Exam for **Fundamental Theory of Statistical Inference, 2023.**
You should answer *all* parts.

(1) Consider point estimation of the value of a parameter θ known to be positive, $\theta > 0$. It is proposed to adopt a loss function which measures the loss in estimating a true θ by a as

$$L(\theta, a) = \frac{a}{\theta} - 1 - \log \frac{a}{\theta}.$$

Why is this a sensible loss function?
Find the general form, in terms of the posterior distribution, of the Bayes estimator for this loss function.

(2) Write $X \sim G(\alpha, \beta)$ to indicate that X has the Gamma distribution with density

$$f_X(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \ x > 0, \ \alpha, \beta > 0.$$

Define $Y = 1/X$ and write $Y \sim IG(\alpha, \beta)$.
Verify that Y has expectation $E(Y) = 1/\{(\alpha - 1)\beta\}, \ \alpha > 1, \ \beta > 0$.
Suppose that we observe $Y = (Y_1, \ldots, Y_n)$ $(n \geq 2)$ with the Y_i independent, identically distributed normal random variables with known mean μ and unknown variance θ, and suppose we adopt a prior distribution on θ which is $IG(\alpha, \beta)$, with known $\alpha > 0$ and $\beta > 0$.
What is the posterior distribution of θ? Find the explicit form of the Bayes estimator, $\delta_1(y)$ say, of θ under the loss function described in (1), and also of the Bayes estimator, $\delta_2(y)$, under squared error loss $L(\theta, a) = (\theta - a)^2$. Show that one estimator is strictly larger than the other, whatever the data outcome y.

(3) Let $Y_1, \ldots, Y_n$ be independent, identically distributed normal random variables, with known common variance 1 and unknown mean θ, and let $\bar{Y} = n^{-1} \sum_{i=1}^n Y_i$.
Consider the estimator of θ defined by

$$\tilde{\theta}_n = \begin{cases} \bar{Y} & \text{if } |\bar{Y}| \geq n^{-1/4} \\ t\bar{Y} & \text{if } |\bar{Y}| < n^{-1/4}, \end{cases}$$

where t is a constant with $|t| < 1$.
What is the distribution of $\sqrt{n}(\tilde{\theta}_n - \theta)$ in the limit as $n \to \infty$? Distinguish between the cases $\theta = 0$ and $\theta \neq 0$.
On the basis of this analysis, is $\tilde{\theta}_n$ preferable to $\bar{Y}$ as an estimator of θ?

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Verify that $R(\theta, \tilde{\theta}_n) = nE_\theta(\tilde{\theta}_n - \theta)^2$ can be expressed as

$$\begin{aligned} R(\theta, \tilde{\theta}_n) &= 1 + \{n\theta^2(t-1)^2 + t^2 - 1\}[\Phi\{a_n(\theta)\} - \Phi\{b_n(\theta)\}] \\ &\quad -2\theta t(t-1)n^{1/2}[\phi\{a_n(\theta)\} - \phi\{b_n(\theta)\}] \\ &\quad +(1-t^2)[a_n(\theta)\phi\{a_n(\theta)\} - b_n(\theta)\phi\{b_n(\theta)\}], \end{aligned}$$

where $a_n(\theta) = n^{1/4} - n^{1/2}\theta, b_n(\theta) = -n^{1/4} - n^{1/2}\theta$, in terms of the $N(0,1)$ density function $\phi(\cdot)$ and distribution function $\Phi(\cdot)$.

[Note the indefinite integral $\int z^2\phi(z)dz = \Phi(z) - z\phi(z) + C$].

Let $\theta_n = n^{-1/4}$. Show that $R(\theta_n, \tilde{\theta}_n) \to \infty$ as $n \to \infty$.

Deduce that

$$\lim_{n\to\infty} \frac{\sup_\theta E_\theta(\tilde{\theta}_n - \theta)^2}{\sup_\theta E_\theta(\bar{Y} - \theta)^2} = \infty.$$

On the basis of this further analysis, is $\tilde{\theta}_n$ preferable to $\bar{Y}$ as an estimator of θ?

[Recall that $z\phi(z) \to 0$ as $z \to \infty$].

(4) 'There are two main roles for conditioning in statistical inference'. Discuss, in no more than about 250–300 words.