Problem 1 Solutions

1) This is a long computation - see klein.py which is attached, but if one runs the computation we see that

$$\mathrm{rk}(H_1(X;\mathbb{Z}_2)) = 2 \quad \& \quad \mathrm{rk}(H_1(X;\mathbb{Z}_3)) = 1$$

This is because the Klein bottle has torsion, so it depends which field we use. An open question is: can torsion appear in natural data.

2) The general formula for the expected Euler characteristic is

$$\mathbb{E}(\chi) = \mathbb{E}\left(\sum_{i=0}^{d} (-1)^i (\# \text{ of } i\text{-simplices})\right)$$

$$= \sum_{i=0}^{d} (-1)^i \,\mathbb{E}(\# \text{ of } i\text{-simplices})$$

(1) In the Linial-Meshulam model (LM), the expected number of simplices is deterministic for $i<k$ & is the number of k-simplices times the probability p. With n-vertices there are $\binom{n}{i+1}$ i-simplices So

$$\mathbb{E}(\chi) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i+1} + p \binom{n}{k+1}$$

(2) The situation is slightly more complicated here

- there are n vertices 0-simplices
- p times $\frac{n(n-1)}{2}$ edges
- a k-simplex corresponds to a $k+1$ clique which has $\frac{k(k+1)}{2}$ edges, all of which must be present. So the probabily a k simplex is included is

$$\mathbb{P}(k\text{-simplex}) = p^{\frac{k(k+1)}{2}}$$

Hence

$$\mathbb{E}(\chi) = n - \frac{n(n-1)}{2} \cdot p + \sum_{i=2}^{n-1} (-1)^i \underbrace{\binom{n}{i+1}}_{\#\text{ of } k\text{-simplices}} p^{\overbrace{\frac{i(i+1)}{2}}^{\text{probability}}}$$

3) The computation can be big but we can do it dimension by dimension

	bc	ae	bf	ce	ef	be	bd	df	de
b	1		1			1	1		
c	1			1					
e		1		1	1	1			1
a		1							
f			1		1			1	
d							1	1	1

		bc	ae (2)	bf (3)	ce	ef	be	bd (4)	df	de (5)
						bf+ce+bc	ce+bc		bd+bf	de → bd+ce+bc
	b	1		1		~~1~~	~~1~~	1	~~1~~	~~1~~
1	c	$\boxed{1}$			1	~~1~~	~~1~~			~~1~~
	e		1		$\boxed{1}$	~~1~~	~~1~~			~~1~~
	a		$\boxed{1}$							
2	f			$\boxed{1}$		~~1~~			~~1~~	
	d							$\boxed{1}$	~~1~~	~~1~~

The barcode for H_0 is $[1,\infty]$, $[1,2]$, $[2,2]$, $[2,3]$, $[1,3]$, $[2,4]$

The 1 dimensional cycles are

$ef + bf + ce + bc$ $\quad$ $df + bd + bf$

$be + ce + bc$ $\quad$ $de + bd + ce + bc$

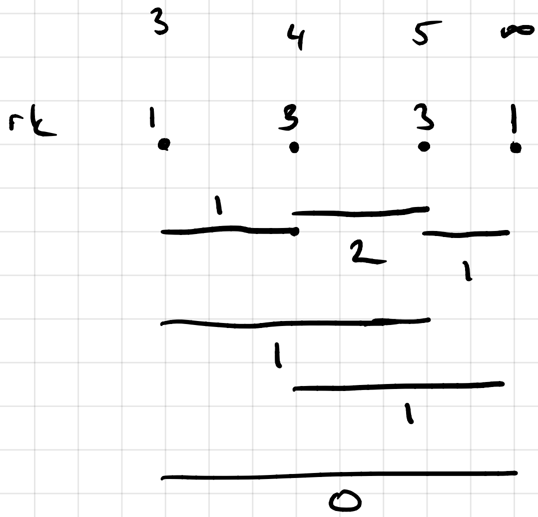
We can just consider the leading terms for computing H_1

		4		5
		cbe	bed	bef
3	ef			$\boxed{1}$
4	be	$\boxed{1}$	1	$\cancel{1}$
	df			
5	de		$\boxed{1}$	

So the H_1 barcode is

$[4,4]$
$[4,\infty]$
$[5,5]$
$[3,5]$

b) Checking for H_1, here we are assuming closed intervals



Checking inclusion-exclusion we see that we get the same answer.