LTCC: Stochastic Processes

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Outline

The Poisson process with rate λ

Let $\{N(t) : t \geq 0\}$ be a Markov process, where $N(t)$ represents the number of events (or births) in the interval $(0, t]$, for example claim times to an insurance company, arrival times of customers in a shop,...

Let the transition probabilities of $\{N(t)\}$ be:

- the probability of one occurrence in the time interval $(t, t+h)$ is $\lambda h + o(h)$:

$$p_{i\,i+1}(h) = \lambda h + o(h)$$

- the probability of more than one occurrence is $o(h)$:

$$p_{i\,j}(h) = o(h), \text{for } j \neq i, i+1$$

- This implies that the probability of no events in $(t, h+t)$ is $1 - \lambda h + o(h)$:

$$p_{i\,i}(h) = 1 - \lambda h + o(h)$$

- the number of occurrences in $(t, t+h)$ is independent of the occurrences in $(0, t]$.

The state space is $\mathcal{S} = \{0, 1, 2, \ldots\}$. Then

$$\begin{aligned}
P(N(t+h) = k \mid N(t) = i) &= \lambda h + o(h), \quad k = i+1 \\
P(N(t+h) = k \mid N(t) = i) &= o(h), \quad k > i+1 \\
P(N(t+h) = k \mid N(t) = i) &= 0, \quad k < i
\end{aligned}$$

The Poisson process has generator

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \ldots \\ 0 & -\lambda & \lambda & 0 & \ldots \\ 0 & 0 & -\lambda & \lambda & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

- A Poisson process $N(t)$ describes the number of occurrences of some event up to time t, i.e. in the interval $(0, t]$.
- We assume $N(0) = 0$, so that the process stays in state 0 for a random time τ_1, before jumping to state 1 and staying there a random time τ_2, then jumping to state 2, *etc.*
- The intervals $\tau_1, \tau_1, \ldots$ are independent, identically distributed (*i.i.d.*) exponential variables with common parameter λ.

Corollary 1 (holding times)

Poisson process, rate λ has holding times $\overset{iid}{\sim} Expon(\lambda)$

Proof From Theorem on Holding Times , $T_i \sim \text{Expon}(-q_{ii})$. Want $\lambda = -q_{ii}$.

- This property provides an alternative definition of the Poisson process as a sequence of point events where the intervals between successive events are i.i.d. exponential variables.
- It follows that the time $T_r = \sum_{i=1}^{r} \tau_i$ to the rth event has a gamma distribution, $\mathrm{Gamma}(r, \lambda)$.
- By the Central Limit Theorem, for large r, this can be approximated by the normal distribution $\mathrm{N}(r/\lambda, r/\lambda^2)$.

- The Poisson process has the property of *independent increments*, that numbers of events in non-overlapping intervals (and, more generally, disjoint sets) are independent.
- It follows that the interval from an arbitrary time point until the next event also has an exponential distribution with parameter λ.
- The chain is not irreducible, since, e.g., $p_{ij}(t) \equiv 0$ if $j < i$.
- Each state is in a separate transient class.
- There is no invariant distribution; the only solution of $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ is $\pi_0 = \pi_1 = \ldots = 0$.

Recall Forward equations: $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

We are interested in $p_{0j}(t) = \mathrm{P}(j \text{ events in } (0, t] \mid N(0) = 0)$.
The forward equations are

$$p'_{00}(t) = -\lambda p_{00}(t), \;\; p'_{0j}(t) = \lambda p_{0,j-1}(t) - \lambda p_{0j}(t) \text{ for } j = 1, 2, \ldots$$

which are most easily solved using generating functions.

Let

$$G(s,t) = \mathrm{E}(s^{N(t)} \mid N(0)=0) = \sum_{j=0} s^j p_{0j}(t)$$

be the probability generating function of $N(t)$ ($|s| \le 1$). Then

$$\begin{aligned}
\frac{\partial G(s,t)}{\partial t} &= \frac{\partial}{\partial t}\sum_{j=0}^{\infty} s^j p_{0j}(t) \\
&= \sum_{j=0}^{\infty} p'_{0j}(t) s^j \\
&= -\lambda p_{00}(t) + \lambda \sum_{j=1}^{\infty} p_{0,j-1}(t) s^j - \lambda \sum_{j=1}^{\infty} p_{0j}(t) s^j \\
&= -\lambda G(s,t) + \lambda s G(s,t) \\
&= -\lambda(1-s)G(s,t).
\end{aligned}$$

The initial condition is $G(s,0) \equiv 1$, so the solution of this equation is $G(s,t) = e^{-\lambda(1-s)t}$, which is the probability generating function of a Poisson variable with mean λt.

It follows that

$$p_{0j}(t) = e^{-\lambda t}(\lambda t)^j/j! \text{ for } j = 0, 1, 2, \ldots.$$

This result provides another means of defining the Poisson process, as a point process for which,

- for all t and τ, $N(t)$ has a Poisson distribution with mean λt,

Corollary 2 ($\mathbb{E}\, N(t) = \operatorname{var} N(t) = \lambda t$)

- the 'rate' at which events occur is $\dfrac{\mathrm{d}\,\mathbb{E}N(t)}{\mathrm{d}t} = \lambda$
- and the numbers of events in $(0, t]$ and $(t, t+\tau]$ are independent variables.

There are many further characterisations of the Poisson process.

An important property of the Poisson process is that, given the number of events in an interval, these events are independently and uniformly distributed over the interval.

Theorem 3

$\{N(t)\}$ *Poisson. Given* $N(t) = n$*, the arrival times* $\Theta_1 < \ldots < \Theta_n \overset{iid}{\sim} Unif(0, t)$.

Sketch Proof The joint density of the arrival times $\Theta_{1:n} := \Theta_1, \dots, \Theta_n$ with n arrivals at t can be written in terms of the (iid Expon) inter-arrival (holding) times $T_{0:n} := T_0, \dots, T_n$, namely (see figure on board):

$$\begin{aligned} \{\Theta_{1:n} = \theta_{1:n}, N(t) = n\} &= \{\Theta_1 = \theta_1, \dots, \Theta_n = \theta_n, N(t) = n\} \\ &= \{T_0 = \theta_1, T_1 = \theta_2 - \theta_1, \dots, T_{n-1} = \theta_n - \theta_{n-1}, T_n > t - \theta_n\}. \end{aligned}$$

Now $f(\Theta_{1:n} = \theta_{1:n} | N(t) = n)$

$$\begin{aligned} &= \frac{f(\Theta_{1:n} = \theta_{1:n}, N(t) = n)}{\mathbb{P}(N(t) = n)} \\ &= \frac{f(T_0 = \theta_1, T_1 = \theta_2 - \theta_1, \dots, T_{n-1} = \theta_n - \theta_{n-1}, T_n > t - \theta_n)}{\mathbb{P}(N(t) = n)} \\ &= \underbrace{\frac{n!}{(\lambda t)^n} e^{\lambda t}}_{1/\mathbb{P}(N(t)=n)} \underbrace{e^{-\lambda(t-\theta_n)}}_{f(T_n > t-\theta_n)} \prod_{i=0}^{n-1} \underbrace{\lambda e^{-\lambda(\theta_{i+1}-\theta_i)}}_{f(T_i = \theta_{i+1}-\theta_i)\text{ ,with } \theta_0 := 0} = \underbrace{\frac{n!}{t^n}}_{\text{joint pdf of ordered unif r.v.s}} \end{aligned}$$

which is the joint density of the order statistics of n independent variables, each uniformly distributed on $(0, t)$.

- We have reviewed homogeneous Poisson Processes where the intensity, λ, is constant across time.
- In practice we often meet inhomogeneous Poisson Processes where the intensity, $\lambda(t)$, varies with time. This may be due to covariates which vary with time.
- In that case, the number of events that occur in $(t_1, t_2]$ will have a Poisson distribution with mean $\Lambda(t_2) - \Lambda(t_1)$ where

$$\Lambda(t_i) = \int_0^{t_i} \lambda(t) \mathrm{d}t$$

Birth Process

A birth process with intensities $\lambda_0, \lambda_1, \ldots$ is a process $\{N(t), t \geq 0\}$ taking values in $\mathcal{S} = \{0, 1, 2, \ldots\}$ such that

- $N(0) = 0$; if $s < t$ then $N(s) \leq N(t)$
- $$\mathbf{P}\big(N(t+h) = n + m \mid N(t) = n\big) = \begin{cases} \lambda_n h + \mathrm{o}(h) & \text{if } m = 1 \\ \mathrm{o}(h) & \text{if } m > 1 \\ 1 - \lambda_n h + \mathrm{o}(h) & \text{if } m = 0 \end{cases}$$
- if $s < t$ then $N(t) - N(s)$ is independent of all arrivals prior to s.

From the definition of the process it is clear that :

$$q_{ii} = -\lambda_i, \quad q_{i,i+1} = \lambda_i, \quad q_{ij} = 0 \text{ if } j < i \text{ or } j > i$$

The generator is then given

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0\cdots & \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0\cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Special Cases

Poisson Process: $\lambda_n = \lambda$ for all n

Simple birth: $\lambda_n = n\lambda$. This models the growth of a population in which each living individual may give birth to a new individual with probability $\lambda h + o(h)$ in the interval $(t, t+h)$. No individuals may die.
The number M of births in the interval $(t, t+h)$ satisfies

$$\begin{aligned} \mathbf{P}(M = m \mid N(t) = n) &= \binom{n}{m} (\lambda h)^m (1-\lambda h)^{n-m} + o(h) \\ &= \begin{cases} 1 - nh\lambda + o(h) & \text{if } m = 0 \\ nh\lambda + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \end{cases} \end{aligned}$$

Simple birth with immigration: $\lambda_n = n\lambda + \nu$. This models a simple birth process which experiences immigration at constant rate ν from elsewhere.

Consider a stochastic process $\{X(t)\}$ ($-\infty < t < \infty$ or $t = \ldots, -2, -1, 0, 1, 2, \ldots$). The process $X(t)$ is *reversible* if its stochastic behaviour remains the same when the direction of time is reversed.

Definition 4

A Markov chain is reversible if

$$(X(t_1), \ldots, X(t_n)) \stackrel{d}{=} (X(\tau - t_1), \ldots, X(\tau - t_n)) \qquad \forall\, t_1, \ldots, t_n, \tau, n$$

Setting τ equal to 0, it follows that $(X(t_1), ..., X(t_n))$ and $(X(-t_1), ..., X(-t_n))$ have the same joint distribution.

Definition 5

$\{X(t)\}$ *is stationary if*

$$(X(t_1), \dots, X(t_n)) \stackrel{d}{=} (X(t_1 + \tau), \dots, X(t_n + \tau)) \qquad \forall\, t_1, \dots, t_n, \tau, n$$

Theorem 6

A reversible process is stationary

Proof

$$\begin{aligned}
(X(t_1), \dots, X(t_n)) &\stackrel{d}{=} (X(\tau - t_1), \dots, X(\tau - t_n)) && \text{reversib.} \\
&\quad \text{Set } \tau = 0 \\
&\stackrel{d}{=} (X(-t_1), \dots, X(-t_n)) \\
&\quad \text{Replace } t_i \text{ by } t_i + \tau \text{ in the def.} \\
(X(t_1 + \tau), \dots, X(t_n + \tau)) &\stackrel{d}{=} X(-t_1), \dots, X(-t_n) \\
&\Downarrow \qquad \text{putting 2 eq together} \\
(X(t_1), \dots, X(t_n)) &\stackrel{d}{=} (X(t_1 + \tau), \dots, X(t_n + \tau)) \quad \blacksquare
\end{aligned}$$

Note that if a Markov chain is reversible it must be stationary.
In other words, the chain must be *in equilibrium*, i.e. not only must an equilibrium distribution $\boldsymbol{\pi}$ exist, but the marginal distribution of X_n must be given by $\boldsymbol{\pi}$ for all n.

Remark 7

X *stationary* $\Rightarrow \exists$ *equilibrium distribution* $\boldsymbol{\pi}$, *i.e.* $\mathbb{P}\big(X(t_n) = j\big) = \pi_j \quad \forall\, n$

Discrete-time Markov Chain

Theorem 8

Let $\{X_n : n = \dots, -2, -1-, 0, 1, 2, \dots\}$ *be irreducible, stationary, discrete time Markov chain with transitions* $\mathbf{P} = (p_{ij})_{ij \in \mathcal{S}}$ *and unique equilibrium distribution* $\boldsymbol{\pi}$*. Then the time reversed process, defined as* $\{X_n^* := X_{-n}\}$ *is also a stationary, irreducible, Markov chain with unique equilibrium distribution* $\boldsymbol{\pi}$ *and transition matrix* $\mathbf{P}^* = (p_{ij}^*)_{ij \in \mathcal{S}}$ *given by*

$$p_{ij}^* = \frac{\pi_j p_{ji}}{\pi_i}$$

Sketch of Proof – Plan:

- X^* Markov
- $p_{ij}^* = \pi_j p_{ji} / \pi_i$
- X^* stationary
- $\boldsymbol{\pi} \mathbf{P}^* = \boldsymbol{\pi}$

Simplify proof: fix a large time $N > 0$ and let $\{X_{0:N}\}$ be irreducible, stationary, discrete time Markov chain and define the time reversed process as $\{X_n^* := X_{N-n}\}_{n=0}^{N}$

Then for $n < N$

$$
\begin{aligned}
& \mathbb{P}\left(X^*_{n+1} = j \middle| X^*_{0:n}\right) \\
= \; & \mathbb{P}\left(X_{N-(n+1)} = j \middle| X_{N-n:N}\right) \\
= \; & \frac{\mathbb{P}\left(X_{N-(n+1):N}\right)}{\mathbb{P}\left(X_{N-n:N}\right)} \\
= \; & \frac{\mathbb{P}\left(X_{N-(n+1)}\right)\mathbb{P}\left(X_{N-n} \middle| X_{N-(n+1)}\right) \prod_{k=0}^{n} \mathbb{P}\left(X_{N-(n-k-1)} \middle| X_{N-(n-k)}\right)}{\mathbb{P}\left(X_{N-n}\right) \prod_{k=0}^{n} \mathbb{P}\left(X_{N-(n-k-1)} \middle| X_{N-(n-k)}\right)} \\
= \; & \frac{\mathbb{P}\left(X_{N-(n+1)} = j\right)\mathbb{P}\left(X_{N-n} = i \middle| X_{N-(n+1)} = j\right)}{\mathbb{P}\left(X_{N-n} = i\right)} \quad [= \pi_j p_{ji} / \pi_i] \\
= \; & \frac{\mathbb{P}\left(X_{N-n} = i, X_{N-(n+1)} = j\right)}{\mathbb{P}\left(X_{N-n} = i\right)} = \frac{\mathbb{P}\left(X^*_n = i, X^*_{n+1} = j\right)}{\mathbb{P}\left(X^*_n = i\right)} \\
= \; & \mathbb{P}\left(X^*_{n+1} = j \middle| X^*_n = i\right) = p^*_{ij}
\end{aligned}
$$

Hence X^* Markov.

The transition matrix $\mathbf{P}^* = (p^*_{ij})$ is given by

$$p^*_{ij} = \frac{\mathrm{P}(X^*_n = i, X^*_{n+1} = j)}{\mathrm{P}(X^*_n = i)} = \frac{\mathrm{P}(X_{-n} = i, X_{-n-1} = j)}{\pi_i} = \frac{\pi_j p_{ji}}{\pi_i}.$$

We have $p^*_{ij} = \pi_j p_{ji}/\pi_i$. Sum both sides over i to get:

$$\begin{aligned} \sum_{i\in\mathcal{S}} \pi_i p^*_{ij} &= \pi_j \sum_{i\in\mathcal{S}} p_{ji} \\ &= \pi_j \\ \Rightarrow \boldsymbol{\pi}\mathbf{P}^* &= \boldsymbol{\pi} \end{aligned}$$

i.e., $\boldsymbol{\pi}$ is the equilibrium distribution of X^*. Also,

$$\mathbb{P}(X^*_n = j) = \mathbb{P}(X_{-n} = j) = \pi_j \qquad \text{for all } n, j = \pi_j$$

Hence, X^* is stationary. ■

Corollary 9

If X is reversible then $\mathbf{P}^* = \mathbf{P}$.

Proof X reversible $\Rightarrow (X_0, X_1) \stackrel{d}{=} (X_1, X_0)$, i.e.

$$\begin{aligned} \mathbb{P}(X_n = i, X_{n+1} = j) &= \mathbb{P}(X_{n+1} = i, X_n = j) \\ \pi_i p_{ij} &= \pi_j p_{ji} \quad \forall\, i, j \end{aligned}$$

Now, from Theorem 8

$$p^*_{ij} = \pi_j p_{ji}/\pi_i = \pi_i p_{ij}/\pi_i = p_{ij} \qquad (**)$$

so that $\mathbf{P}^* = \mathbf{P}$ and the chain has an identical probabilistic structure in both forward and reversed time. ■

The equations (**) are called the *detailed balance equations*; in words, they say that, for all pairs of states, the rate at which transitions occur from state i to state j ($\pi_i\, p_{ij}$) balances the rate at which transitions occur from j to i ($\pi_j\, p_{ji}$). The converse of this result also holds.

Recall that reversibility $\Rightarrow$ stationarity. The reverse comes with extra conditions attached.

Theorem 10

An irreducible, stationary Markov chain is reversible iff $\exists$ probability row vector $\boldsymbol{\pi}$ s.t. the following 'detailed balance equations' hold

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall\, i, j \in \mathcal{S}$$

Furthermore, if $\boldsymbol{\pi}$ exists then it is an equilibrium distribution.

Proof (partial) ($\Longleftarrow$) If $\boldsymbol{\pi}$ (above) exists then

$$\begin{aligned} \pi_i p_{ij} &= \pi_j p_{ji} \\ \pi_i \sum_{j \in \mathcal{S}} p_{ij} &= \sum_{j \in \mathcal{S}} \pi_j p_{ji} \\ \pi_i &= \sum_{j \in \mathcal{S}} \pi_j p_{ji} \\ \boldsymbol{\pi} &= \boldsymbol{\pi} \mathbf{P} \end{aligned}$$

I.e. if $\boldsymbol{\pi}$ exists it is an equilibrium distribution.

($\Longrightarrow$) Now show detailed balance equations imply irreducible, stationary X is reversible. Since X is Markov we have

$$\mathbb{P}(X_n = j_0, \ldots, X_{n+N} = j_N) = \pi_{j_0} \prod_{n=1}^{N} p_{j_{n-1}, j_n} \tag{1}$$

$$\mathbb{P}(X_m = j_N, \ldots, X_{m+N} = j_0) = \pi_{j_N} \prod_{n=1}^{N} p_{j_n, j_{n-1}} \tag{2}$$

where stationarity has been used above to write $\pi_j = \mathbb{P}(X_n = j)$.
RHS Eq (2) is

$$\begin{aligned}
\pi_{j_N} \prod_{n=1}^{N} p_{j_n, j_{n-1}} &= \pi_{j_N} \prod_{n=1}^{N} \frac{\pi_{j_{n-1}}}{\pi_{j_n}} p_{j_{n-1}, j_n} && \text{[from dbe]} \\
& \qquad \text{cancel out} \\
&= \pi_{j_0} \prod_{n=1}^{N} p_{j_{n-1}, j_n} && \text{[= RHS Eq (1)]}
\end{aligned}$$

I.e. $(X_n, \ldots, X_{n+N}) \stackrel{d}{=} (X_m, \ldots, X_{m+N})$. Now put $m = k - n$ to get $(X_n, \ldots, X_{n+N}) \stackrel{d}{=} (X_{k-n}, \ldots, X_{k-n-N})$. Hence, X is reversible. ■

Remark 11

The detailed balance equations of a (stationary) Makov chain X

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall\, i,j \in \mathcal{S}$$

imply, in particular that

$$\begin{aligned} \mathbb{P}\big(X_0 = i\big)\,\mathbb{P}\big(X_1 = j \big| X_0 = i\big) &= \mathbb{P}\big(X_0 = j\big)\,\mathbb{P}\big(X_1 = i \big| X_0 = j\big) \\ \mathbb{P}\big(X_1 = j, X_0 = i\big) &= \mathbb{P}\big(X_1 = i, X_0 = j\big) \\ \mathbb{P}\big(\textit{observing } i \to j\big) &= \mathbb{P}\big(\textit{observing } j \to i\big) \end{aligned}$$

I.e. probability of observing transition from i *to* j *is the same as observing a transition from* j *to* i*. If, furthermore* $i \leftrightarrow j$ *(chain is irreducible) then time reversed process has the same distribution as the original process.*

The detailed balance equations provide a useful way of finding the equilibrium distributions of reversible chains.

However, in general, if a chain is stationary, solving the detailed balance equations only determines the equilibrium distribution *if a solution exists*.

If a chain is stationary, i.e. has an equilibrium distribution, but there is no solution to the detailed balance equations, then the chain is not reversible.

Example 12 (urn model)

Consider 2 urns. Urn A contains N white balls. Urn B contains N black balls. At each turn (time index $n = 1, 2, \ldots$) a ball is chosen at random from each urn and the two balls are interchanged. Denote the # of black balls in urn A, after nth interchange, by $\{X_n, n \in \mathbb{N}\}$.

X is Markov. $X_0 = 0$ (urn A starts out with 0 black balls.) State space: $\mathcal{S} = \{0, \ldots, N\}$. Transition probabilities:

- one more black ball in A: $q_i* := p_{i,i+1} = \mathbb{P}\big(X_{n+1} = i + 1 \big| X_n = i\big)$
- one less black ball in A: $q_i := p_{i,i-1} = \mathbb{P}\big(X_{n+1} = i - 1 \big| X_n = i\big)$
- no change in # black balls in A: $p_{i,i} = \mathbb{P}\big(X_{n+1} = i \big| X_n = i\big)$
  $= 1 - q_i - q_i^*$

The urn model is irreducible, positive recurrent (and aperiodic), with a unique invariant distribution.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & & & & \\ q_1 & 1-q_1-q_1^* & q_1^* & & & \\ & q_2 & 1-q_2-q_2^* & q_2^* & & \\ & & \ddots & \ddots & \ddots & \\ & & & q_{N-1} & 1-q_{N-1}-q_{N-1}^* & q_{N-1}^* \\ & & & & & 0 \end{bmatrix}$$

Solving (to see what happens!) $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ for $\boldsymbol{\pi} = [\pi_0, \ldots, \pi_{N-1}]$.

$$\cancel{\pi_i} = q_{i-1}^*\pi_{i-1} + (\cancel{1} - q_i^* - q_i)\pi_i + q_{i+1}\pi_{i+1}, \quad i = 1, \ldots, N-1$$

and $\pi_0 = q_1\pi_1$ and $\pi_N = q_{N-1}^*\pi_{N-1}$. We set $q_0^* = 1, q_0 = 0, q_N^* = 0, q_N = 1$.

Now define

$$\begin{aligned} g(i) &:= q_i\pi_i - q_{i-1}^*\pi_{i-1} \\ &= q_{i+1}\pi_{i+1} - q_i^*\pi_i = g(i+1) \qquad i = 1, \ldots, N-1 \end{aligned}$$

Boundary Conditions:

$$g(1) = q_1\pi_1 - q_0^*\pi_0 = q_1\pi_1 - \pi_0 = q_1\pi_1 - q_1\pi_1 = 0$$

Hence $g(1) = 0, g(i+1) = g(i) \Rightarrow g(i) = 0, i = 1, \dots N$ and we have

$$\begin{aligned} q_i \pi_i &= q^*_{i-1} \pi_{i-1} \\ \pi_i &= \frac{q^*_{i-1}}{q_i} \pi_{i-1} \end{aligned}$$

which is the detailed balance eq.

Now recall from Lecture 1 that

$$\begin{cases} q^*_i &= (1 - i/N)^2 \\ q_i &= (i/N)^2 \end{cases}$$

i.e.

$$\begin{aligned} \pi_i &= \frac{\left(1 - \frac{i-1}{N}\right)^2}{(i/N)^2} \pi_{i-1} \\ &= \left(\frac{N - i + 1}{i} \right)^2 \pi_{i-1} \end{aligned}$$

Hence

$$\begin{aligned}
\pi_1 &= N^2\pi_0 \\
\pi_2 &= \left(\frac{N-1}{2}\right)^2\pi_1 = N^2\left(\frac{N-1}{2}\right)^2\pi_0 \\
\pi_3 &= \left(\frac{N(N-1)(N-2)}{3\cdot 2\cdot 1}\right)^2\pi_0 \\
\vdots \quad & \vdots \qquad \vdots \\
\pi_i &= \left(\frac{N!}{(N-i)!\,i!}\right)^2\pi_0 = \binom{N}{i}^2\pi_0
\end{aligned}$$

Summing both sides over i:

$$1 = \sum_{i\in\mathcal{S}}\pi_i = \sum_{i\in\mathcal{S}}\binom{N}{i}^2\pi_0$$

and we have that $\pi_0 = \left(\sum_{i\in\mathcal{S}}\binom{N}{i}^2\right)^{-1}$ which gives us invariant (and equilibrium) distribution of the urn chain (and urn is reversible):

$$\pi_i = \binom{N}{i}^2\left(\sum_{i\in\mathcal{S}}\binom{N}{i}^2\right)^{-1}$$

In summary:

- The detailed balance equations provide a useful way of finding the equilibrium distributions of reversible (and therefore irreducible) chains.
- if the chain is reversible, then the detailed balance equations determine the equilibrium distribution, if it exists;
- if the chain is stationary and the detailed balance equations hold, then these equations determine the equilibrium distribution, if it exists;
- if the chain is stationary and the detailed balance equations do not hold, then the chain is not reversible.

Now, finally back to ctMc. Perhaps unsurprisingly, the detailed balance equations can be written in terms of the generator (transition rate) matrix.

Theorem 13

$\{X(t)\}$ stationary, irreducible, ctMc. Then X is reversible iff $\exists$ probability row vector $\boldsymbol{\pi}$ s.t. the detailed balance equations

$$\pi_i q_{ij} = \pi_j q_{ji} \qquad \forall\ i, j \in \mathcal{S}$$

hold. If $\exists \boldsymbol{\pi}$. Then $\boldsymbol{\pi}$ is equilibrium distribution.

Proof Omitted (see Ross, Section 5.6.1).

We know that, for an irreducible continuous-time Markov chain, if an invariant distribution can be found then it is the equilibrium distribution. Assuming stationarity, if the process is reversible, we can solve the detailed balance equations to find the equilibrium distribution.

If there is no solution of the detailed balance equations then the process is not stationary.

Example 14 (reversible ctMc process)

Consider the birth-death process $\{X(t), t \in \mathbb{R}^+\}$ *with* $\mathcal{S} = \mathbb{N}$.
$X(t)$ *denotes the number of individuals alive at time* t *in some population.*
$X(t)$ *evolves continuously in the following way:*

(a) $\{X(t)\}$ *is a Markov chain taking values in* $S = (0, 1, 2, \ldots)$.

(b) *For small* δ *the transition probabilities are given by:*

$$
\begin{aligned}
\mathbb{P}\big(X(t+\delta) = n+1 \big| X(t) = n\big) &= \lambda_n \delta + o(\delta), \quad n \in \{0, 1, 2, \ldots\} && (3)\\
\mathbb{P}\big(X(t+\delta) = n-1 \big| X(t) = n\big) &= \mu_n \delta + o(\delta), \quad n \in \mathbb{N}_* = \{1, 2, \ldots\} && (4)\\
\mathbb{P}\big(X(t+\delta) = n+m \big| X(t) = n\big) &= o(\delta), \quad n \in \mathbb{N}, |m| > 1 && (5)
\end{aligned}
$$

- Eq. (3): births occur has a rate of λ_n in a population of size n
- Eq. (4): deaths occur has a rate of μ_n in a population of size n
- Eq. (5): prob. more than one birth or death is order δ

We assume $\mu_0 = 0, \lambda_0 > 0$ and $\lambda_n > 0, \mu_n > 0$ for $n > 0$.

The generator is

$$\mathbf{Q} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

All jumps are between adjacent states and all rates of transitions between adjacent states are positive, so the chain is irreducible.

Assuming the process is in equilibrium, the Markov process is reversible and the detailed balance equations hold:

$$\pi_i q_{ij} = \pi_j q_{ji}$$

This implies:

$$\pi_i q_{i,i-1} = \pi_{i-1} q_{i-1,i} \quad \Longrightarrow \quad \pi_i \mu_i = \pi_{i-1} \lambda_{i-1}$$

Thus

$$\begin{aligned}
[i=0, j=1]: \quad \pi_0 \lambda_0 &= \pi_1 \mu_1 \Longrightarrow \pi_1 = \frac{\pi_0 \lambda_0}{\mu_1} \\
[i=1, j=2]: \quad \pi_1 \lambda_1 &= \pi_2 \mu_2 \frac{\pi_1 \lambda_1}{\mu_2} \Longrightarrow \pi_2 = \frac{\pi_0 \lambda_0 \lambda_1}{\mu_1 \mu_2} \\
\vdots \quad & \vdots \quad \vdots \\
\pi_j &= \frac{\lambda_{j-1}}{\mu_j} \pi_j = \pi_0 \prod_{k=1}^{j} \frac{\lambda_{k-1}}{\mu_k} \qquad (6)
\end{aligned}$$

where π_0 is determined by the requirement that the π_js sum to 1. If no such π_0 exists then our assumption is false, i.e. the process cannot be in equilibrium.

The case $\lambda_n = \lambda, \mu_n = \mu$ gives the $M/M/1$ process. From Eq (6)

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \pi_0$$

Summing over j (and assuming $\mu > \lambda$):

$$\begin{aligned} 1 = \sum_{j=0}^{\infty} \pi_j &= \pi_0 \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = \pi_0 (1 - \lambda/\mu)^{-1} \quad \Rightarrow \pi_0 = 1 - \frac{\lambda}{\mu} \\ \pi_j &= \left(\frac{\lambda}{\mu}\right)^j (1 - \lambda/\mu), \qquad j = 0, 1, 2, \ldots \end{aligned}$$

i.e. equilibrium distribution is $\text{geo}(\lambda/\mu)$.
If $\lambda \geq \mu$ the process does not have an equilibrium (in the long term the population will explode).

N.b., q'Kendall notation' $M/M/n$ above comes from interpretation of birth/death processes as a queue.
This birth-death process is used as a model for the number of customers in a single server queue (known as the $M/M/1$) , in which

- arrivals occur in a Poisson process of rate λ:

$$M = \text{arrivals is Markov (Poisson)}$$

- the service times of the customers are i.i.d exponentially-distributed variables with mean $1/\mu$ that are independent of the arrival sequence.

$$M = \text{service times is Markov (i.e.} \sim \text{Expon)}$$

- Then $X(t)$ is the total number of customers in the system at time t including the customer being served, if any

$$n = \# \text{ of servers}$$

Remark 15

Recall, from Theorem 10: for an irreducible, stationary Markov chain:

$$\begin{array}{ccc} \textit{reversibility} & \Leftrightarrow & \textit{d.b.e} \\ & \Downarrow & \\ & \boldsymbol{\pi} \textit{ is e.d.} & \end{array}$$

An iff statement can be achieved with a weaker version of d.b.e

Consider irreducible, ctMc $\{X(t)\}$ with equilibrium distribution $\boldsymbol{\pi}$. I.e. $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, or:

$$\sum_{j \in \mathcal{S}} \pi_j q_{ji} = 0 \quad \forall\ i \in \mathcal{S}$$

then $\sum_{j \in \mathcal{S}\setminus\{i\}} \pi_j q_{ji} = -\pi_i q_{ii}$. But, (recall)

$$\sum_{j \in \mathcal{S}} q_{ij} = 0 \Leftrightarrow \sum_{j \in \mathcal{S}\setminus\{i\}} q_{ij} = -q_{ii}$$

then

$$\sum_{j \in \mathcal{S}\setminus\{i\}} \pi_j q_{ji} = \pi_i \sum_{j \in \mathcal{S}\setminus\{i\}} q_{ij} \qquad \text{[full balance eqns.]} \quad \Longrightarrow$$

For each state i, rate of transitions out of i = rate of transitions into i.

In discrete time we have a similar situation. The equilibrium distribution satisfies $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. For every i:

$$\pi_i = \pi_i \sum_{j \in \mathcal{S}} p_{ij}$$

since $\sum_{j \in \mathcal{S}} p_{ij} = 1$. We have

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \Leftrightarrow \pi_i = \sum_{j \in \mathcal{S}} \pi_j p_{ji}$$

i.e.

$$\begin{aligned} \pi_i \sum_{j \in \mathcal{S}} p_{ij} &= \sum_{j \in \mathcal{S}} \pi_j p_{ji} \\ \pi_i \sum_{j \in \mathcal{S} \setminus \{i\}} p_{ij} &= \sum_{j \in \mathcal{S} \setminus \{i\}} \pi_j p_{ji} \qquad \text{[full balance eqns.]} \end{aligned}$$

This shows that the probability of leaving state i exactly balances the probability of entering state i.

- The solutions of the detailed balance equations ($\pi_i\, q_{ij} = \pi_j\, q_{ji}$ or $\pi_i\, p_{ij} = \pi_j\, p_{ji}$) necessarily satisfy the full balance equations;
- these follow immediately by summing over j.

The connections between the properties of an irreducible Markov chain can be summarised as follows:

Theorem 16 (balance equations, equilibrium, and reversibility)

$$\begin{array}{ccc} \boldsymbol{\pi} \text{ is e.d} & \Leftrightarrow & \text{f.b.e} \\ \Uparrow & & \Uparrow \\ X \text{ is rev.} & \Leftrightarrow & \text{d.b.e.} \end{array}$$

In discrete time, an equilibrium distribution may not exist, even if the system is reversible. However, if it does, then the dbes determine it.