LTCC Geometry and Physics: Exercise Sheet 4

Homotopy theory

1. Let α be the loop

$$\alpha : [0,1] \to \mathbb{R}^2 \setminus \{0\} : t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}.$$

By constructing a suitable homotopy show that $[\alpha * \alpha^{-1}] = [c]$ where

$$c : [0,1] \to \mathbb{R}^2 \setminus \{0\} : t \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2. Let α, β and γ be three loops. Show that

$$[(\alpha * \beta) * \gamma] = [\alpha * (\beta * \gamma)].$$

3. Consider the equivalence relation in the $\mathbb{R}^2$-plane

$$\mathbf{x} \sim \mathbf{y} \quad \text{if} \quad \mathbf{x} = \mathbf{y} + \begin{pmatrix} k \\ l \end{pmatrix}, \quad \text{where} \quad k, l \in \mathbb{Z}.$$

The equivalence classes form the torus T^2. Let

$$\alpha : [0,1] \to T^2 : t \mapsto \begin{pmatrix} 0 \\ t \end{pmatrix} \quad \text{and} \quad \beta : [0,1] \to T^2 : t \mapsto \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

   (a) Think of the torus as a donut and sketch the loops α and β.

   (b) Show that for the torus $[\alpha * \beta * \alpha^{-1} * \beta^{-1}] = [c]$ for a suitable constant path c. What does this mean? (Hint: $[\alpha * \beta] =?$)

Vortices

5. Show that

$$V = \frac{1}{2} \int \left(B^2 + \overline{D_i \phi} D_i \phi + \frac{\lambda}{4} \left(1 - \overline{\phi} \phi\right)^2 \right) \mathrm{d}^2 x,$$

is gauge invariant under

$$\begin{aligned} \phi &\mapsto e^{i\alpha} \phi \\ a_i &\mapsto a_i + \partial_i \alpha \end{aligned}$$

where $D_i = \partial_i \phi - i a_i \phi$, and $B = \partial_1 a_2 - \partial_2 a_1$.

6. The vector potential is a one form

$$a = a_1 dx + a_2 dy = a_\rho d\rho + a_\theta d\theta.$$

Show how a_ρ and a_θ are related to a_1 and a_2. For the field strength $f = da$ calculate f_{12} and $f_{\rho\theta}$ where

$$f = f_{12} dx \wedge dy = f_{\rho\theta} d\rho \wedge d\theta.$$

7. Show that the gauge transformation

$$\phi \to e^{i\theta}\phi$$

is not continuous.

8. Derive the equations of motions for $L = T - V$

   (a) for the relativistic Lagrangian

   $$T = \frac{1}{2}\int \left(e_1^2 + e_2^2 + \overline{D_0\phi}D_0\phi\right) \mathrm{d}^2x,$$

   where $e_i = \partial_0 a_i - \partial_i a_0$ are the components of the electric field.

   (b) For the Schrödinger-Chern-Simons Lagrangian

   $$T = \int \left(\frac{i}{2}\left(\bar{\phi}D_0\phi - \phi\overline{D_0\phi}\right) + Ba_0 + e_1 a_2 - e_2 a_1 - a_0\right) \mathrm{d}^2x.$$

9. Show that inserting $h = 2g + 2\log\left(\frac{1}{2}(1 - |z|^2)\right)$, into

$$\nabla^2 h + \Omega - \Omega e^h = 4\pi \sum_{r=1}^{N} \delta^2(z - Z_r)$$

where

$$\Omega = \frac{8}{(1 - |z|^2)^2}.$$

gives rise to Liouville's equation

$$\nabla^2 g - e^{2g} = 2\pi \sum_{r=1}^{N} \delta^2(z - Z_r).$$

The solution of Liouville's equation is

$$g = -\log\left(\frac{1}{2}(1 - |f|^2)\right) + \frac{1}{2}\log\left|\frac{df}{dz}\right|^2,$$

where $f(z)$ is an arbitrary complex function.

Take

$$f(z) = \left(\frac{z - Z}{1 - \bar{Z}z}\right)^2$$

and calculate the metric for a single vortex using the formula from lecture 5. Show that the Kähler potential for this metric is proportional to $\log(1 - |z|^2)$.

Dr Steffen Krusch, February 2019.