theory of von Neumann algebras. On the other hand, C*-algebras do not always have non-trivial projections, for instance, the algebra $C[0, 1]$ of continuous functions on $[0, 1]$, but the representation theory plays an important role in C*-algebras.

Given a self-adjoint operator $T \in B(H)$, we have $|T| = (T^2)^{1/2}$ by functional calculus in Chapter 3. For any $T \in B(H)$, we define

$$|T| = (T^*T)^{1/2}.$$

The spectrum $\sigma(|T|)$ of $|T|$ is contained in $[0,\infty)$ because

$$\sigma(|T|) = \sigma(((T^*T)^{1/2})^2) = \{\alpha^2 : \alpha \in \sigma((T^*T)^{1/2})\}.$$

By a remark following Definition **6.7**, we have $[T^*] = [|T|\,]$.

Lemma 7.2. *Let $T \in B(H)$ and let*

$$T_n = \left(\frac{\mathbf{1}}{n} + |T|\right)^{-1} |T| \qquad (n = 1, 2, \ldots).$$

Then $T_n \leq \mathbf{1}$ and the sequence (T_n) is increasing.

Proof. This follows from functional calculus by considering the C*-subalgebra of $B(H)$ generated by T^*T and $\mathbf{1}$. □

By Proposition **6.5**, the sequence (T_n) above converges to some $S \in B(H)$ in the strong operator topology. It follows that the sequence $(T_n - T_m)^2$ converges to 0 in the strong, and hence, weak operator topology since for each $x \in H$, we have

$$\|(T_n - T_m)^2 x\| \leq \|T_n - T_m\| \|(T_n - T_m)x\| \leq 2\|(T_n - T_m)x\| \longrightarrow 0 \quad \text{as} \quad n, m \to \infty.$$

We now show that every element T in a von Neumann algebra admits a *polar decomposition* analogous to the polar decomposition of a complex number $z = e^{i\theta}|z|$.

Proposition 7.3. *Let $\mathcal{M} \subset B(H)$ be a von Neumann algebra and let $T \in \mathcal{M}$. Then $T = U|T|$ for some $U \in \mathcal{M}$ such that U^*U is the range projection of $|T|$.*

Proof. We have $|T| \in \mathcal{M}$ and $[|T|\,] \in \mathcal{M}$. For $n = 1, 2, \ldots$, let

$$U_n = T\left(\frac{\mathbf{1}}{n} + |T|\right)^{-1} \in \mathcal{M}.$$

Since the range projection $[|T|\,]$ commutes with $(\frac{\mathbf{1}}{n} + |T|)^{-1}$ and $T = T[|T|\,]$, we have $U_n = U_n[|T|\,]$ and

$$\begin{aligned}
&(U_n - U_m)^*(U_n - U_m)\\
&= \left(\left(\frac{\mathbf{1}}{n} + |T|\right)^{-1} - \left(\frac{\mathbf{1}}{m} + |T|\right)^{-1}\right) T^*T \left(\left(\frac{\mathbf{1}}{n} + |T|\right)^{-1} - \left(\frac{\mathbf{1}}{m} + |T|\right)^{-1}\right)\\
&= \left(\left(\frac{\mathbf{1}}{n} + |T|\right)^{-1} |T| - \left(\frac{\mathbf{1}}{m} + |T|\right)^{-1} |T|\right)\left(|T|\left(\frac{\mathbf{1}}{n} + |T|\right)^{-1} - |T|\left(\frac{\mathbf{1}}{m} + |T|\right)^{-1}\right)\\
&= (T_n - T_m)^2
\end{aligned}$$

with T_n defined in Lemma **7.2**. Hence

$$\|(U_n - U_m)x\|^2 = \langle (T_n - T_m)^2 x, x\rangle \longrightarrow 0 \quad \text{as} \quad n, m \to \infty.$$

It follows that the sequence (U_n) converges to some $U \in \mathcal{M}$ in the strong operator topology and $U = U[|T|\,]$.

Since $U_n|T| = T\left(\frac{\mathbf{1}}{n} + |T|\right)^{-1}|T|$ converges to T in the strong operator topology, we have $T = U|T|$.

Finally, we have $U^*U = |T|U^*U|T|$ which gives $(|T|U^*U - |T|)|T| = 0$ and hence $(|T|U^*U - |T|)[|T|\,] = 0$. It follows that $|T|U^*U[|T|\,] - |T| = 0$ and $[|T|\,]U^*U|T| - |T| = 0$. Therefore $([|T|\,]U^*U - \mathbf{1})[|T|\,] = 0$ and $[|T|\,]U^*U[|T|\,] = [|T|\,]$, giving $U^*U = [|T|\,]$ since $U = U[|T|\,]$. □

Definition 7.4. An element u in a C*-algebra $\mathcal{A}$ is called a *partial isometry* if u^*u is a projection.

A partial isometry $u \in \mathcal{A} \subset B(H)$ has the polar decomposition

$$u = uu^*u$$

since $|u| = u^*u$. The self-adjoint element uu^* is also a projection since $\sigma(uu^*) \cup \{0\} = \sigma(u^*u) \cup \{0\} = \{1, 0\}$. We call u^*u the *initial projection* of u, and uu^* the *final projection*.

Two projections $p, q \in \mathcal{M}$ are said to be *equivalent*, in symbol $p \sim q$, if there is a partial isometry $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$. If $p \leq q$, we say that q *contains* p. If $p \sim z \leq q$ for some subprojection z of q, we write $p \preceq q$.

We now classify von Neumann algebras using projections. As usual, the centre $\mathcal{Z}$ of a C*-algebra $\mathcal{A}$ is the subalgebra of $\mathcal{A}$ consisting of elements which commute with every element in $\mathcal{A}$. Given a von Neumann algebra $\mathcal{M} \subset B(H)$, we define its *commutant* $\mathcal{M}'$ by

$$\mathcal{M}' = \{T \in B(H) : TS = ST\, \forall S \in \mathcal{M}\}.$$

The centre of $\mathcal{M}$ is $\mathcal{M} \cap \mathcal{M}'$. If the centre of $\mathcal{M}$ is trivial, that is, if the centre consists of only scalar multiples of the identity, then $\mathcal{M}$ is called a *factor*.

A projection in the centre of $\mathcal{M}$ is called a *central projection*. The identity is the only nonzero central projection in $B(H)$.

Given a projection p in a von Neumann algebra $\mathcal{M}$, it is evident that the reduced algebra $p\mathcal{M}p$ is also a von Neumann algebra.

Definition 7.5. A projection p in a von Neumann algebra $\mathcal{M}$ is called *abelian* if the algebra $p\mathcal{M}p$ is commutative.

Definition 7.6. A von Neumann algebra $\mathcal{M}$ is said to be of *type I* if every nonzero central projection $p \in \mathcal{M}$ contains a nonzero abelian projection.

Evidently every abelian von Neumann algebra is of type I. Also, $B(H)$ is of type I since every projection $p \in B(H)$ with $\dim p(H) = 1$ is abelian as $pB(H)p = \mathbb{C}p$.

Example 7.7. Let $\mathcal{M}$ be a von Neumann algebra. A nonzero projection $p \in \mathcal{M}$ is called *minimal* if $p\mathcal{M}p = \mathbb{C}p$. Trivially minimal projections are abelian. If $\mathcal{M}$ is a factor and contains a minimal projection, it must be of type I.

Let $\dim \mathcal{M} < \infty$. We may assume $\mathcal{M} \subset B(H)$ with $\dim H < \infty$ by the universal representation of $\mathcal{M}$. Then $\dim p(H) < \infty$ for every $p \in \mathcal{M}$ and, by a simple dimension argument, $\mathcal{M}$ contains a nonzero projection p such that $0 \neq q \leq p \Rightarrow q = p$ for any projection $q \in \mathcal{M}$. We must have $p\mathcal{M}p = \mathbb{C}p$, that is, p is a minimal projection in $\mathcal{M}$. Indeed, the condition on p implies that every nonzero element in $p\mathcal{M}p$ has range projection p. If $T \in p\mathcal{M}p$ and if $\alpha p - T \neq 0$ for some $\alpha \in \mathbb{C}\backslash\{0\}$, then $(\alpha p - T)^{-1}(0) = p^{-1}(0)$ and hence α is not an eigenvalue of T since $(\alpha\mathbf{1} - T)(x) = 0$ for $x \in H$ implies $(\alpha p - T)(x) = 0$, giving $p(x) = 0$ and

$$\alpha(\mathbf{1} - p)(x) = (\alpha - T)(\mathbf{1} - p)(x) = 0.$$

Hence, if T has a nonzero eigenvalue α, then $T = \alpha p$.

It follows that $\mathcal{M}$ is of type I since $z\mathcal{M}z$ is finite dimensional and contains a minimal projection for every central projection $z \in \mathcal{M}$.

Definition 7.8. A projection p in a von Neumann algebra $\mathcal{M}$ is called *finite* if $p = q$ for any projection q satisfying $p \sim q \leq p$.

In other words, a finite projection is one which is not equivalent to any of its proper subprojection, in analogy to the concept of a finite set. A von Neumann algebra $\mathcal{M}$ is called *finite* if the identity $\mathbf{1}$ is a finite projection.

A finite type I von Neumann algebra is said to be of *type* I_f.

Proposition 7.9. *In a von Neumann algebra* $\mathcal{M}$*, every abelian projection is finite.*

Proof. Let p be abelian and let $p \sim q \leq p$. Then there is a partial isometry $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$. We have $pvp = pvv^*v = pqv = qv = vv^*v = v$ and hence v is in the abelian algebra $p\mathcal{M}p$. Therefore $p = vv^* = q$. □

Example 7.10. On any Hilbert space H, the *rank* of an operator $T \in B(H)$ is defined to be the dimension $\dim T(H)$ of its range. Two projections p and q are equivalent in $B(H)$ if, and only if, they have the same rank, in which case the partial isometry implementing the equivalence is the natural extension of the isometry between $p(H)$ and $q(H)$. It follows that the finite projections in $B(H)$ are exactly the finite rank projections. In particular, every finite dimensional von Neumann algebra is of type I_f.

Example 7.11. In contrast to the case of the full algebra $B(H)$, a finite projection p in a von Neumann algebra $\mathcal{M}$ need not have finite rank. Let $\ell_2(\mathbb{N})$ be the Hilbert space of square-summable sequences. An operator $T \in B(\ell_2(\mathbb{N}))$ can be represented as an infinite matrix (a_{ij}) with

$$a_{ij} = \langle Te_j, e_i \rangle$$

where $\{e_1, e_2, \ldots\}$ is the standard basis in $\ell_2(\mathbb{N})$, namely, e_i is the sequence whose terms are 0 except the i-th term which is 1.

Let $\mathcal{M} \subset B(\ell_2(\mathbb{N}))$ be the abelian von Neumann subalgebra consisting of the diagonal matrices. The $\mathcal{M}$ contains the projection

$$p = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} : (x_1, x_2, x_3, \cdots) \in \ell_2(\mathbb{N}) \mapsto (0, x_2, x_3, \cdots) \in \ell_2(\mathbb{N})$$

which is a finite projection in $\mathcal{M}$, but has infinite rank and is not a finite projection in $B(\ell_2(\mathbb{N}))$.

Definition 7.12. A von Neumann algebra $\mathcal{M}$ is said to be of *type II* if it has no nonzero abelian projection and every nonzero central projection in $\mathcal{M}$ contains a nonzero finite projection. A finite type II von Neumann algebra is said to be of *type* II_1.

Definition 7.13. A von Neumann algebra $\mathcal{M}$ is said to be of *type III* if it contains no nonzero finite projection.

Definition 7.14. A von Neumann algebra is said to be *properly infinite* if it contains no nonzero finite central projection.

Definition 7.15. A properly infinite type I von Neumann algebra is said to be of type I_∞. A properly infinite type II von Neumann algebra is said to be of *type* II_∞.

Theorem 7.16. *A von Neumann algebra $\mathcal{M}$ decomposes uniquely into five direct summands:*

$$\mathcal{M} = \bigoplus_j \mathcal{M}_j$$

where $\mathcal{M}_j$ is either $\{0\}$ or of type j, for $j = \mathrm{I}_f, \mathrm{I}_\infty, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$.

A factor has one and only one of the above types.

We omit the proof of the above theorem which can be found in books on operator algebras, for instance, [6, p.422], [8, p.174], [10, p.86], [11, p.296] and [12, p.25]. We have to leave out, due to limited time and scope, the discussion of two celebrated density theorems in operator algebras, namely, *Kaplansky's density theorem* and *von Neumann's double commutant theorem* which can also be found in the books mentioned above.

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Sample Examination Question

Question (a) Let $\mathcal{A}$ be an abelian C*-algebra. Explain, with sufficient details, how the Gelfand transform identifies $\mathcal{A}$ with an algebra of continuous functions on a locally compact Hausdorff space.

Infer from this fact that $\mathbf{1} + a^*a$ is always invertible for each element a in *any* C*-algebra with identity $\mathbf{1}$.

(b) Let p be an abelian projection in a von Neumann algebra M, and let q be a projection in M equivalent to p. Show that q is abelian.