LTCC Geometry and Physics: Exercise Sheet 1

1. On S^n define the coordinate neighbourhoods

$$\begin{aligned} U_{i+} &= \{(x_0, x_1, \ldots, x_n) \in S^n : x_i > 0\}, \\ U_{i-} &= \{(x_0, x_1, \ldots, x_n) \in S^n : x_i < 0\}. \end{aligned}$$

   Define the coordinate maps $\phi_{i\pm} \to \mathbb{R}^n$ via

$$\phi_{i\pm} = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

   For $n = 2$ calculate $\phi_{y-} \circ \phi_{x+}^{-1}$.

2. If f is a function on an open set $U \subset \mathbb{R}^n$, show that $d(df) = 0$.

3. Let $F\colon M = \mathbb{R}^2 \to N = \mathbb{R}^2$ be defined by $F(x_1, x_2) = (x_2 - x_1^3, x_1)$ (or, if you prefer, by saying $y_1 = x_2 - x_1^3$, etc.). If $\omega = y_1 dy_1 \wedge dy_2$, compute $F^*\omega$ (which should be a 2-form on M expressed using $dx_1 \wedge dx_2$—differentiating something may help).

4. Recall the pullback formula for metrics:

$$g^M_{\mu\nu}(x) = g^N_{\alpha\beta}(F(x)) \frac{\partial F^\alpha}{\partial x^\mu} \frac{\partial F^\beta}{\partial x^\nu}.$$

   Let $F\colon M = \mathbb{R}^2 \to N = \mathbb{R}^2$ be defined by $F(r, \phi) = (r\cos\phi, r\sin\phi)$. Compute the induced metric on M. Also calculate the induced map of the sphere $S^2 \subset \mathbb{R}^3$ for both polar coordinates and stereographic coordinates.

5. Recall that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

   is a basis of $(T_pM)^{\mathbb{C}}$ (I omit indices z^μ on the variables). Show that $dz = dx + idy$ and $d\bar{z} = dx - idy$ make a dual basis.

6. Show how the Hopf bundle can be derived by considering $z_1, z_2 \in \mathbb{C}$, with $|z_1|^2 + |z_2|^2 = 1$, as the total space, and $[z_1, z_2]$ as homogeneous coordinates of the base space $\mathbb{C}P^1 \cong S^2$. Here homogeneous coordinates are defined via the equivalence relation

$$[z_1, z_2] = [\lambda z_1, \lambda z_2] \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

   Find two suitable coordinate charts for S^2 and write down the corresponding local trivialisations. Give the projections. What are the transition functions?

Dr Steffen Krusch, 2019.

LTCC Geometry and Physics: Exercise Sheet 2

(Note: Unless otherwise stated, the Einstein summation convention is assumed throughout.)

1. Given a Riemannian manifold M with metric g, consider the action $S = \int_{t_0}^{t_1} L\, dt$ with the purely kinetic Lagrangian

$$L = \frac{1}{2} g(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = \frac{1}{2} g_{jk} \dot{x}^j \dot{x}^k, \tag{1}$$

where $g = (\, g_{jk}(\mathbf{x})\,)$ is the metric given in terms of local coordinates x^j for a point $\mathbf{x} \in M$ with tangent vector $\dot{\mathbf{x}}$; the dot denotes differentiation w.r.t. the parameter t along a curve.
(i) Applying the principle of least action, $\delta S = 0$, calculate the Euler-Lagrange equations for this Lagrangian, and show that they can be written in the form

$$\ddot{x}^j + \Gamma^j_{kl} \dot{x}^k \dot{x}^l = 0, \tag{2}$$

where Γ^j_{kl} (the Christoffel symbols) should be found in terms of g and its derivatives.
(ii) Perform a Legendre transformation and hence reformulate the geodesic equations as an Hamiltonian system on T^*M.

2. Write down the Euclidean metric on $\mathbb{R}^3$ in Cartesian coordinates, and the invariant volume form. Use this to calculate the invariant 2-form on the sphere S^2, and show that this is a symplectic form.

3.(i) For a pair of vector fields X, Y on a smooth manifold M of dimension d, define their commutator $[X, Y]$ (or Lie bracket) by the commutator of the corresponding differential operators. Show that with this definition the vector fields form a Lie algebra.
(ii) Now suppose that $d = 2n$ and (M, ω) is a symplectic manifold. For Hamiltonian vector fields X_G, X_H corresponding to smooth functions G, H, show that the following formula holds:

$$[X_G, X_H] = -X_{\{G,H\}},$$

where $\{,\}$ denotes the Poisson bracket. (Hint: Use Darboux's theorem.)
(iii) Prove that the Hamiltonian vector fields form a Lie subalgebra of the vector fields.

4.(i) Check that the formula

$$\{\pi_j, \pi_k\} = -\epsilon_{jkl} \pi_l$$

defines a Poisson bracket on $\mathbb{R}^3$ with coordinates (π_1, π_2, π_3), and show that (on linear functions) this is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of the rotation group in three dimensions.
(ii) Write down the Poisson tensor for the above bracket and calculate its rank. Show that $C = \pi_1^2 + \pi_2^2 + \pi_3^2$ is a Casimir for this bracket, and describe the symplectic leaves.
(iii) Consider the Hamiltonian system on $\mathbb{R}^3$ defined by the Hamiltonian

$$H = \frac{\pi_1^2}{2I_1} + \frac{\pi_2^2}{2I_2} + \frac{\pi_3^2}{2I_3}.$$

Write down the equations of motion, and show that they are equivalent to the Euler top (free motion of a rigid body about a fixed point) in terms of the angular momentum $\boldsymbol{\pi}$ and angular velocity $\boldsymbol{\omega}$, where $\boldsymbol{\pi} = I\boldsymbol{\omega}$ with the inertia tensor $I = \mathrm{diag}(I_1, I_2, I_3)$. Can you describe the orbits?

Professor Andy Hone, January 2013.